Geometric discord for non-X states*

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The level surfaces of geometric discord for a class of two-qubit non-X states are investigated when the Bloch vectors are in arbitrary directions. The level surfaces of constant geometric discord are formed by three intersecting open tubes along three orthogonal directions. When Bloch vectors increase, the tubes along one or two directions shrink towards the center and may either totally disappear or the open tubes may become closed tubes when the Bloch vectors reach a critical value. In the generalized amplitude damping channel, the evolution of geometric discord shows double sudden changes when the parameter γ increases. In the phase damping channel, the freezing phenomenon of geometric discord also exists.

Keywords: geometric discord, non-X states, Bloch vectors, generalized amplitude damping channel, phase damping channel

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1. Introduction

Quantum correlation has received considerable attention in recent years because it plays an important role in quantum information processing. Among all of the quantum correlations, quantum entanglement[1] has been considered as the quantum correlation itself because numerous quantum information processing tasks could be realized based on entanglement; for example, quantum teleportation,[2] remote state preparation,[3] quantum cloning,[4] and quantum cryptography,[5] etc. However it has been demonstrated that the absence of quantum entanglement is not a sign of the locality of a quantum system. In the DQC1 model,[6] the deterministic quantum computation can be realized when the entanglement is almost free. Although the quantum entanglement in this system is negligibly weak, there is still enough quantum correlation to realize the quantum computation with one qubit.

Thus, more adequate measures of quantum correlation are necessary. Quantum discord, which was first introduced by Ollivier and Zurek[7] and independently by Henderson and Vedral,[8] is an entropic measure of quantum correlation. It calculates the distance of the mutual information of the original state and its nearest classical state. Following this work, Luo obtained the analytical solution of quantum discord for two-qubit Bell-diagonal states.[9] Subsequently, the result was improved to calculate quantum discord for general two-qubit X-states.[10,11] Meanwhile, difficulties have arisen because different correlation measures cannot be compared directly. Modi *et al*. have successfully dealt with this problem by using relative entropy, which put all of the correlations on an equal footing,[12] including entanglement, quantum discord, and quantum dissonance. Compared with quantum entanglement, quantum discord has a lot of different properties that are quite interesting. For example, the dynamic behavior of quantum discord shows that quantum discord is more robust than quantum entanglement. It is found that quantum discord does not suffer from a sudden death phenomenon in different physical systems or under different environments.[13–17] In particular, the dynamics of quantum discord exhibit a freezing phenomenon of Bell-diagonal states under a phase damping channel.[18,19] The necessary and sufficient conditions of this freezing phenomenon were derived by You *et al*.[20] Following these theories, experimental applications of quantum discord are also carried out. Thus, quantum discord has become a practical quantum resource.[21–28]

Although quantum discord is an important measure of quantum correlations, it is quite difficult to obtain an analytical expression for a general two-qubit state. The algorithm for calculating quantum discord involves one numerical minimum optimization over the von Neumann measure. For some complicated systems, rather than Bell-diagonal states or X-states, it is very hard to calculate the quantum discord, even in numerical simulations. To reduce the difficulty in the calculations, Dakic *et al*. introduced a new measure of quantum correlations that they called geometric discord,[29] which uses the Hilbert–Schmidt distance instead of the distance of the mutual information. Although it takes the same minimum optimization, it uses different quantities and offers analytical solutions in more general conditions.[30–32] To reveal the difference between the entropic measure and the geometric measure, Bellomo *et al*. made a comparison between quantum discord (en-

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tropic measure) and geometric discord.[33] It was found that they could no longer satisfy a closed additivity relation. In addition, it was shown that the decoherence effect on geometric discord was quite different with the original quantum discord.[34,35] Subsequently, the operational meaning of geometric discord was revealed,[36–39] especially in remote state preparation (RSP) since the lowest fidelity of RSP is equal to the geometric discord. Meanwhile, Lang and Caves and other groups investigated the level surfaces of entanglement and discord on a three-dimensional geometry since a class of two-qubit states can be depicted as a tetrahedron in three dimensions.[11,20,34,40,41] They provided a pictorial approach and presented a complete explanation of the structure of entanglement and nonclassicality as well as the dynamic behaviors of entanglement and discord under decoherence. In most of these previous works, the quantum and geometric discords of Bell-diagonal states and X-states were investigated. However, to see more clearly the general properties of discord, the discord of two-qubit arbitrary states needs to be further studied.

In this paper, the geometric discord of two-qubit non-X states with more general conditions and arbitrary Bloch vectors is investigated. In Section 2, the geometric discord is briefly reviewed. The level surfaces of a class of non-X states are demonstrated when the Bloch vectors change. Different measures of discord are compared. In Section 3, the dynamics of the geometric discord of non-X states under decoherence channels is presented. A discussion concludes the paper.

2. Level surfaces of two-qubit non-X states

2.1. Geometric discord

For a geometric measure of discord, even complicated non-X states have analytic solutions. Some interesting phenomena of geometric discord can be found from the graph of a level surface.[34]

Any two-qubit system can be represented by pairs of Bloch vectors and a matrix of correlation tensor. The density matrix is in the form of

$$\rho = \frac{1}{4}\left[I\otimes I + \boldsymbol{x}'\cdot\boldsymbol{\sigma}\otimes I + I\otimes \boldsymbol{y}'\cdot\boldsymbol{\sigma} + \sum_{i,j=1}^{3} t_{ij}\sigma_i\otimes\sigma_j\right], \quad (1)$$

where $\{\sigma_i\}$ are the Pauli spin matrices, and $t_{ij} = \mathrm{Tr}\,\rho(\sigma_k\otimes\sigma_j)$ are elements of the correlation tensor matrix. The vectors $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are the local Bloch vectors of the reduced density operator of two subsystems, with components $x'_k = \mathrm{Tr}(\rho_A\sigma_k)$, $y'_k = \mathrm{Tr}(\rho_B\sigma_k)$, and ρ_A, ρ_B are reduced density operators of ρ. The Bell diagonal state, X-state, and non-X state depend on different values of Bloch vectors. Equation (1) can be further simplified by local unitary transformation from the theorem of singular value decomposition.[9] After the transformation, the correlation matrix can be diagonalized and the x and y vectors in Eq. (1) are also changed. Then the density matrix can be written as

$$\rho = \frac{1}{4}\left[I\otimes I + \boldsymbol{x}\cdot\boldsymbol{\sigma}\otimes I + I\otimes \boldsymbol{y}\cdot\boldsymbol{\sigma} + \sum_{i=1}^{3} c_i\sigma_i\otimes\sigma_i\right]. \quad (2)$$

The parameters of Eq. (2) should obey the semi-positive definition of the density matrix, which ensures that the density matrix is a physical one. In other words, the eigenvalues of Eq. (2) need to be kept greater than, or equal to, zero. The trace of the state should also be one when one studies the quantum discord or geometric discord with a level surface. This restricting condition will be represented geometrically by a tetrahedron. States inside the tetrahedron are physically allowed. For comparison, the Bell diagonal states and X-states are also presented.

In Fig. 1, the deformation of the tetrahedron is plotted when the local Bloch vectors are increased from $\boldsymbol{x} = (0,0,0)$, $\boldsymbol{x} = (0.3,0,0)$, $\boldsymbol{x} = (0.3,0.3,0)$, to $\boldsymbol{x} = (0.3,0.3,0.3)$ with $\boldsymbol{y} = (0,0,0)$. The Bell-diagonal state is shown in Fig. 1(a) with $\boldsymbol{x} = (0,0,0), \boldsymbol{y} = (0,0,0)$. While the X-state is shown in Fig. 1(b) with $\boldsymbol{x} = (0.3,0,0)$, $\boldsymbol{y} = (0,0,0)$. The non-X states with arbitrary Bloch vectors are shown in Figs. 1(c) and 1(d) with $\boldsymbol{x} = (0.3,0.3,0)$, $\boldsymbol{y} = (0,0,0)$, and $\boldsymbol{x} = (0.3,0.3,0.3)$, $\boldsymbol{y} = (0,0,0)$. From Fig. 1, one can see that the tetrahedron shrinks and its vertices and edges become smoother when the local Bloch vectors increase.

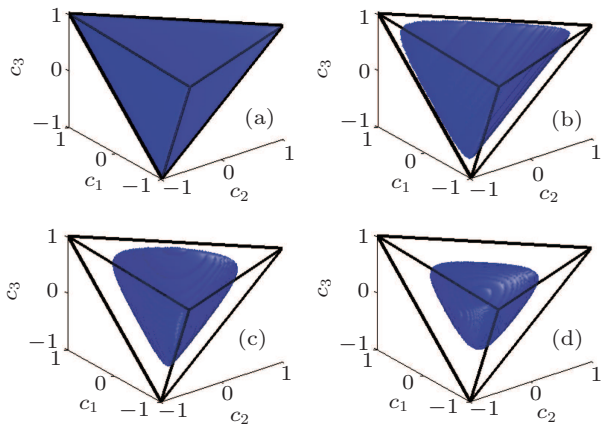


Fig. 1. (color online) Tetrahedron as a function of the local Bloch vectors when $\boldsymbol{y} = (0,0,0)$. (a) $\boldsymbol{x} = (0,0,0)$; (b) $\boldsymbol{x} = (0.3,0,0)$; (c) $\boldsymbol{x} = (0.3,0.3,0)$; (d) $\boldsymbol{x} = (0.3,0.3,0.3)$.

It is noted that the eigenvalue of density matrix Eq. (2) can be very complicated with arbitrary values of Bloch vectors. Therefore, previous discussion of entropic measure of quantum discord are mainly focused on Bell diagonal states and X-states.[34] On the contrary, the conceptual expression of geometric discord can be written as

$$D_{\mathrm{g}} := \min_{\chi\in\Omega_0} ||\rho - \chi||_2^2, \quad (3)$$

where Ω_0 denotes the set of classical states, which are all zero-discord states, and $||X||^2 = \mathrm{Tr}(X)^2$ is square norm in the

Hilbert–Schmidt space. It can be easily seen that the analytical expression can be derived in more general conditions when we calculate the quantum correlation of a two-qubit system with geometric discord. Equation (3) can be deformed as follows:[23]

$$D_{\rm g} = \frac{1}{4}[||\boldsymbol{x}||^2 + ||T||^2 - k_{\max}], \tag{4}$$

where $||x||^2 = \sum_i x_i^2$, $T = (t_{ij})$, and $k_{\max}$ is the largest eigenvalue of the matrix $K = \boldsymbol{x}\boldsymbol{x}^{\rm T} + TT^{\rm T}$.

2.2. Level surfaces of geometric discord

For X-state with local Bloch vectors $\boldsymbol{x} = (x,0,0)$, $\boldsymbol{y} = (0,0,0)$, the geometric discord can be written as

$$D_{\rm g} = \frac{1}{4}[c_1^2 + c_2^2 + c_3^2 + x^2 - \max\{c_1^2 + x^2, c_2^2, c_3^2\}]. \tag{5}$$

While for X-states with Bloch vectors $\boldsymbol{x} = (0,x,0), \boldsymbol{y} = (0,0,0)$ and $\boldsymbol{x} = (0,0,x)$, $\boldsymbol{y} = (0,0,0)$, similar results can be obtained.

For non-X states with Bloch vectors $\boldsymbol{x} = (x,x,0)$, $\boldsymbol{y} = (0,0,0)$, the geometric discord can be written as

$$D_{\rm g} = \frac{1}{4}[c_1^2 + c_2^2 + c_3^2 + 2x^2 - \max\{\lambda_1, \lambda_2, \lambda_3\}], \tag{6}$$

where λ_1, λ_2, λ_3 are the eigenvalues of K in Eq. (4) and are given by

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}\left[2x^2 + c_1^2 + c_2^2 - \sqrt{4x^4 + (c_1^2 - c_2^2)^2}\right], \\
\lambda_2 &= \frac{1}{2}\left[2x^2 + c_1^2 + c_2^2 + \sqrt{4x^4 + (c_1^2 - c_2^2)^2}\right], \\
\lambda_3 &= c_3^2.
\end{aligned} \tag{7}$$

For non-X states with Bloch vectors $\boldsymbol{x} = (0,x,x)$, $\boldsymbol{y} = (0,0,0)$, and $\boldsymbol{x} = (x,0,x)$, $\boldsymbol{y} = (0,0,0)$, similar results can also be obtained.

For non-X states with Bloch vectors $\boldsymbol{x} = (x,x,x)$, $\boldsymbol{y} = (0,0,0)$, the geometric discord can be written as

$$D_{\rm g} = \frac{1}{4}[c_1^2 + c_2^2 + c_3^2 + 3x^2 - \max\{\lambda_1, \lambda_2, \lambda_3\}], \tag{8}$$

where λ_1, λ_2, λ_3 are the eigenvalues of K in Eq. (4) and satisfy the following third order equation

$$\begin{aligned}
&\lambda^3 - (3x^2 - c_1^2 + c_2^2 + c_3^2)\lambda^2 \\
&- (2c_1^2x^2 - 2c_2^2x^2 - 2c_3^2x^2 + c_1^2c_2^2 - c_2^2c_3^2 + c_1^2c_3^2)\lambda \\
&- c_1^2(c_2^2 + c_3^2)x^2 - c_1^2c_2^2c_3^2 = 0.
\end{aligned} \tag{9}$$

Since the general solutions of Eq. (9) are very complicated, the eigenvalues λ_1, λ_2, and λ_3 are not explicitly expressed here.

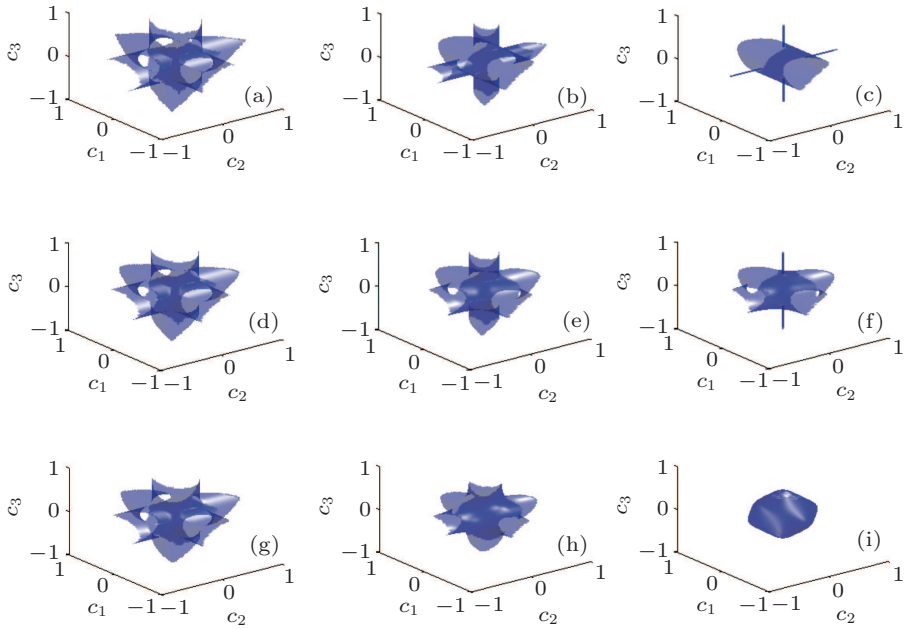


Fig. 2. (color online) Level surfaces of geometric discord of $D_{\rm g} = 0.03$ when the Bloch vectors are $\boldsymbol{y} = (0,0,0)$ and (a) $\boldsymbol{x} = (0,0,0)$; (b) $\boldsymbol{x} = (0.3,0,0)$; (c) $\boldsymbol{x} = (\sqrt{0.12},0,0)$; (d) $\boldsymbol{x} = (0.1,0.1,0)$; (e) $\boldsymbol{x} = (0.2,0.2,0)$; (f) $\boldsymbol{x} = (\sqrt{0.06},\sqrt{0.06},0)$; (g) $\boldsymbol{x} = (0.1,0.1,0.1)$; (h) $\boldsymbol{x} = (0.2,0.2,0.2)$; (i) $\boldsymbol{x} = (\sqrt{0.075},\sqrt{0.075},\sqrt{0.075})$.

The level surfaces of $D_{\rm g} = 0.03$ are plotted in Fig. 2 for different values of Bloch vectors. In Figs. 2(a), 2(b), and 2(c), the level surfaces are plotted when the Bloch vectors are increased from $\boldsymbol{x} = (0,0,0)$ to $\boldsymbol{x} = (\sqrt{0.12},0,0)$ with $\boldsymbol{y} = (0,0,0)$. It is clear that the tubes along c_2 and c_3 directions shrink more quickly and disappear. The three intersecting tubes become one tube along c_1 direction when $\boldsymbol{x} = (\sqrt{0.12},0,0)$. In Figs. 2(d), 2(e), and 2(f), the level

surfaces are plotted when the Bloch vectors are increased from $\boldsymbol{x} = (0.1, 0.1, 0)$ to $\boldsymbol{x} = (\sqrt{0.06}, \sqrt{0.06}, 0)$ with $\boldsymbol{y} = (0,0,0)$. It is noted that the tube along c_3 direction shrinks more quickly and disappears. The three intersecting tubes become two intersecting tubes along c_1 and c_2 directions when $\boldsymbol{x} = (\sqrt{0.06}, \sqrt{0.06}, 0)$. In Figs. 2(g), 2(h), and 2(i), the level surfaces are plotted when the Bloch vectors are increased from $\boldsymbol{x} = (0.1, 0, 1, 0.1)$ to $\boldsymbol{x} = (\sqrt{0.075}, \sqrt{0.075}, \sqrt{0.075})$ with $\boldsymbol{y} = (0,0,0)$. It is seen that none of these three tubes will disappear, even when the Bloch vectors are increased. While the length or height and the diameters of the mouth of the three tubes are reduced. The three open tubes will eventually be closed and the shape of the level surface will become a polyhedron with smooth vertices and edges. From Fig. 2, it is seen that the tubes shrink and may disappear in some directions or three open tubes may become closed tubes when the local Bloch vectors are increased. There are some threshold values where the open tubes of the level surfaces may disappear or close.

2.3. Threshold of level surfaces

To control the geometric discord, the threshold values of the level surfaces need to be calculated.

For X-states with local Bloch vectors $\boldsymbol{x} = (x,0,0)$, $\boldsymbol{y} = (0,0,0)$, or $\boldsymbol{x} = (0,x,0)$, $\boldsymbol{y} = (0,0,0)$, or $\boldsymbol{x} = (0,0,x)$, $\boldsymbol{y} = (0,0,0)$, two of the three tubes will eventually disappear on axes when the threshold value of the Bloch vector is chosen. So one can easily set $c_1 = c_2 = 0$, $|c_3| = c$ and regard x as a constant since in the level surface x does not change. Then, K can be written as

$$K_1 = \begin{bmatrix} x^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^2 \end{bmatrix} = \begin{bmatrix} x^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^2 \end{bmatrix}. \quad (10)$$

Similarly, when $c_1 = c_3 = 0$ and $|c_2| = c$ are chosen, one has

$$K_2 = \begin{bmatrix} x^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x^2 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (11)$$

The eigenvalues of both K_1 and K_2 are the same and are given by $\lambda_1 = 0$, $\lambda_2 = x^2$, $\lambda_3 = c^2$. From Eq. (5), one can get $D_g = \frac{1}{4}\min\{x^2, c^2\}$. Since x is a constant, when c satisfies $x^2 < c^2$, D_g is also a constant with $D_g = \frac{1}{4}x^2$. This means that the D_g will be the same on the axes c_2 and c_3 above the critical value $|c_2| = |c_3| = \sqrt{x^2}$ and D_g will vary below the critical value $|c_2| = |c_3| = \sqrt{x^2}$. Therefore, the shape of level surface of the two tubes will shrink to two straight lines in c_2 and c_3 directions at the critical value of $x = \sqrt{4D_g}$ and the tubes will disappear for $x > \sqrt{4D_g}$, as shown in Fig. 2(c), while the shape of the tube along c_1 direction remains the same. That is, when c_3 is large enough that $c_3^2 + x^2$ become the largest eigenvalue of K, D_g can keep the formation with $D_g = \frac{1}{4}\left(c_1^2 + c_2^2\right)$. It is obvious that the level surface $D_g = 0.03$ will be a circle for $c_1^2 + c_2^2 = 0.12$.

Similarly, for non-X states with local Bloch vectors $\boldsymbol{x} = (x,x,0)$, $\boldsymbol{y} = (0,0,0)$, or $\boldsymbol{x} = (0,x,x)$, $\boldsymbol{y} = (0,0,0)$, or $\boldsymbol{x} = (x,0,x)$, $\boldsymbol{y} = (0,0,0)$, one of the three tubes will eventually disappear on an axis when the threshold values of the Bloch vectors are chosen. One may set $c_1 = c_2 = 0$, $|c_3| = c$ and regard x as a constant. Then K can be written as

$$K = \begin{bmatrix} x^2 & x^2 & 0 \\ x^2 & x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^2 \end{bmatrix} = \begin{bmatrix} x^2 & x^2 & 0 \\ x^2 & x^2 & 0 \\ 0 & 0 & c^2 \end{bmatrix}. \quad (12)$$

The eigenvalues of Eq. (12) are $\lambda_1 = 0$, $\lambda_2 = 2x^2$, and $\lambda_3 = c^2$. From Eq. (6) one has $D_g = \frac{1}{4}\min\{2x^2, c^2\}$. When $2x^2 < c^2$, D_g is also a constant with $D_g = x^2$. This means that the D_g will be the same on axis c_3 above the critical value $|c_3| = \sqrt{2x^2}$ and the D_g will vary below the critical value $|c_3| = \sqrt{2x^2}$. The shape of the level surface of one tube will shrink to a straight line in c_3 direction and eventually disappear when the Bloch vectors are increased above the critical values with $x > \sqrt{2D_g}$, as shown in Fig. 2(f). The two tubes in other directions remain unchanged [Fig. 2(f)].

While for non-X states with local Bloch vectors $\boldsymbol{x} = (x,x,x)$, $\boldsymbol{y} = (0,0,0)$, the three open tubes will be eventually closed and the shape of the level surface will become a polyhedron with smooth vertices and edges. Without loosing generality, one can choose $c_1 = c_2 = 0$, $|c_3| = c$ and regard x as a constant for convenience of the symmetry. Then, K can be written as

$$K = \begin{bmatrix} x^2 & x^2 & x^2 \\ x^2 & x^2 & x^2 \\ x^2 & x^2 & x^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^2 \end{bmatrix} = \begin{bmatrix} x^2 & x^2 & x^2 \\ x^2 & x^2 & x^2 \\ x^2 & x^2 & x^2 + c^2 \end{bmatrix}. \quad (13)$$

The eigenvalues of Eq. (13) are $\lambda_1 = 0$,

$$\lambda_2 = \frac{1}{2}[3x^2 + c^2 + \sqrt{c^4 - 2c^2x^2 + 9x^4}],$$

and

$$\lambda_3 = \frac{1}{2}[3x^2 + c^2 - \sqrt{c^4 - 2c^2x^2 + 9x^4}].$$

It can be easily seen that λ_2 is always the largest among the three eigenvalues. Therefore, from Eq. (8) one has

$$D_g = \frac{1}{8}[3x^2 + c^2 + \sqrt{c^4 - 2c^2x^2 + 9x^4}].$$

This indicates that the three tubes of the level surfaces will not shrink along any direction while instead the open tubes will become closed tubes. When x is small enough, D_g cannot reach the value of 0.03 due to the restriction of the tetrahedron. If x is large enough, the tetrahedron shrinks but D_g on the boundary of tetrahedron grows even faster. Therefore, when x reaches the threshold value of $\sqrt{0.075}$, it can satisfy both $D_g = 0.03$ and the restriction of tetrahedron.

2.4. Comparison between different discord measures

Piani and Tufarelli *et al*. and Hu *et al*.,[42–44] have recently pointed out that geometric discord is not a good measure for the quantumness of correlations. It has an ancilla problem that geometric discord may change arbitrarily and reversibly through some trivial actions of the unmeasured subsystem. To deal with this problem, many remedies on geometric discord have been proposed.[41,45] The square root of matrix was introduced into the definition formula and could be regarded as more strict definitions of geometric discord. Therefore, it is interesting to compare the quantum discord and geometric discord based on different measures.

In particular, a two-qubit Werner state with the parameter $p \in [-1/3, 1]$ can be considered as follows:

$$\rho = \frac{1-p}{4} I + p|\Psi^+\rangle\langle\Psi^+|, \tag{14}$$

where $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. It is noted that ρ is not a density matrix when $p < -1/3$ since it is out of the tetrahedron. The different discord measures are plotted in Fig. 3. In Fig. 3(a), discord measures based on different distances of Eq. (14) are plotted as a function of p. One can see among all different measures, the one-norm geometric discord in Ref. [41] deviates from quantum discord more than others since it uses the intermediate values of $|c_1|$, $|c_2|$, and $|c_3|$ as the discord. It keeps two straight lines at either side of $p = 0$. The comparison of the geometric discord D_g and the geometric discord D_h using Hellinger distance is the same as that in Ref. [45]. When D_g and D_h are compared with quantum discord D_q, it is seen that the geometric discord D_g is closer to D_q in the range of $p \in [0, 1]$ while D_h is closer to D_q in the range of $p \in [-1/3, 0]$.

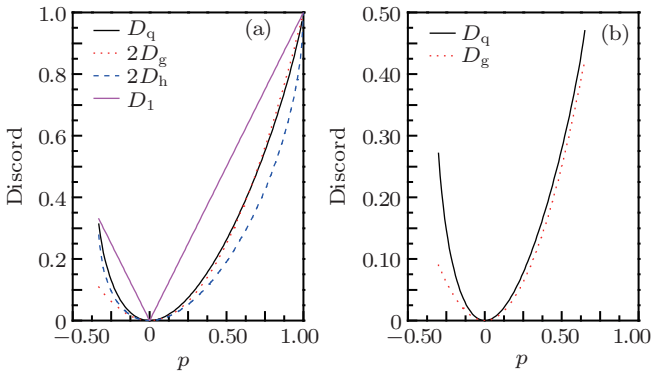


Fig. 3. (color online) Different measures of two-qubit Werner state. (a) Quantum discord (solid black line); original geometric discord (red dotted line); geometric discord using Hellinger distance (blue dashed line); and geometric discord using Schatten one-norm (purple dash–dotted line). (b) Quantum discord (solid black line); original geometric discord (red dashed line).

For non-X states, the quantum discord[25] and geometric discord are plotted in Fig. 3(b) as a function of p using Werner-like states, which are the states of Eq. (14) when the Bloch vectors are $\boldsymbol{x} = (0.2, 0.2, 0.2)$ and $\boldsymbol{y} = (0, 0, 0)$. It is clear that the two measures of D_q and D_g keep similar tendency as that in Fig. 3(a) and matches quite well when $p \in [0, 1]$ and deviates slightly more when $p \in [-1/3, 0]$. This is similar to the level surfaces for Bell-diagonal states in Ref. [34].

3. Geometric discord of non-X states with decoherence

In a real physical system, the system will interact with the environment. Thus, the dynamics of two-qubit geometric discord under local Markovian environments need to be investigated. The evolution of a quantum state under a trace-preserving quantum operator $\varepsilon(\rho)$ can be written as

$$\varepsilon(\rho) = \sum_{i,j} E_{i,j} \rho(0) E_{i,j}^\dagger, \tag{15}$$

where $\rho(0)$ represents the original state, $E_{i,j} = E_i \otimes E_j$ and they satisfy $\sum_{i,j} E_{i,j} E_{i,j}^\dagger = I$.

The Kraus operators for some typical kinds of decoherence channels can be specifically studied. The geometric discord for non-X states under these channels can be evaluated.

3.1. Generalized amplitude damping channel

For a generalized amplitude damping channel with one qubit, the operation elements can be given by

$$\begin{aligned} E_0 &= \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \quad E_1 = \sqrt{p}\begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}, \\ E_2 &= \sqrt{1-p}\begin{bmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{bmatrix}, \\ E_3 &= \sqrt{1-p}\begin{bmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{bmatrix}, \end{aligned} \tag{16}$$

where γ is the probability that a photon from the system has been scattered. For a two-qubit system, equation (15) is used to operate the combination of Kraus operators on the density matrix. Under the operation of generalized amplitude damping channel, components of the original states will be deformed into

$$\begin{aligned} c_1' &= (1-\gamma)c_1, \quad c_2' = (1-\gamma)c_2, \quad c_3' = (1-\gamma)^2 c_3, \\ x_1' &= \sqrt{(1-\gamma)}x_1, \quad x_2' = \sqrt{(1-\gamma)}x_2, \quad x_3' = (1-\gamma)x_3. \end{aligned} \tag{17}$$

It is seen that the components of local Bloch vectors and the corresponding correlation components decay in different rates. Therefore, the evolution of geometric discord exhibits different phenomena in contrast with the Bell-diagonal states.

The evolution of geometric discord and the eigenvalues k_i of K in Eq. (4) are plotted in Fig. 4 with $\boldsymbol{c} = (0.1, 0.4, 0.5)$ and $\boldsymbol{x} = (0.3, 0, 0)$. In Fig. 4(a), the evolution of geometric discord is plotted as a function of γ. It is seen that there are two sudden changes in D_g when γ is increased. The corresponding sudden changes appear at $\gamma_1 = 1 - c_2/c_3 = 0.2$ and $\gamma_2 = 1 - x^2/|c_2^2 - c_1^2| = 0.4$. The double sudden changes are

demonstrated in Ref. [41] in one-norm geometric discord, although the definition of geometric discord is different. It is interesting to note that the double sudden changes of geometric discord can also be found here in non-X states. In Fig. 4(b), the largest eigenvalues k_i are plotted as a function of γ. From Fig. 4(b), the largest eigenvalues k_i can be written as

$$k_{\max} = \begin{cases} c_3'^2, & \gamma < 0.2. \\ c_2'^2, & 0.2 \le \gamma \le 0.4. \\ c_1'^2 + x_1'^2, & 0.4 < \gamma. \end{cases} \tag{18}$$

It is clear that the double sudden changes in D_g appear at the cross points of $k_{\max}$ and the largest eigenvalue $k_{\max}$ is totally different in three regions as given by Eq. (18). From Eq. (17) one can find that the decay rate of c_3' is higher than c_1' and c_2'. Therefore, the double sudden changes will happen under the conditions of $c_3 > c_1$ and $c_3 > c_2$. Because c_1' and c_2' decay at the same rate without loss of generality, one can set $\boldsymbol{x} = (x,0,0)$, while c_1' and c_2' will not be equivalent. To witness the double sudden changes, one must keep the initial condition $c_1^2 + x_1^2 < c_2^2$ and the crosses of c_3' and c_2' ahead of the crosses of c_1' and c_2'. That is, the original parameters must satisfy $c_3 > c_2$ and

$$c_1 < \sqrt{\left(1 - \frac{c_2}{c_3}\right)^2 c_3^2 + \frac{1}{(1 - c_2/c_3)} x_1^2}.$$

The two vertical dashed lines in Fig. 4 show the positions of double sudden changes.

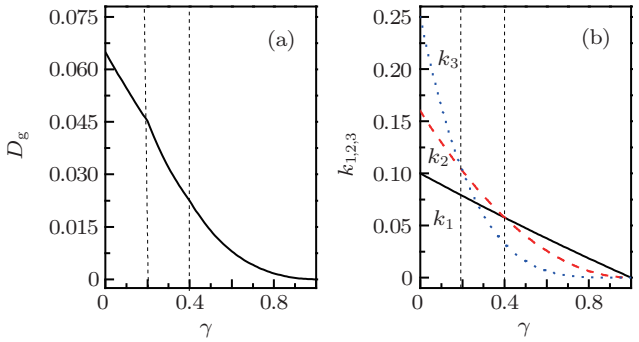


Fig. 4. (color online) (a) Evolution of geometric discord as a function of γ for $\boldsymbol{c} = (0.1, 0.4, 0.5)$ and $\boldsymbol{x} = (0.3, 0, 0)$. The double sudden changes accompany with two crosses in eigenvalues k_i of K. (b) Evolution of eigenvalues k_i of K as a function of γ.

3.2. Phase damping channel

For Bell-diagonal states, it is known that the freezing phenomenon of quantum discord under phase damping channel occurs for a certain time interval. For one qubit under a phase damping channel, the operation elements can be given by

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\gamma} \end{bmatrix}. \tag{19}$$

For a two-qubit system under the phase damping channel, the components are

$$\begin{aligned} c_1' &= (1-\gamma)c_1, & c_2' &= (1-\gamma)c_2, & c_3' &= c_3, \\ x_1' &= \sqrt{(1-\gamma)}x_1, & x_2' &= \sqrt{(1-\gamma)}x_2, & x_3' &= x_3. \end{aligned} \tag{20}$$

From Eq. (2), if the parameters are chosen as $\boldsymbol{c} = (0.8, 0, 0.3)$ and $\boldsymbol{x} = (0.2, 0, 0)$, $k_1 = c_1^2 + x_1^2$ will remain the largest eigenvalue of K. The geometric discord will keep $D_g = \frac{1}{4}(c_2^2 + c_3^2)$ before γ reaches the sudden change point at around 0.41. The geometric discord under the operation of phase damping channel is plotted in Fig. 5. It is clearly shown that the freezing phenomenon is a non-X state. This is similar to the results in Ref. [34], except for different values of Bloch vectors.

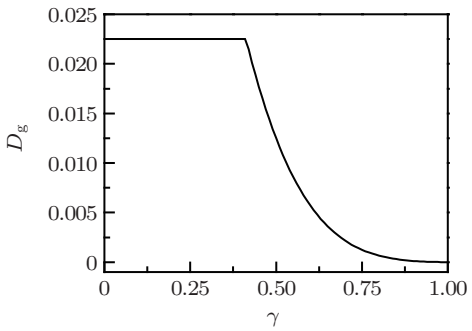


Fig. 5. Geometric discord as a function of γ. The freezing phenomenon occurs when the parameters are chosen as $\boldsymbol{c} = (0.8, 0, 0.3)$ and $\boldsymbol{x} = (0.2, 0, 0)$.

4. Conclusion

In this paper, the geometric discord of non-X states is investigated with different values of Bloch vectors. The level surfaces of the geometric discord are formed by three intersecting open tubes along orthogonal directions and they shrink toward the center when the values of Bloch vectors are increased. One or two of the three tubes may disappear, or the three open tubes may become closed tubes and the level surfaces may become a polyhedron with smooth vertices and edges if the Bloch vectors are increased to threshold values. For non-X states under generalized amplitude damping channel, there are two sudden changes in the curve of geometric discord. The sudden changes appear at the cross points of the maximum eigenvalues. The freezing phenomenon in geometric discord also occurs in the phase damping channel. These phenomena may be helpful in controlling the quantum correlation.

References

[1] Hill S and Wootters W K 1997 *Phys. Rev. Lett.* **78** 5022
[2] Bennett C H, Brassard G, Crepeau C, Jozsa R, Peres A and Wootters W K 1993 *Phys. Rev. Lett.* **70** 1895
[3] Bennett C H, DiVicenzo D P, Shor P W, Smolin J A, Terhal B M and Wootters W K 2001 *Phys. Rev. Lett.* **87** 077902
[4] Scarani V, Lblisdir S, Gisin N and Acin A 2005 *Rev. Mod. Phys.* **77** 1225
[5] Ekert A K 1991 *Phys. Rev. Lett.* **67** 661
[6] Knill E and Laflamme R 1998 *Phys. Rev. Lett.* **81** 5672
[7] Ollivier H and Zurek W H 2001 *Phys. Rev. Lett.* **88** 017901
[8] Henderson L and Vedral V 2001 *J. Phys. A: Math. Gen.* **34** 6899

[9] Luo S 2008 *Phys. Rev. A* **77** 022301
[10] Ali M, Rau A R P and Alber G 2010 *Phys. Rev. A* **81** 042105
[11] Li B, Wang Z X and Fei S M 2011 *Phys. Rev. A* **83** 022321
[12] Modi K, Paterek T, Son W, Vedral V and Williamson M 2010 *Phys. Rev. Lett.* **104** 080501
[13] Werlang T, Souza S, Fanchini F F and Villas Boas C J 2009 *Phys. Rev. A* **80** 024103
[14] Wang B, Xu Z Y, Chen Z Q and Feng M 2010 *Phys. Rev. A* **81** 014101
[15] Xu J W and Chen Q H 2012 *Chin. Phys. B* **21** 040302
[16] Qian Y and Xu J B 2012 *Chin. Phys. B* **21** 030305
[17] Su X L 2013 *Chin. Phys. B* **22** 080304
[18] Auccaise R, Celeri L C, Soares-Pinto D O, deAzevedo E R, Maziero J, Souza A M, Bonagamba T J, Sarthour R S, Oliveira I S and Serra R M 2011 *Phys. Rev. Lett.* **107** 140403
[19] Mazzola L, Piilo J and Maniscalco S 2010 *Phys. Rev. Lett.* **104** 200401
[20] You B and Cen L 2012 *Phys. Rev. A* **86** 012102
[21] Piani M, Horodecki P and Horodecki R 2008 *Phys. Rev. Lett.* **100** 090502
[22] Luo S and Sun W 2010 *Phys. Rev. A* **82** 012338
[23] Madhok V and Datta A 2011 *Phys. Rev. A* **83** 032323
[24] Gu M, Chrzanowski H M, Assad S M, Symul T, Modi K, Ralph T C, Vedral V and Lam P K 2012 *Nat. Phys.* **8** 671
[25] Yao J Y, Dong Y L and Zhu S Q 2013 *Commun. Theor. Phys.* **59** 693
[26] Chen Q Y, Fang M F, Xiao X and Zhou X F 2011 *Chin. Phys. B* **20** 050302
[27] Wang L C, Shen J and Yi X X 2011 *Chin. Phys. B* **20** 050306
[28] Wang C and Chen Q H 2013 *Chin. Phys. B* **22** 040304
[29] Dakic B, Vedral V and Brukner C 2010 *Phys. Rev. Lett.* **105** 190502
[30] Luo S and Fu S 2010 *Phys. Rev. A* **82** 034302
[31] Rana S and Parashar P 2012 *Phys. Rev. A* **85** 024102
[32] Hassan A S M, Lari B and Joag P S 2012 *Phys. Rev. A* **85** 024302
[33] Bellomo B, Giorgi G L, Galve F, Franco R L, Compagno G and Zambrini R2012 *Phys. Rev. A* **85** 032104
[34] Yao Y, Li H W, Yin Z Q and Han Z F 2012 *Phys. Lett. A* **376** 358
[35] Mohamed A B A 2013 *Quantum. Inf. Process* **12** 1141
[36] Adhikari S and Banerjee S 2012 *Phys. Rev. A* **86** 062313
[37] Xiao R L, Xiao X and Zhong W J 2013 *Chin. Phys. B* **22** 080306
[38] Dakic B, Lipp Y O, Ma X, Ringbauer M, Kropatschek S, Barz S, Paterek T, Vedral V, Zeilinger A, Brukner C and Walther P 2012 *Nat. Phys.* **8** 666
[39] Giorgi G L 2013 *Phys. Rev. A* **88** 022315
[40] Lang M D and Caves C M 2010 *Phys. Rev. Lett.* **105** 150501
[41] Montealegre J D, Paula F M, Saguia A and Sarandy M S 2013 *Phys. Rev. A* **87** 042115
[42] Piani M 2012 *Phys. Rev. A* **86** 034101
[43] Tufarelli T, Girolami D, Vasile R, Bose S and Adesso G 2012 *Phys. Rev. A* **86** 052326
[44] Hu X, Fan H, Zhou D L and Liu W M 2012 arXiv: 1203.6149
[45] Chang L and Luo S 2013 *Phys. Rev. A* **87** 062303