Fusion, fission, and annihilation of complex waves for the (2+1)-dimensional generalized Calogero–Bogoyavlenskii–Schiff system*

Zhu Wei-Ting(朱维婷), Ma Song-Hua(马松华)†, Fang Jian-Ping(方建平), Ma Zheng-Yi(马正义) and Zhu Hai-Ping(朱海平)

College of Science, Lishui University, Lishui 323000, *China*

(Received 12 November 2013; revised manuscript received 12 December 2013; published online 20 April 2014)

With the help of the symbolic computation system, Maple and Riccati equation ($\xi' = a_0 + a_1\xi + a_2\xi^2$), expansion method, and a linear variable separation approach, a new family of exact solutions with $q = lx + my + nt + \Gamma(x,y,t)$ for the (2+1)-dimensional generalized Calogero–Bogoyavlenskii–Schiff system (GCBS) are derived. Based on the derived solitary wave solution, some novel localized excitations such as fusion, fission, and annihilation of complex waves are investigated.

Keywords: Riccati equation expansion method, GCBS system, exact solutions, fusion, fission and annihilation of complex waves

PACS: 05.45.Yv, 03.65.Ge **DOI:** 10.1088/1674-1056/23/6/060505

1. Introduction

The interactions between solitons of integrable models are usually considered to be completely elastic. That is to say, the amplitude, velocity, and wave shape of solitons are not changed after nonlinear interaction.[1] However, for some specific solutions of (2+1)-dimensional models, the interactions among solitonic excitations are not completely elastic since their shapes are changed after their collisions.[2] Furthermore, for some nonlinear models, two or more solitons may fuse into one soliton at a specific time while sometimes one soliton may fission into two or more solitons at another specific time.[3] These phenomena are often called soliton fusion and soliton fission, respectively, and they have been observed in many kinds of physical systems, such as some organic membranes and macromolecular materials, and many physical fields like plasma physics, nuclear physics, and hydrodynamics.[4,5] Recently, Wang *et al*. were the first to discuss two (1+1)-dimensional models, the Burgers equation, and Sharma–Tasso–Olver (STO) equation, via Hirota's direct method and the Bäcklund transformation, and they found the soliton fission and soliton fusion phenomena.[3] Furthermore, Ma *et al*. also found these phenomena in the (2+1)-dimensional higher-order Broek–Kaup (HBK) system.[6] Along this line, in this paper, we further study the fusion, fission, and annihilation phenomena of complex waves in the following (2+1)-dimensional generalized Calogero–Bogoyavlenskii–Schiff (GCBS) system

$$\alpha V_{xt} + \gamma V_x V_{xy} + \beta V_y V_{xx} + V_{xxxy} = 0, \tag{1}$$

where, $V \equiv V(x,y,t)$, $\alpha \neq 1$, γ, β are the two arbitrary constants. This system has been studied by many scholars with different methods.[7,8] When $\alpha = 4$, $\gamma = 4$, $\beta = 2$, equation (1) can be changed to Bogoyavlenskii–Schiff system.[9] When $\alpha = 4$, $\gamma = 8$, $\beta = 4$, equation (1) can be changed to Calogero–Bogoyavlenskii–Schiff system.[10] Zhang, Chen and Li obtained the integrability condition ($\gamma = 2\beta$) of GCBS system by Painlevé test. Based on this condition and Painlevé test, the bilinear form of GCBS system is found.[11]

In the following discussion, we will investigate the case $\gamma = 2\beta$ for Eq. (1), namely

$$\alpha V_{xt} + 2\beta V_x V_{xy} + \beta V_y V_{xx} + V_{xxxy} = 0. \tag{2}$$

2. Exact solutions of the GCBS system

As is well known, to search for the solitary wave solutions of a nonlinear physical model, we can apply several different approaches.[12–27] One of the most efficient methods of finding soliton excitations of a physical model is the so-called mapping approach.[24,28–38] Previously, Mei and Zhang have obtained exact traveling wave solutions for the Gross–Pitavskii system with the Riccari equation ($\xi' = a_0 + a_1\xi + a_2\xi^2$) expansion method.[39] In this section, we will extend this method to construct complex wave solutions with $q = lx + my + nt + \Gamma(x,y,t)$ of Eq. (2). The basic ideal of the algorithm is as follows. For a given nonlinear partial differential equation (NPDE) with the independent variables $x = (x_0 = t, x_1, x_2, \ldots, x_m)$, and the dependent variable u, in the form

$$P(u, u_t, u_{x_i}, u_{x_i x_j}, \ldots) = 0, \tag{3}$$

*Project supported by the National Natural Science Foundation of China (Grant No. 11375079), the Scientific Research Fund of Zhejiang Provincial Education Department of China (Grant No. Y 201120994), and the Natural Science Foundation of Zhejiang Province, China (Grant Nos. Y6100257, LY14A010005, and Y6110140).

†Corresponding author. E-mail: msh6209@aliyun.com

© 2014 Chinese Physical Society and IOP Publishing Ltd

where P is in general a polynomial function of its arguments, and the subscripts denote the partial derivatives, the solution can be assumed to be in the form

$$u = A(x) + \sum_{i=1}^{n} \left\{ B_i(x)\xi^i[q(x)] + \frac{C_i(x)}{\xi^i[q(x)]} \right\}, \tag{4}$$

with

$$\xi' = a_0 + a_1\xi + a_2\xi^2, \tag{5}$$

and

$$q = lx + my + nt + \Gamma(x,y,t), \tag{6}$$

where a_0, a_1, a_2, l, m, n are constants, $\Gamma \equiv \Gamma(x,y,t)$ is an arbitrary function of (x,y,t). Substituting Eqs. (4), (5), and (6) into the given NPDE and collecting coefficients of polynomials of ξ, and then eliminating each coefficient to derive a set of equations of A, B_i, C_i, and q, and solving these equations to obtain A, B_i, C_i, and q, finally, as equation (5) is known to possess the solutions,[39] one obtains the complex solutions to the given NPDE.

Now we apply the Riccati equation expansion method to Eq. (2). By the balancing procedure, ansatz (4) becomes

$$V = A(x,y,t) + B(x,y,t)\xi(q(x,y,t)) + \frac{C(x,y,t)}{\xi(q(x,y,t))}, \tag{7}$$

where A, B, C, and q are functions of (x,y,t) to be determined. By substituting Eqs. (7) and (5) into Eq. (2) and collecting coefficients of polynomials of ξ, then setting each coefficient to be zero, we have

$$A = \frac{1}{2\beta}\int \frac{-\alpha q_x^2 q_t + 16a_0a_2q_x^4q_y + 4a_1q_x^2q_yq_{xx} + a_1^2q_x^4q_y + 2q_xq_yq_{xxx} - q_{xx}q_y}{q_x^2q_y}\,\mathrm{d}x,$$

$$B = -\frac{4a_2q_x}{\beta}, \quad C = \frac{4a_0q_x}{\beta}, \tag{8}$$

with

$$q = lx + my + nt + \Gamma, \quad \Gamma = \chi(x) + \varphi(y - ct). \tag{9}$$

Based on the solutions of Eq. (5), we can derive the following complex wave solutions of Eq. (2).

Case 1 Solitary wave solutions

$$V_1 = A + B\tanh(q) + \frac{C}{\tanh(q)}, \tag{10}$$

$$V_2 = A + B\coth(q) + \frac{C}{\coth(q)}, \tag{11}$$

$$V_3 = A + B(\coth(q) + \operatorname{csch}(q)) + \frac{C}{\coth(q) + \operatorname{csch}(q)}, \tag{12}$$

$$V_4 = A + B(\tanh(q) + I\operatorname{sech}(q)) + \frac{C}{\tanh(q) + I\operatorname{sech}(q)}, \tag{13}$$

$$V_5 = A + B\frac{\coth(q)}{1 + I\operatorname{csch}(q)} + \frac{C}{\dfrac{\coth(q)}{1 + I\operatorname{csch}(q)}}, \tag{14}$$

$$V_6 = A + B\frac{\tanh(q)}{1 + \tanh^2(q)} + \frac{C}{\dfrac{\tanh(q)}{1 + \tanh^2(q)}}, \tag{15}$$

with $q = lx + my + nt + \Gamma$, $\Gamma = \chi(x) + \varphi(y - ct)$. A, B, C are the same as Eq. (8).

Case 2 Periodic wave solutions

$$V_{16} = A + B\tan(q) + \frac{C}{\tan(q)}, \tag{16}$$

$$V_8 = A + B\cot(q) + \frac{C}{\cot(q)}, \tag{17}$$

$$V_9 = A + B(\tan(q) + \sec(q)) + \frac{C}{\tan(q) + \sec(q)}, \tag{18}$$

$$V_{10} = A + B(\csc(q) - \cot(q)) + \frac{C}{\csc(q) - \cot(q)}, \tag{19}$$

$$V_{11} = A + B\frac{\cot(q)}{1 - \csc(q)} + \frac{C}{\dfrac{\cot(q)}{1 - \csc(q)}}, \tag{20}$$

$$V_{12} = A + B\frac{\tan(q)}{1 - \tan^2(q)} + \frac{C}{\dfrac{\tan(q)}{1 - \tan^2(q)}}, \tag{21}$$

with $q = lx + my + nt + \Gamma$, $\Gamma = \chi(x) + \varphi(y - ct)$. A, B, C are the same as Eq. (8).

Case 3 Variable separated solution

$$V_{13} = A - B\frac{1}{aq + k} - \frac{C}{1/(aq + k)}, \tag{22}$$

with $q = lx + my + nt + \Gamma$, $\Gamma = \chi(x) + \varphi(y - ct)$. A, B, C are the same as Eq. (8), a and k are two constants.

3. Fusion, fission, and annihilation of complex waves

Due to the functions $q = lx + my + nt + \Gamma$, $\Gamma = \chi(x) + \varphi(y - ct)$ included in the above cases, the physical quantities V may possess rich localized structures. In this section, we mainly discuss some complex wave localized coherent excitations from the potential of solitary wave solutions determined by Eq. (12) in Case 1 and rewrite them in a simple form, namely

$$V = V_{3x} = \frac{(q_{xx}q_y - q_x^2q_t - 2q_yq_xq_{xxx})\cosh^2(q) + 8q_{xx}q_x^2q_y\sinh(q)\cosh(q) - \alpha q_xq_t - q_{xx}^2q_y}{2\beta q_x^2q_y(\cosh^2(q) - 1)}, \tag{23}$$

here, $q = lx + my + nt + \Gamma$, $\Gamma = \chi(x) + \varphi(y - ct)$.

Now we focus our attention on these intriguing fusion, fission, and annihilation phenomena for the complex wave solution V, which may exist under certain circumstances. For instance, when we select the arbitrary functions χ and φ to be

$$\chi = 1 + \tanh(x), \quad \varphi = 1 + \exp(y + ct) + \operatorname{sech}(y - ct), \quad (24)$$

and substitute Eq. (24) into Eq. (23), we can obtain a new kind of solitary solution of Eq. (2). Figure 1 shows an evolutional profile corresponding to the physical quantity V of the two dromion complex waves expressed by Eq. (23) with fixed parameters $\alpha = 2,\ \beta = 1,\ c = 1,\ l = 0.5,\ m = 0.5,\ n = 0.5$ at times $t = -10,\ -5,\ 0,\ 3, 10$, exhibiting a fusion phenomenon for the two dromion complex waves. From Fig. 1, we can clearly see that the two single dromion complex waves finally fuse to one complex wave.

Along this line, since we consider χ and φ to be,

$$\chi = 1 + \operatorname{sech}(x), \quad \varphi = 1 + 2\operatorname{sech}(y + ct) + \exp(y - ct), \quad (25)$$

we can obtain another new type of solitary excitation with apparently different properties presented in Fig. 2 with fixed parameters $\alpha = 2, \beta = 1, c = 1, l = 0.6, m = 0.6, n = 0.6$ at times $t = -10,\ 0,\ 2,\ 6$, comparing with the fusion phenomenon in Fig. 1. From Fig. 2, we can find that one single dromion complex wave fissions into two complex waves.

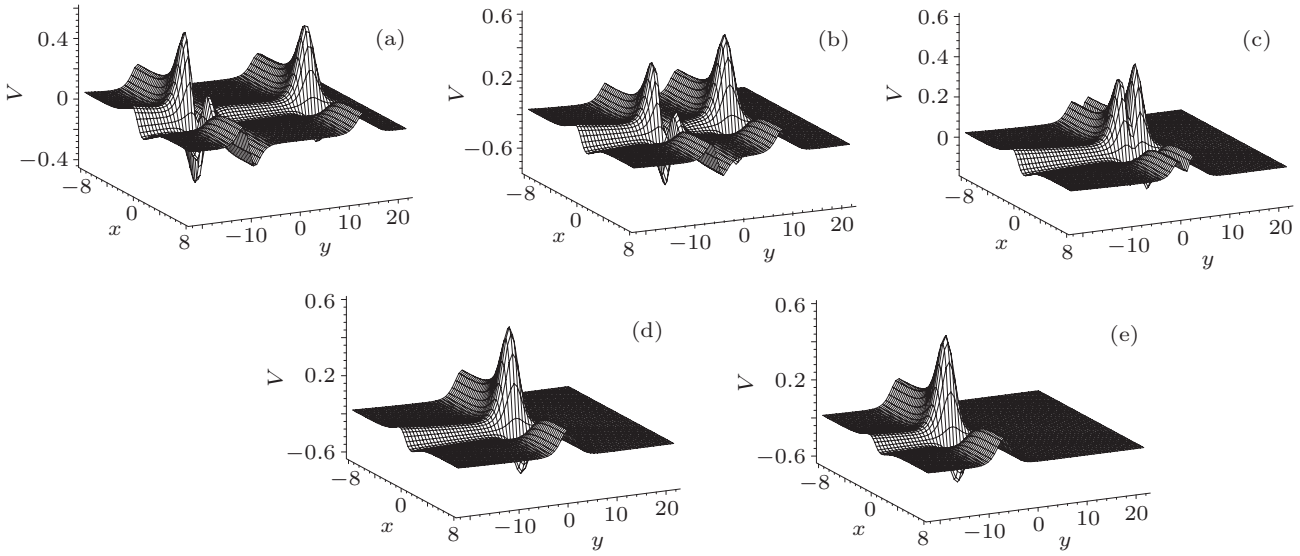


Fig. 1. The evolutional profile of two dromion complex waves fusion into one dromion complex wave for the solution V expressed by Eq. (23) with condition (24) at different times (a) $t = -10$, (b) $t = -5$, (c) $t = 0$, (d) $t = 3$, (e) $t = 10$, respectively.

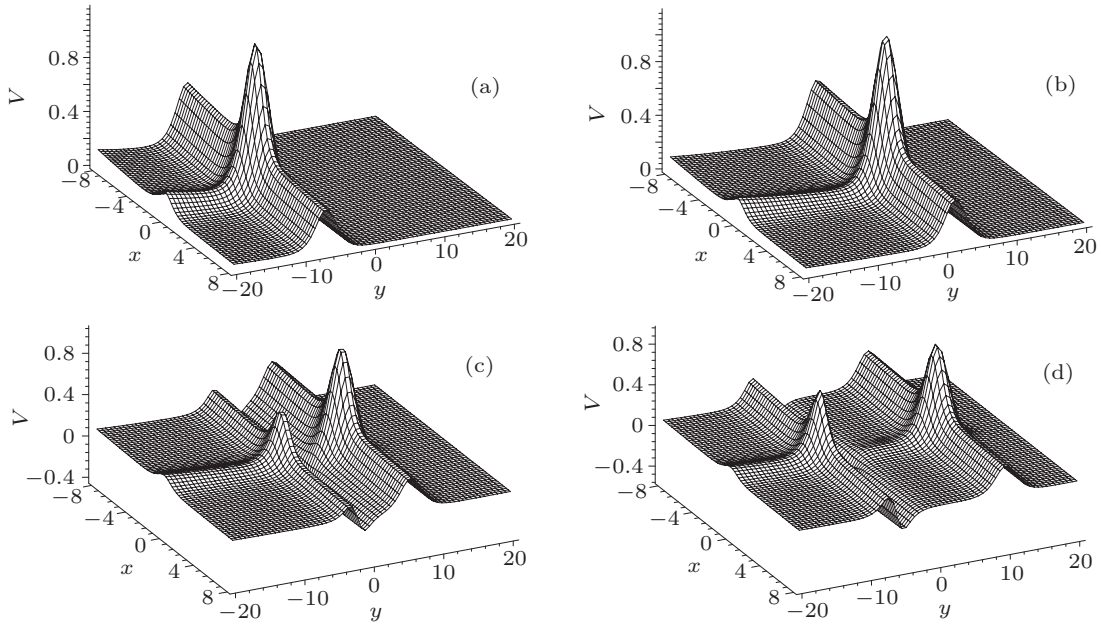


Fig. 2. One single dromion complex wave fissions into two dromion complex waves evolutional plot of the solution V expressed by Eq. (23) with condition (25) at different times (a) $t = -10$, (b) $t = 0$, (c) $t = 2$, (d) $t = 6$, respectively.

Similar to other particles, solitons can also be annihilated in some appropriate conditions. For example, by choosing χ and φ in solution (23) to be

$$\chi = 1 + 2\mathrm{sech}(x^2), \quad \varphi = 1 + 2\mathrm{sech}(y - ct)^2, \tag{26}$$

we can see the annihilation of dromion complex waves for the physical quantity V of Eq. (23) under the condition (26) presented in Fig. 3 with fixed parameters $\alpha = 2$, $\beta = 1$, $c = 1$, $l = 0.8$, $m = 0.8$, $n = 0.8$ at times $t = -10, -5, -2, 0, 2, 10$. From Fig. 3, we find that the amplitude and shape of the dromion complex waves become smaller and smaller after interactions. Finally, they reduce to zero.

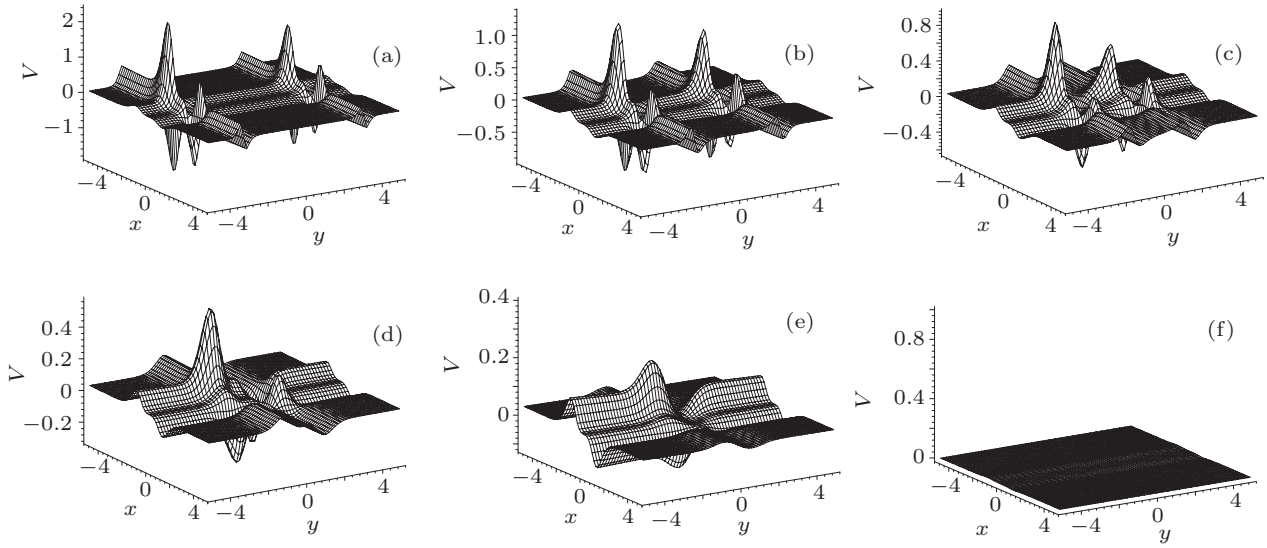


Fig. 3. A plot of the annihilation of dromion complex waves for the solution V by Eq. (23) under the condition (26) at different times (a) $t = -10$, (b) $t = -5$, (c) $t = -2$, (d) $t = 0$, (e) $t = 3$, (f) $t = 10$, respectively.

4. Summary and discussion

Previously, Mei and Zhang have obtained the exact traveling wave solutions for nonlinear evolution equations, such as Gross–Pitavskii equation with the Riccati equation expansion method.[39] In this paper, we extend this approach and construct complex wave solutions of the (2+1)-dimensional generalized Calogero–Bogoyavlenskii–Schiff system. Based on the derived potential of solitary wave solution (12), we have studied the fusion, fusion, and annihilation phenomena of complex waves. We can see that solitons have the same characteristics of other particles in many aspects. Because of the wide applications of the soliton theory, to learn more about the localized excitations and their applications in reality is worth further study.

Acknowledgments

The authors would like to thank professors Lou Sen-Yue, Hu Xing-Biao, Zhang Da-Jun and Chen Yong for fruitful and helpful suggestions.

References

[1] Ma S H, Fang J P and Zhu H P 2007 *Acta Phys. Sin.* **56** 4319 (in Chinese)
[2] Ma S H, Ren Q B, Fang J P and Zheng C L 2007 *Acta Phys. Sin.* **56** 6777 (in Chinese)
[3] Wang S, Tang X Y and Lou S Y 2004 *Chaos, Solitons and Fractals* **19** 769
[4] Serkin V N 2001 *Opt. Commun.* **192** 237
[5] Stoitchena G 2001 *Math. Comput. Simul.* **55** 621
[6] Ma Z Y and Zheng C L 2005 *Commun. Theor. Phys.* **43** 994
[7] Toda K, Yu S J and Fukuyama F 1999 *Rep. Math. Phys.* **44** 247
[8] Yan Z Y 2003 *Czech. J. Phys.* **53** 89
[9] Bogoyavlenskii O I 1990 *Math. USSR. Izv.* **34** 245
[10] Estvez P G and Hernaez G A 2000 *J. Phys. A: Math. Gen.* **33** 2131
[11] Zhang H P, Chen Y and Li B 2009 *Acta Phys. Sin.* **58** 7393 (in Chinese)
[12] Lou S Y 1995 *J. Phys. A: Math. Gen.* **28** 7227
[13] Lou S Y, Tang X Y and Li J 2001 *Euro. Phys. J. B* **22** 473
[14] Zhang S L, Zhu X N, Wang Y M and Lou S Y 2008 *Commun. Theor. Phys.* **49** 829
[15] Zhang S L and Lou S Y 2007 *Commun. Theor. Phys.* **48** 385
[16] Zhang J F, Huang W H and Zheng C L 2002 *Acta Phys. Sin.* **51** 2676 (in Chinese)
[17] Zhang J F 2002 *Commun. Theor. Phys.* **37** 277
[18] Dai C Q and Ni Y Z 2006 *Phys. Scr.* **74** 584
[19] Dai C Q and Zhou G Q 2007 *Chin. Phys.* **16** 1201
[20] Dai C Q and Zhu H P 2013 *J. Opt. Soc. Am. B* **30** 3291
[21] Ma S H and Fang J P 2006 *Acta Phys. Sin.* **55** 5611 (in Chinese)
[22] Ma S H, Wu X H, Fang J P and Zheng C L 2006 *Z. Natur. Forsch.* **61a** 249
[23] Ma S H, Fang J P and Zheng C L 2008 *Chin. Phys. B* **17** 2767
[24] Ma S H, Wu X H, Fang J P and Zheng C L 2008 *Acta Phys. Sin.* **57** 11 (in Chinese)
[25] Ma S H, Qiang J Y and Fang J P 2007 *Acta Phys. Sin.* **56** 620 (in Chinese)
[26] Taogetusang and Sirendaoerji 2009 *Acta Phys. Sin.* **58** 2121 (in Chinese)

[27] Taogetusang and Sirendaoerji 2009 *Acta Phys. Sin.* **58** 5887 (in Chinese)
[28] Li B Q, Ma Y L and Xu M B 2010 *Acta Phys. Sin.* **59** 1409 (in Chinese)
[29] Ma Y L, Li B Q and Sun J Z 2009 *Acta Phys. Sin.* **58** 7042 (in Chinese)
[30] Fang J P, Zheng C L and Zhu J M 2005 *Acta Phys. Sin.* **54** 2990 (in Chinese)
[31] Fang J P and Zheng C L 2005 *Acta Phys. Sin.* **54** 670 (in Chinese)
[32] Ma S H, Fang J P, Hong B H and Zheng C L 2009 *Chaos, Solitons and Fractals* **40** 1352
[33] Ma S H and Fang J P 2009 *Z. Naturforsch* **64a** 37
[34] Ma S H, Fang J P and Zheng C L 2009 *Chaos, Solitons and Fractals* **40** 210
[35] Ma S H, Fang J P and Ren Q B 2010 *Acta Phys. Sin.* **59** 4420 (in Chinese)
[36] Yang Z, Ma S H and Fang J P 2011 *Chin. Phys. B* **20** 040301
[37] Ma S H, Fang J P, Ren Q B and Yang Z 2012 *Chin. Phys. B* **21** 050511
[38] Ma S H, Fang J P and Wu H Y 2013 *Z. Natur Forsch.* **68a** 350
[39] Mei J Q and Zhang H Q 2005 *Commun. Theor. Phys.* **44** 209