Abundant solutions of Wick-type stochastic fractional 2D KdV equations

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A modified fractional sub-equation method is applied to Wick-type stochastic fractional two-dimensional (2D) KdV equations. With the help of a Hermit transform, we obtain a new set of exact stochastic solutions to Wick-type stochastic fractional 2D KdV equations in the white noise space. These solutions include exponential decay wave solutions, soliton wave solutions, and periodic wave solutions. Two examples are explicitly given to illustrate our approach.

Keywords: 2D KdV equations, fractional calculus, white noise, Hermite transform

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1. Introduction

In this paper, we use the Hermite transform and modified fractional sub-equation method to study Wick-type stochastic fractional 2D KdV equations, as follows:

$$D_x^{\alpha}(D_t^{\alpha}U+6U\diamond D_x^{\alpha}U+D_x^{3\alpha}U)+A(t)\diamond D_x^{\alpha}U+B(t)\diamond D_y^{2\alpha}U=0, \tag{1}$$

where $(x,y,t)\in\mathbb{R}^2\times\mathbb{R}_+$, $0<\alpha\leq 1$, D_x^{α}, D_y^{α}, and D_t^{α} are the modified Riemann–Liouville derivatives with respect to x,y,t, respectively, "$\diamond$" is the Wick product on the Kondratiev distribution space $(\mathcal{S})_{-1}$, which was defined in Ref. [1] and $A(t)$, $B(t)$ are nonzero white noise functions. Moreover, when Wick product "$\diamond$" is replaced by ordinary product in Eq. (1), we obtain the variable coefficients fractional 2D KdV equations:

$$D_x^{\alpha}(D_t^{\alpha}u+6uD_x^{\alpha}u+D_x^{3\alpha}u)+a(t)D_x^{\alpha}u+b(t)D_y^{2\alpha}u=0,\quad 0<\alpha\leq 1, \tag{2}$$

where $a(t),b(t)$ are bounded measurable or integrable functions on $\mathbb{R}_+$. Equation (2) describes the anomalous propagation of quasi-one-dimensional waves in a nonlinear weakly dispersive medium; the sign of the function $b(t)$ coincides with the sign of the dispersion.[2] Moreover, if the problem is considered in a random environment, we can get random fractional 2D KdV equations. In order to give the exact stochastic solutions of the random fractional 2D KdV equations, we only consider this problem in a white noise environment, that is, we will study the variable coefficients Wick-type stochastic fractional 2D KdV equations (1).

The analytical and approximate solutions of deterministic fractional partial differential equations attract great attention and are becoming a considerably interesting subject in mathematical physics. There are many methods for calculating the approximate solutions for nonlinear fractional partial differential equations, such as the variational iterations method,[3] Adomian decomposition method,[4,5] the homotopy perturbation method,[6,7] and the expansion-function method.[8–12] The exact solutions for nonlinear fractional partial differential equations are still under study. Li and He[13] introduced complex transformation for reducing fractional differential equations into ordinary differential equations, so that all analytical methods for advanced calculus can be easily applied to fractional calculus. Since Wadati first introduced and studied stochastic KdV equation,[14] many authors, such as, Xie,[15–18] Chen and Xie,[19–21] Ghany,[22] Ghany and Saad,[23] and many other authors,[24,25] have more intensively investigated stochastic partial differential equations (SPDE).

Broadly speaking, we generalize the work done by Elwakil *et al.*[26] and Zhu *et al.*[27] on the variable coefficients 2D KdV equations to new strategies that give white noise functional solutions of the variable coefficients Wick-type stochastic fractional 2D KdV equations. The strategies that will be pursued in our work rest mainly on Hermite transform, white noise theory, and modified fractional sub-equation method, all of which are employed to find white noise functional solutions of Eq. (1). The proposed schemes, as we believe, are entirely new and introduce new solutions in addition to the well-known traditional solutions. The ease of using these methods, to determine shock or solitary type of solutions, shows their power.

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2. Preliminaries

In this section, we recall the definition and some properties of the modified Riemann–Liouville derivative and introduce the modified fractional sub-equation method. There are different definitions for fractional derivatives, for more details see Ref. [28]. In our paper we use the modified Riemann–Liouville derivative as defined by Jumarie[29]

$$D_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^x (x-y)^{-\alpha-1}[f(y)-f(0)]\,\mathrm{d}y, \\ \qquad\qquad \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_0^x (x-y)^{-\alpha}[f(y)-f(0)]\,\mathrm{d}y, \\ \qquad\qquad 0 < \alpha < 1, \\ \left[f^{(\alpha-n)}(x)\right]^{(n)}, \quad n \le \alpha < n+1, \ n \in \mathbb{N}, \end{cases} \tag{3}$$

which has merits over the original; for example, the α-order derivative of a constant is zero. Some properties of the modified Riemann–Liouville derivative were summarized in Ref. [29], three useful formulas from this reference are given below

$$\begin{cases} D_x^\alpha x^\beta = \dfrac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \quad \beta > 0, \\ D_x^\alpha [u(x)v(x)] = u(x) D_x^\alpha v(x) + v(x) D_x^\alpha u(x), \\ D_x^\alpha [f(u(x))] = \dfrac{\mathrm{d}f}{\mathrm{d}u} D_x^\alpha u(x) = \left(\dfrac{\mathrm{d}u}{\mathrm{d}x}\right)^\alpha D_u^\alpha f(u). \end{cases} \tag{4}$$

In the following, we outline the main idea of the modified fractional sub-equation method. Consider a fractional partial differential equation, say, in three variables x, y, and t:

$$F(u, u_x, u_y, u_t, D_x^\alpha u, D_y^\alpha u, D_t^\alpha u, \ldots) = 0, \quad 0 < \alpha \le 1, \tag{5}$$

where F is a nonlinear function with respect to the indicated variables. To determine the solution $u = u(x,y,t)$ explicitly, we first introduce the following transformation:

$$u = u(\xi), \quad \xi = \xi(x,y,t), \tag{6}$$

which converts Eq. (5) into a fractional ordinary differential equation:

$$G(u, u', u'', D_\xi^\alpha u, D_\xi^{2\alpha} u, \ldots) = 0. \tag{7}$$

Next we introduce a new variable $\Phi = \Phi(\xi)$ which is a solution of the fractional Riccati equation:

$$D_\xi^\alpha \Phi(\xi) = \lambda_1 + \lambda_2 \Phi(\xi) + \lambda_3 \Phi^2(\xi), \quad 0 < \alpha \le 1, \tag{8}$$

where λ_1, λ_2, and λ_3 are arbitrary constants. Equation (8) is the fractional Riccati differential equation, where α is a parameter describing the order of the fractional derivative. In the case of $\alpha = 1$, equation (8) is reduced to the classical Riccati differential equation. The importance of this equation usually arises in the optimal control problems. The feedback gain of the linear quadratic optimal control depends on a solution of a Riccati differential equation, which has to be found for the whole time horizon of the control process.[30,31] We next suppose that equation (5) has a series expansion solution in the form:

$$u(x,y,t) = u(\xi) = \sum_{i=0}^{n} p_i(t) \Phi^i(\xi) + \sum_{i=1}^{n} q_i(t) \Phi^{-i}(\xi), \tag{9}$$

where p_i, q_i are functions to be determined later, n is a positive integer that can be determined via the balancing of the highest derivative term with the nonlinear term in Eq. (7). Inserting Eq. (9) into Eq. (7) and using Eq. (8) will give an algebraic equation in powers of Φ. Since all coefficients of Φ^i must vanish, this will give a system of algebraic equations with respect to p_i, q_i. With the aid of Mathematica, we can determine p_i, q_i. From the recent paper by Zhang *et al.*,[31] we can deduce the following set of solutions of Eq. (8):

$$\begin{cases} \Phi(\xi) = E_\alpha(\xi) - 1, \qquad \lambda_1 = \lambda_2 = 1, \ \lambda_3 = 0, \\ \Phi(\xi) = \coth_\alpha(\xi) \pm \mathrm{csch}_\alpha(\xi), \\ \Phi(\xi) = \tanh_\alpha(\xi) \pm \mathrm{i}\,\mathrm{sech}_\alpha(\xi), \\ \qquad\qquad \lambda_1 = -\lambda_3 = \frac{1}{2}, \ \lambda_2 = 0, \\ \Phi(\xi) = \frac{1}{2}\tan_\alpha(2\xi), \ \Phi(\xi) = \frac{1}{2}\cot_\alpha(2\xi), \\ \qquad\qquad \lambda_1 = \frac{1}{4}\lambda_3 = 1, \ \lambda_2 = 0, \end{cases} \tag{10}$$

where $\tanh_\alpha(x)$, $\coth_\alpha(x)$, $\mathrm{csch}_\alpha(x)$, $\mathrm{sech}_\alpha(x)$, $\tan_\alpha(x)$, and $\cot_\alpha(x)$ are the generalized hyperbolic and trigonometric functions which were defined in Ref. [32] by:

$$\tanh_\alpha(x) = \frac{\sinh_\alpha(x)}{\cosh_\alpha(x)}, \ \coth_\alpha(x) = \frac{\cosh_\alpha(x)}{\sinh_\alpha(x)},$$
$$\mathrm{csch}_\alpha(x) = \frac{1}{\sinh_\alpha(x)}, \ \mathrm{sech}_\alpha(x) = \frac{1}{\cosh_\alpha(x)},$$
$$\sinh_\alpha(x) = \frac{E_\alpha(x^\alpha) - E_\alpha(-x^\alpha)}{2},$$
$$\cosh_\alpha(x) = \frac{E_\alpha(x^\alpha) + E_\alpha(-x^\alpha)}{2},$$
$$\tan_\alpha(x) = \frac{\sin_\alpha(x)}{\cos_\alpha(x)}, \ \cot_\alpha(x) = \frac{\cos_\alpha(x)}{\sin_\alpha(x)},$$
$$\sin_\alpha(x) = \frac{E_\alpha(\mathrm{i}x^\alpha) - E_\alpha(-\mathrm{i}x^\alpha)}{2i},$$
$$\cos_\alpha(x) = \frac{E_\alpha(\mathrm{i}x^\alpha) + E_\alpha(-\mathrm{i}x^\alpha)}{2},$$

where

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+k\alpha)}$$

is the Mittag–Leffler function.

3. Exact stochastic solutions of Eq. (1)

In this section, we apply Hermite transform, white noise theory, and modified fractional sub-equation method to explore exact stochastic solutions of Eq. (1). Taking the Hermite transform of Eq. (1), we get the deterministic equation:

$$D_x^\alpha [D_t^\alpha \tilde{U}(x,y,t,z) + 6\tilde{U}(x,y,t,z) D_x^\alpha \tilde{U}(x,y,t,z) + D_x^{3\alpha} \tilde{U}(x,y,t,z)]$$

$$+\tilde{A}(t,z)D_x^{\alpha}\tilde{U}(x,y,t,z)+\tilde{B}(t,z)D_y^{2\alpha}\tilde{U}(x,y,t,z)=0, \quad (11)$$

where $z=(z_1,z_2,\ldots)\in(\mathbb{C}^{\mathbb{N}})_c$ is a vector parameter. To look for the travelling wave solutions of Eq. (11), we put $\tilde{A}(t,z):=A(t,z)$, $\tilde{B}(t,z):=B(t,z)$, $\tilde{U}(x,y,t,z):=u(x,y,t,z)$ and make the transformation $u(x,y,t,z)=u(\xi(x,y,t,z))$, with

$$\xi(x,y,t,z)=k_1x+k_2y+s\int_0^t l(\tau,z)\,\mathrm{d}\tau+c,$$

where k_1, k_2, s, and c are arbitrary constants which satisfy $k_1k_2s\neq 0$, $l(t,z)$ is a nonzero function of the indicated variables to be determined later. Then equation (11) can be transformed into a fractional ordinary differential equation:

$$k_1^{4\alpha}D_{\xi}^{4\alpha}u+6k_1^{2\alpha}(D_{\xi}^{\alpha}u)^2+6k_1^{2\alpha}uD_{\xi}^{2\alpha}u +(k_1^{\alpha}s^{\alpha}l^{\alpha}+k_2^{2\alpha}B)D_{\xi}^{2\alpha}u+k_1^{\alpha}AD_{\xi}^{\alpha}u=0. \quad (12)$$

Balancing $D_{\xi}^{4\alpha}u$ with $uD_{\xi}^{2\alpha}u$ gives $n=2$. Hence, we set the solution of Eq. (11) by:

$$u(x,y,t,z) = p_0(t,z)+p_1(t,z)\Phi(\xi)+p_2(t,z)\Phi^2(\xi) +q_1(t,z)\Phi^{-1}(\xi)+q_2(t,z)\Phi^{-2}(\xi), \quad (13)$$

where $\Phi(\xi)$ satisfies the fractional Riccati equation (8). By substituting Eq. (13) along with Eq. (8) into Eq. (12), and collecting the coefficients of Φ^i $(i=-6,-5,\ldots,6)$ and setting them to be zero, we obtain the following system of algebraic equations in the unknowns $p_i\,(i=0,1,2)$, $q_i\,(i=1,2)$, and l:

$$k_1^{4\alpha}H_i+6k_1^{2\alpha}(G_i+F_i)+(k_1^{\alpha}s^{\alpha}l^{\alpha}+k_2^{2\alpha}B)M_i+k_1^{\alpha}AL_i=0, \quad i=0,1,2,3, \quad (14)$$

$$k_1^{4\alpha}H_{i+7}+6k_1^{2\alpha}(G_{i+7}+F_{i+7})+(k_1^{\alpha}s^{\alpha}l^{\alpha}+k_2^{2\alpha}B)M_{i+5} +k_1^{\alpha}AL_{i+4}=0, \quad i=0,1,2, \quad (15)$$

$$k_1^{4\alpha}H_{i+5}+6k_1^{2\alpha}(G_{i+5}+F_{i+5})=0, \quad i=0,1, \quad (16)$$

$$k_1^{4\alpha}H_{i+11}+6k_1^{2\alpha}(G_{i+11}+F_{i+11})=0, \quad i=0,1, \quad (17)$$

$$k_1^{4\alpha}H_4+6k_1^{2\alpha}(G_4+F_4)+(k_1^{\alpha}s^{\alpha}l^{\alpha}+k_2^{2\alpha}B)M_4=0, \quad (18)$$

$$k_1^{4\alpha}H_{10}+6k_1^{2\alpha}(G_{10}+F_{10})+(k_1^{\alpha}s^{\alpha}l^{\alpha}+k_2^{2\alpha}B)M_8=0, \quad (19)$$

where

$$L_0=p_1\lambda_1-q_1\lambda_3,\ L_1=p_1\lambda_2-2p_2\lambda_1,$$
$$L_2=p_1\lambda_3+2p_2\lambda_2,\ L_3=2p_2\lambda_3,\ L_4=-(q_1\lambda_2+2q_2\lambda_3),$$
$$L_5=-(q_1\lambda_1+2q_2\lambda_2),\ L_6=-2q_2\lambda_1,\ M_0=\lambda_1L_1-2\lambda_3L_5,$$
$$M_1=2\lambda_1L_2+\lambda_2L_1,\ M_2=3\lambda_1L_3+2\lambda_2L_2+\lambda_3L_1,$$
$$M_3=3\lambda_2L_3+2\lambda_3L_2,\ M_4=3\lambda_3L_3,\ M_5=-(\lambda_2L_4+2\lambda_3L_5),$$
$$M_6=-(\lambda_1L_4+2\lambda_2L_5+3\lambda_3L_6),\ M_7=-(2\lambda_1L_5+3\lambda_2L_6),$$
$$M_8=-3\lambda_1L_6,\ N_0=\lambda_1M_1-\lambda_3M_5,\ N_1=2\lambda_1M_2+\lambda_2M_1,$$
$$N_2=3\lambda_1M_3+2\lambda_2M_2+\lambda_3M_1,$$
$$N_3=4\lambda_1M_4+3\lambda_2M_3+2\lambda_3M_2,$$
$$N_4=4\lambda_2M_4+3\lambda_3M_3,\ N_5=4\lambda_3M_4,$$
$$N_6=-(\lambda_2M_5+2\lambda_3M_6),\ N_7=-(\lambda_1M_5+2\lambda_2M_6+3\lambda_3M_7),$$
$$N_8=-(2\lambda_1M_6+3\lambda_2M_7+4\lambda_3M_8),$$
$$N_9=-(3\lambda_1M_7+4\lambda_2M_8),\ N_{10}=-4\lambda_1M_8,$$
$$H_0=\lambda_1N_1-\lambda_3N_6,\ H_1=2\lambda_1N_2+\lambda_2N_1,$$
$$H_2=3\lambda_1N_3+2\lambda_2N_2+\lambda_3N_1,\ H_3=4\lambda_1N_4+3\lambda_2N_3+2\lambda_3N_2,$$
$$H_4=5\lambda_1N_5+4\lambda_2N_4+3\lambda_3N_3,\ H_5=5\lambda_2N_5+4\lambda_3N_4,$$
$$H_6=5\lambda_3N_5,\ H_7=-(\lambda_2N_6+2\lambda_3N_7),$$
$$H_8=-(\lambda_1N_6+2\lambda_2N_7+3\lambda_3N_8),$$
$$H_9=-(2\lambda_1N_7+3\lambda_2N_8+4\lambda_3N_9),$$
$$H_{10}=-(3\lambda_1N_8+4\lambda_2N_9+5\lambda_3N_{10}),$$
$$H_{11}=-(4\lambda_1N_9+5\lambda_2N_{10}),\ H_{12}=-5\lambda_1N_{10},$$
$$G_0=L_0^2+2L_1L_4+2L_2L_5+2L_3L_6,$$
$$G_1=2L_0L_1+2L_2L_4+2L_3L_5,\ G_2=2L_0L_2+L_1^2+2L_3L_4,$$
$$G_3=2L_0L_3+2L_1L_2,\ G_4=2L_1L_3+L_2^2,\ G_5=2L_2L_3,$$
$$G_6=L_3^2,\ G_7=2L_0L_4+2L_1L_5+2L_2L_6,$$
$$G_8=2L_0L_5+2L_1L_6+L_4^2,\ G_9=2L_0L_6+2L_4L_5,$$
$$G_{10}=2L_4L_6+L_5^2,\ G_{11}=2L_5L_6,\ G_{12}=L_6^2,$$
$$F_0=p_0M_0+p_1M_5+p_2M_6+q_1M_1+q_2M_2,$$
$$F_1=p_0M_1+p_1M_0+p_2M_5+q_1M_2+q_2M_3,$$
$$F_2=p_0M_2+p_1M_1+p_2M_0+q_1M_3+q_2M_4,$$
$$F_3=p_0M_3+p_1M_2+p_2M_1+q_1M_4,$$
$$F_4=p_0M_4+p_1M_3+p_2M_2,\ F_5=p_1M_4+p_2M_3,\ F_6=p_2M_4,$$
$$F_7=p_0M_5+p_1M_6+p_2M_7+q_1M_0+q_2M_1,$$
$$F_8=p_0M_6+p_1M_7+p_2M_8+q_1M_5+q_2M_0,$$
$$F_9=p_0M_7+p_1M_8+q_1M_6+q_2M_5,$$
$$F_{10}=p_0M_8+q_1M_7+q_2M_6,$$
$$F_{11}=q_1M_8+q_2M_7,\quad \text{and}\quad F_{12}=q_2M_8.$$

In the remaining part of this section we investigate and solve our problem for some particular cases for the fractional Riccati equation (8), as follows:

Case A If we set $\lambda_1=\lambda_2=1$, $\lambda_3=0$ in Eq. (8), and use Mathematica to solve the resulting system, we will obtain the following set of solutions:

$$p_0=\frac{1}{432k_1^{2\alpha}}(36k_1^{\alpha}A+32k_2^{2\alpha}B),\ p_1=p_2=0,$$
$$q_1=-q_2=2k_1^{2\alpha},\ l=\frac{1}{sk_1}(k_1^{\alpha}A+8k_2^{2\alpha}B)^{1/\alpha}.$$

By substituting these values into Eq. (13) and using Eq. (10), we obtain the following exponential decay wave solution of Eq. (11):

$$u_1(x,y,t,z) = \frac{1}{432k_1^{2\alpha}}[36k_1^{\alpha}A(t,z)+32k_2^{2\alpha}B(t,z)] +\frac{2k_1^{2\alpha}}{E_{\alpha}[\xi_1(x,y,t,z)]-1}$$

$$-\frac{2k_1^{2\alpha}}{(E_\alpha[\xi_1(x,y,t,z)]-1)^2}, \tag{20}$$

with

$$\xi_1(x,y,t,z)=k_1x+k_2y+\frac{1}{k_1}\int_0^t[k_1^\alpha A(\tau,z)+8k_2^{2\alpha}B(\tau,z)]^{1/\alpha}\,\mathrm{d}\tau+c_1.$$

Case B If we set $\lambda_1=-\lambda_3=\frac{1}{2}$, $\lambda_2=0$ in Eq. (8), and use Mathematica to solve the resulting system, we will obtain the following set of solutions:

$$p_0=\frac{221k_1^{3\alpha}-12A}{36k_1^\alpha},$$
$$p_1=\frac{1}{12}\left(-k_1^{2\alpha}+1326k_1^{3\alpha}-\sqrt{4k_1^\alpha A+k_1^{4\alpha}(1-1326k_1^\alpha)^2}\right),$$
$$p_2=-\frac{k_1^{2\alpha}}{12},\quad q_1=q_2=0,\quad l=\frac{1}{sk_1}(2k_1^\alpha A-k_2^{2\alpha}B)^{1/\alpha}.$$

Substituting these values into Eq. (13) and using Eq. (10), we obtain the following soliton wave solutions of Eq. (11):

$$u_2(x,y,t,z)=\frac{221k_1^{3\alpha}-12A(t,z)}{36k_1^\alpha}+\frac{1}{12}\left(-k_1^{2\alpha}+1326k_1^{3\alpha}-\sqrt{4k_1^\alpha A(t,z)+k_1^{4\alpha}(1-1326k_1^\alpha)^2}\right)\times(\coth_\alpha[\xi_2(x,y,t,z)]\pm\operatorname{csch}_\alpha[\xi_2(x,y,t,z)])-\frac{k_1^{2\alpha}}{12}(\coth_\alpha[\xi_2(x,y,t,z)]\pm\operatorname{csch}_\alpha[\xi_2(x,y,t,z)])^2, \tag{21}$$

$$u_3(x,y,t,z)=\frac{221k_1^{3\alpha}-12A(t,z)}{36k_1^\alpha}+\frac{1}{12}\left(-k_1^{2\alpha}+1326k_1^{3\alpha}-\sqrt{4k_1^\alpha A(t,z)+k_1^{4\alpha}(1-1326k_1^\alpha)^2}\right)\times(\tanh_\alpha[\xi_2(x,y,t,z)]\pm\operatorname{isech}_\alpha[\xi_2(x,y,t,z)])-\frac{k_1^{2\alpha}}{12}(\tanh_\alpha[\xi_2(x,y,t,z)]\pm\operatorname{isech}_\alpha[\xi_2(x,y,t,z)])^2, \tag{22}$$

with

$$\xi_2(x,y,t,z)=k_1x+k_2y+\frac{1}{k_1}\int_0^t[2k_1^\alpha A(\tau,z)-k_2^{2\alpha}B(\tau,z)]^{1/\alpha}\,\mathrm{d}\tau+c_2.$$

Case C If we set $\lambda_1=\frac{1}{4}\lambda_3=1$, $\lambda_2=0$ in Eq. (8), and use Mathematica to solve the resulting system, we will obtain the following set of solutions:

$$p_0=\frac{410k_1^{2\alpha}A+551k_2^{2\alpha}B}{60k_1^{3\alpha}A},$$
$$p_1=\frac{1}{6A}\left(B+\sqrt{8891k_1^{6\alpha}A^2+14k_2^{2\alpha}B^2}\right),$$
$$q_1=-q_2=60k_1^{2\alpha}\frac{B}{A},\quad p_2=\frac{(3k_1^{4\alpha}+6k_1^\alpha)A+404k_2^{2\alpha}B}{27k_1^{6\alpha}A},$$
$$l=\frac{1}{sk_1}\left(\frac{706k_2^{2\alpha}B^2-k_1^{2\alpha}A^2}{104A}\right)^{1/\alpha}.$$

Substituting these values into Eq. (13) and using Eq. (10), we obtain the following periodic wave solution of Eq. (11):

$$u_4(x,y,t,z)=\frac{410k_1^{2\alpha}A(t,z)+551k_2^{2\alpha}B(t,z)}{60k_1^{3\alpha}A(t,z)}+\frac{1}{3A(t,z)}\left(B(t,z)+\sqrt{8891k_1^{6\alpha}A^2(t,z)+14k_2^{2\alpha}B^2(t,z)}\right)\times\tan_\alpha[2\xi_3(x,y,t,z)]+\frac{(3k_1^{4\alpha}+6k_1^\alpha)A+404k_2^{2\alpha}B(t,z)}{108k_1^{6\alpha}A(t,z)}\tan_\alpha^2[2\xi_3(x,y,t,z)]+\frac{120k_1^{2\alpha}B}{A}(\cot_\alpha[2\xi_3(x,y,t,z)]-2\cot_\alpha^2[2\xi_3(x,y,t,z)]), \tag{23}$$

$$u_5(x,y,t,z)=\frac{410k_1^{2\alpha}A(t,z)+551k_2^{2\alpha}B(t,z)}{60k_1^{3\alpha}A(t,z)}+\frac{1}{6A(t,z)}\left(B(t,z)+\sqrt{8891k_1^{6\alpha}A^2(t,z)+14k_2^{2\alpha}B^2(t,z)}\right)\times\cot_\alpha[2\xi_3(x,y,t,z)]+\frac{(3k_1^{4\alpha}+6k_1^\alpha)A+404k_2^{2\alpha}B(t,z)}{108k_1^{6\alpha}A(t,z)}\cot_\alpha^2[2\xi_3(x,y,t,z)]+\frac{120k_1^{2\alpha}B}{A}(\tan_\alpha[2\xi_3(x,y,t,z)]-2\tan_\alpha^2[2\xi_3(x,y,t,z)]), \tag{24}$$

with

$$\xi_3=k_1x+k_2y+\frac{1}{k_1}\int_0^t\left(\frac{706k_2^{2\alpha}B^2(\tau,z)-k_1^{2\alpha}A^2(\tau,z)}{104A(\tau,z)}\right)^{1/\alpha}\mathrm{d}\tau+c_3.$$

Obviously, there are an infinite number of particular solutions for the system Eqs. (14)–(19) together with the fractional Riccati equation (8) resulting in many different cases. The properties of generalized exponential, hyperbolic and trigonometric functions yield that there exists a bounded open set $\mathbf{D}\subset\mathbb{R}^2\times\mathbb{R}_+, q<\infty, r>0$ such that the solution $u(x,y,t,z)$ of Eq. (11) and all its fractional derivatives that are involved in Eq. (11) are uniformly bounded for $(x,y,t,z)\in\mathbf{D}\times K_q(r)$, continuous with respect to $(x,y,t)\in\mathbf{D}$ for all $z\in K_q(r)$ and analytic with respect to $z\in K_q(r)$, for all $(x,y,t)\in\mathbf{D}$. From Ref. [1] Theorem 4.1.1, there exist $U(x,y,t,z)\in(\mathcal{S})_{-1}$ such that $u(x,y,t,z)=\tilde{U}(x,y,t)(z)$ for all $(x,y,t,z)\in\mathbf{D}\times K_q(r)$ and $U(x,y,t)$ solves Eq. (1) in $(\mathcal{S})_{-1}$. Hence, by applying the inverse Hermite transform to Eqs. (20)–(24), we will obtain the solutions of Eq. (1), as follows:

(i) Stochastic exponential decay wave solution

$$U_1(x,y,t)=\frac{1}{432k_1^{2\alpha}}[36k_1^\alpha A(t)+32k_2^{2\alpha}B(t)]+\frac{2k_1^{2\alpha}}{E_\alpha^\diamond[\Xi_1(x,y,t)]-1}-\frac{2k_1^{2\alpha}}{(E_\alpha^\diamond[\Xi_1(x,y,t)]-1)^{\diamond 2}}, \tag{25}$$

with

$$\Xi_1(x,y,t)=k_1x+k_2y+\frac{1}{k_1}\int_0^t[k_1^\alpha A(\tau)+8k_2^{2\alpha}B(\tau)]^{\diamond(1/\alpha)}\,\mathrm{d}\tau+c_1.$$

(ii) Stochastic soliton wave solutions

$$U_2(x,y,t)=\frac{221k_1^{3\alpha}-12A(t)}{36k_1^{\alpha}}+\frac{1}{12}\left(-k_1^{2\alpha}+1326k_1^{3\alpha}-\sqrt{4k_1^{\alpha}A(t)+k_1^{4\alpha}(1-1326k_1^{\alpha})^2}\right)\diamond(\coth_\alpha^{\diamond}[\Xi_2(x,y,t)]\pm\operatorname{csch}_\alpha^{\diamond}[\Xi_2(x,y,t)])-\frac{k_1^{2\alpha}}{12}(\coth_\alpha^{\diamond}[\Xi_2(x,y,t)]\pm\operatorname{csch}_\alpha^{\diamond}[\xi_2(x,y,t)])^{\diamond 2},\quad(26)$$

$$U_3(x,y,t)=\frac{221k_1^{3\alpha}-12A(t)}{36k_1^{\alpha}}+\frac{1}{12}\left(-k_1^{2\alpha}+1326k_1^{3\alpha}-\sqrt{4k_1^{\alpha}A(t)+k_1^{4\alpha}(1-1326k_1^{\alpha})^2}\right)\diamond(\tanh_\alpha^{\diamond}[\Xi_2(x,y,t)]\pm\mathrm{i}\ \operatorname{sech}_\alpha^{\diamond}[\Xi_2(x,y,t)])-\frac{k_1^{2\alpha}}{12}(\tanh_\alpha^{\diamond}[\Xi_2(x,y,t)]\pm\mathrm{i}\ \operatorname{sech}_\alpha^{\diamond}[\Xi_2(x,y,t)])^{\diamond 2},\quad(27)$$

with

$$\Xi_2(x,y,t)=k_1x+k_2y+\frac{1}{k_1}\int_0^t[2k_1^{\alpha}A(\tau)-k_2^{2\alpha}B(\tau)]^{\diamond(1/\alpha)}\,\mathrm{d}\tau+c_2.$$

(iii) Stochastic periodic wave solutions

$$U_4(x,y,t)=\frac{410k_1^{2\alpha}A(t)+551k_2^{2\alpha}B(t)}{60k_1^{3\alpha}A(t)}+\frac{1}{6A(t)}\diamond\left(B(t)+\sqrt{8891k_1^{6\alpha}A^{\diamond 2}(t)+14k_2^{2\alpha}B^{\diamond 2}(t,z)}\right)\diamond\tan_\alpha^{\diamond}[2\Xi_3(x,y,t)]+\frac{(3k_1^{4\alpha}+6k_1^{\alpha})A(t)+404k_2^{2\alpha}B(t)}{108k_1^{6\alpha}A(t)}\diamond\tan_\alpha^{\diamond 2}[2\Xi_3(x,y,t)]+\frac{120k_1^{2\alpha}B(t)}{A(t)}\diamond(\cot_\alpha^{\diamond}[2\Xi_3(x,y,t)]-2\cot_\alpha^{\diamond 2}[2\Xi_3(x,y,t)]),\quad(28)$$

$$U_5(x,y,t)=\frac{410k_1^{2\alpha}A(t)+551k_2^{2\alpha}B(t)}{60k_1^{3\alpha}A(t)}+\frac{1}{6A(t)}\diamond\left(B(t)+\sqrt{8891k_1^{6\alpha}A^{\diamond 2}(t)+14k_2^{2\alpha}B^{\diamond 2}(t,z)}\right)\diamond\cot_\alpha^{\diamond}[2\Xi_3(x,y,t)]+\frac{(3k_1^{4\alpha}+6k_1^{\alpha})A(t)+404k_2^{2\alpha}B(t)}{108k_1^{6\alpha}A(t)}\diamond\cot_\alpha^{\diamond 2}[2\Xi_3(x,y,t)]+\frac{120k_1^{2\alpha}B(t)}{A(t)}\diamond(\tan_\alpha^{\diamond}[2\Xi_3(x,y,t)]-2\tan_\alpha^{\diamond 2}[2\Xi_3(x,y,t)]),\quad(29)$$

with

$$\Xi_3=k_1x+k_2y+\frac{1}{k_1}\int_0^t\left(\frac{706k_2^{2\alpha}B^{\diamond 2}(\tau)-k_1^{2\alpha}A^{\diamond 2}(\tau,z)}{104A(\tau)}\right)^{\diamond(1/\alpha)}\mathrm{d}\tau+c_3.$$

4. Examples and concluding remarks

We observe that for different forms of $A(t)$ and $B(t)$, we can get different solutions of Eq. (1) from Eqs. (25)–(29). We give some examples in the following.

Example 1 Let $\dot{W}_t$ be the Gaussian white noise, where W_t is the Brownian motion. We have the Hermite transform $\tilde{W}_t(z)=\sum_{i=1}^{\infty}z_i\int_0^t\eta_i(s)\,\mathrm{d}s$. Since $E_\alpha^{\diamond}(W_t)=E_\alpha(W_t-t^2/2)$, we have $\tan_\alpha^{\diamond}(W_t)=\tan_\alpha(W_t-t^2/2)$, $\cot_\alpha^{\diamond}(W_t)=\cot_\alpha(W_t-t^2/2)$, $\tanh_\alpha^{\diamond}(W_t)=\tanh_\alpha(W_t-t^2/2)$, $\coth_\alpha^{\diamond}(W_t)=\coth_\alpha(W_t-t^2/2)$, $\operatorname{sech}_\alpha^{\diamond}(W_t)=\operatorname{sech}_\alpha(W_t-t^2/2)$, and $\operatorname{csch}_\alpha^{\diamond}(W_t)=\operatorname{csch}_\alpha(W_t-t^2/2)$. Suppose $B(t)=\mu A(t)$ and $A(t)=(f(t)+\nu\dot{W}_t)^{\diamond\alpha}$, where μ, ν are arbitrary constants and $f(t)$ is integrable or bounded measurable function on $\mathbb{R}_+$.[33] Then the white noise functional solutions of Eq. (1) are as follows:

$$U_6(x,y,t)=\frac{(36k_1^{\alpha}+32\mu k_2^{2\alpha})(f(t)+\nu\dot{W}_t)^{\diamond\alpha}}{432k_1^{2\alpha}}+\frac{2k_1^{2\alpha}(E_\alpha[\Lambda_1(x,y,t)]-2)}{(E_\alpha[\Lambda_1(x,y,t)]-1)^2},\quad(30)$$

with

$$\Lambda_1(x,y,t)=k_1x+k_2y+\left(1+\frac{8\mu k_2^{2\alpha}}{k_1^{\alpha}}\right)^{1/\alpha}\times\left(\int_0^t f(\tau)d\tau+\nu W_t-\frac{\nu t^2}{2}\right)+c_1,$$

$$U_7(x,y,t)=\frac{221k_1^{3\alpha}-12(f(t)+\nu\dot{W}_t)^{\diamond\alpha}}{36k_1^{\alpha}}+\frac{1}{12}\left(-k_1^{2\alpha}+1326k_1^{3\alpha}-\sqrt{4k_1^{\alpha}(f(t)+\nu\dot{W}_t)^{\diamond\alpha}+k_1^{4\alpha}(1-1326k_1^{\alpha})^2}\right)\times(\coth_\alpha[\Lambda_2(x,y,t)]\pm\operatorname{csch}_\alpha[\Lambda_2(x,y,t)])-\frac{k_1^{2\alpha}}{12}(\coth_\alpha[\Lambda_2(x,y,t)]\pm\operatorname{csch}_\alpha[\Lambda_2(x,y,t)])^2,\quad(31)$$

$$U_8(x,y,t)=\frac{221k_1^{3\alpha}-12(f(t)+\nu\dot{W}_t)^{\diamond\alpha}}{36k_1^{\alpha}}+\frac{1}{12}\left(-k_1^{2\alpha}+1326k_1^{3\alpha}-\sqrt{4k_1^{\alpha}(f(t)+\nu\dot{W}_t)^{\diamond\alpha}+k_1^{4\alpha}(1-1326k_1^{\alpha})^2}\right)\times(\tanh_\alpha[\Lambda_2(x,y,t)]\pm\mathrm{i}\ \operatorname{sech}_\alpha[\Lambda_2(x,y,t)])-\frac{k_1^{2\alpha}}{12}(\tanh_\alpha[\Lambda_2(x,y,t)]\pm\mathrm{i}\ \operatorname{sech}_\alpha[\Lambda_2(x,y,t)])^2,\quad(32)$$

with

$$\Lambda_2(x,y,t)=k_1x+k_2y+\left(2-\frac{\mu k_2^{2\alpha}}{k_1^{\alpha}}\right)^{1/\alpha}\times\left(\int_0^t f(\tau)\,\mathrm{d}\tau+\nu W_t-\frac{\nu t^2}{2}\right)+c_2,$$

$$U_9(x,y,t)=\frac{410k_1^{2\alpha}+551\mu k_2^{2\alpha}}{60k_1^{3\alpha}}+\frac{1}{6}\left(\mu+\sqrt{8891k_1^{6\alpha}+14k_2^{2\alpha}\mu^2}\right)\tan_\alpha[2\Lambda_3(x,y,t)]+\frac{3k_1^{4\alpha}+6k_1^{\alpha}+404\mu k_2^{2\alpha}}{108k_1^{6\alpha}}\tan_\alpha^2[2\Lambda_3(x,y,t)]+120\mu k_1^{2\alpha}(\cot_\alpha[2\Lambda_3(x,y,t)]-2\cot_\alpha^2[2\Lambda_3(x,y,t)]),\quad(33)$$

$$U_{10}(x,y,t)=\frac{410k_1^{2\alpha}+551\mu k_2^{2\alpha}}{60k_1^{3\alpha}}+\frac{1}{6}\left(\mu+\sqrt{8891k_1^{6\alpha}+14k_2^{2\alpha}\mu^2}\right)\cot_\alpha[2\Lambda_3(x,y,t)]+\frac{3k_1^{4\alpha}+6k_1^{\alpha}+404\mu k_2^{2\alpha}}{108k_1^{6\alpha}}\cot_\alpha^2[2\Lambda_3(x,y,t)]+120\mu k_1^{2\alpha}(\tan_\alpha[2\Lambda_3(x,y,t)]-2\tan_\alpha^2[2\Lambda_3(x,y,t)]),\quad(34)$$

with

$$\Lambda_3(x,y,t)=k_1x+k_2y+\left(2-\frac{706\mu^2k_2^{2\alpha}-k_1^{2\alpha}}{104k_1^{\alpha}}\right)^{1/\alpha}\times\left(\int_0^t f(\tau)\,\mathrm{d}\tau+\nu W_t-\frac{\nu t^2}{2}\right)+c_3.$$

Example 2 Suppose $B(t)=\rho A(t)$ and $A(t)=(f(t)+\delta W_t\dot{W}_t)^{\diamond\alpha}$, where ρ,δ are arbitrary constants and $f(t)$ has the same property as in Example 1. By using $\int_0^t W_\tau\dot{W}_\tau\,\mathrm{d}\tau=W_t^2-t/2$ and Eqs. (25)–(29), we have the following white noise functional solutions of Eq. (1):

$$U_{11}(x,y,t)=\frac{(36k_1^{\alpha}+32\rho k_2^{2\alpha})(f(t)+\delta W_t\dot{W}_t)^{\diamond\alpha}}{432k_1^{2\alpha}}+\frac{2k_1^{2\alpha}(E_\alpha^{\diamond}[\Theta_1(x,y,t)]-2)}{(E_\alpha^{\diamond}[\Theta_1(x,y,t)]-1)^{\diamond 2}},\quad(35)$$

with

$$\Theta_1(x,y,t)=k_1x+k_2y+\left(1+\frac{8\rho k_2^{2\alpha}}{k_1^{\alpha}}\right)^{1/\alpha}\times\left(\int_0^t f(\tau)\,\mathrm{d}\tau+\delta W_t^2-\frac{\delta t}{2}\right)+c_1,$$

$$U_{12}(x,y,t)=\frac{221k_1^{3\alpha}-12(f(t)+\delta W_t\dot{W}_t)^{\diamond\alpha}}{36k_1^{\alpha}}+\frac{1}{12}\Bigg(-k_1^{2\alpha}+1326k_1^{3\alpha}-\sqrt{4k_1^{\alpha}(f(t)+\delta W_t\dot{W}_t)^{\diamond\alpha}+k_1^{4\alpha}(1-1326k_1^{\alpha})^2}\Bigg)\diamond(\coth_\alpha^{\diamond}[\Theta_2(x,y,t)]\pm\operatorname{csch}_\alpha^{\diamond}[\Theta_2(x,y,t)])-\frac{k_1^{2\alpha}}{12}(\coth_\alpha^{\diamond}[\Theta_2(x,y,t)]\pm\operatorname{csch}_\alpha^{\diamond}[\Theta_2(x,y,t)])^{\diamond 2},\quad(36)$$

$$U_{13}(x,y,t)=\frac{221k_1^{3\alpha}-12(f(t)+\delta W_t\dot{W}_t)^{\diamond\alpha}}{36k_1^{\alpha}}+\frac{1}{12}\Bigg(-k_1^{2\alpha}+1326k_1^{3\alpha}-\sqrt{4k_1^{\alpha}(f(t)+\delta W_t\dot{W}_t)^{\diamond\alpha}+k_1^{4\alpha}(1-1326k_1^{\alpha})^2}\Bigg)\diamond(\tanh_\alpha^{\diamond}[\Theta_2(x,y,t)]\pm\mathrm{i}\,\operatorname{sech}_\alpha^{\diamond}[\Theta_2(x,y,t)])-\frac{k_1^{2\alpha}}{12}(\tan_\alpha^{\diamond}[\Theta_2(x,y,t)]\pm\mathrm{i}\,\operatorname{sech}_\alpha^{\diamond}[\Theta_2(x,y,t)])^{\diamond 2},\quad(37)$$

with

$$\Theta_2(x,y,t)=k_1x+k_2y+\left(2-\frac{\rho k_2^{2\alpha}}{k_1^{\alpha}}\right)^{1/\alpha}\times\left(\int_0^t f(\tau)d\tau+\delta W_t^2-\frac{\delta t}{2}\right)+c_2,$$

$$U_{14}(x,y,t)=\frac{410k_1^{2\alpha}+551\rho k_2^{2\alpha}}{60k_1^{3\alpha}}+\frac{1}{6}\left(\rho+\sqrt{8891k_1^{6\alpha}+14\rho^2k_2^{2\alpha}}\right)\tan_\alpha^{\diamond}[2\Theta_3(x,y,t)]+\frac{3k_1^{4\alpha}+6k_1^{\alpha}+404\rho k_2^{2\alpha}}{108k_1^{6\alpha}}\tan_\alpha^{\diamond 2}[2\Theta_3(x,y,t)]+120\rho k_1^{2\alpha}(\cot_\alpha^{\diamond}[2\Theta_3(x,y,t)]-2\cot_\alpha^{\diamond 2}[2\Theta_3(x,y,t)]),\quad(38)$$

$$U_{15}(x,y,t)=\frac{410k_1^{2\alpha}+551\rho k_2^{2\alpha}}{60k_1^{3\alpha}}+\frac{1}{6}\left(\rho+\sqrt{8891k_1^{6\alpha}+14\rho^2k_2^{2\alpha}}\right)\cot_\alpha^{\diamond}[2\Theta_3(x,y,t)]+\frac{3k_1^{4\alpha}+6k_1^{\alpha}+404\rho k_2^{2\alpha}}{108k_1^{6\alpha}}\cot_\alpha^{\diamond 2}[2\Theta_3(x,y,t)]+120\rho k_1^{2\alpha}(\tan_\alpha^{\diamond}[2\Theta_3(x,y,t)]-2\tan_\alpha^{\diamond 2}[2\Theta_3(x,y,t)]),\quad(39)$$

with

$$\Theta_3(x,y,t)=k_1x+k_2y+\left(2-\frac{706\rho^2k_2^{2\alpha}-k_1^{2\alpha}}{104k_1^{\alpha}}\right)^{1/\alpha}\times\left(\int_0^t f(\tau)\,\mathrm{d}\tau+\delta W_t^2-\frac{\delta t}{2}\right)+c_3.$$

Remark 1 If $\alpha=1$, then the set of solutions U_i $(i=1,2,\ldots,15)$ give a new set of white noise functional solutions for the Wick-type stochastic 2D KdV equations.[27] Moreover the set of Hermit transforms $u_i=\tilde{U}_i(z)$ $(i=1,2,\ldots,15)$ give a new set of exact solutions of the variable coefficients 2D KdV equation.[26]

Remark 2 We have only discussed the solutions of stochastic fractional 2D KdV equations driven by Gaussian white noise. There is a unitary mapping between the Gaussian white noise space and the Poisson white noise space, this connection was given by Benth and Gjerde.[34] Hence, with the help of this connection, we can derive some Poisson white noise functional solutions, if the coefficients $A(t)$, $B(t)$ are Poisson white noise functions in Eq. (1).

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