Entanglement and non-Markovianity of a spin-S system in a dephasing environment*

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(Received 9 August 2013; revised manuscript received 13 December 2013; published online 10 April 2014)

We study the entanglement (measured by negativity) evolution and the non-Markovianity for the dynamical process of a spin-S system embedded in dephasing environments. The exact analytical solution is presented, which shows that the decoherence function governs the evolutions of coherence, entanglement, and the non-Markovianity of the corresponding dynamical processes. For Ohmic and sub-Ohmic reservoirs, the negativity decreases monotonically in time and the corresponding dynamics is Markovian. While for super-Ohmic reservoirs with non-monotonic decoherence function, the negativity appears as the phenomenon of revival and the corresponding dynamics is non-Markovian. The relation between non-Markovianity and the system dimension is studied.

Keywords: entanglement, non-Markovianity, spin-S system, dephasing

PACS: 03.65.Yz, 42.50.Dv, 42.50.Lc **DOI:** 10.1088/1674-1056/23/6/060303

1. Introduction

During the last few years, there has been an increasing interest in open quantum systems with memory; that is, non-Markovian dynamical systems. On the one hand, the memory effect is crucial in many realistic systems, such as quantum optical systems,[1–3] solid systems,[4,5] quantum chemistry,[6,7] and biological systems.[8,9] On the other hand, it can induce some interesting phenomena, such as entanglement trapping,[10–13] correlation quantum beat,[14] and impossible thermalization.[15] More importantly, non-Markovianity may be used as a useful resource in the context of quantum metrology,[16,17] quantum key distribution,[18] quantum teleportation,[19] atomic squeezing,[20,21] and quantum information transmission.[22–24]

Identifying and quantifying the non-Markovianity of an open quantum process is a fundamental problem in the study of open quantum systems. The existing measures of non-Markovianity may be grouped into two types. One is mainly from the physical point of view, such as the trace distance,[25] fidelity, or relative entropy[26] which is based on the distinguishability of quantum states, the quantum entanglement,[27] and quantum mutual information,[28] which are based on the quantum correlation property. The other starts from more mathematical points of view, such as the semigroup property[29] or the divisibility of dynamical maps.[27] These measures capture the intrinsic characteristics of non-Markovianity of quantum processes from different aspects and do not coincide exactly.[30–32] In this paper, we will use the measure based on the entanglement negativity[33,34] to study the non-Markovianity of the considered system because negativity can be straightforwardly extended to high-dimensional systems and the calculation is feasible.

In the previous works about non-Markovian quantum processes, much attention has been poured into the study of two-level open quantum systems that has seldom involved their higher-dimensional versions. However, high-spin systems or effective multi-level systems extensively exist in nature.[35–37] This can increase security[38,39] in quantum cryptography by exploiting the high-dimensional systems. Furthermore, high-dimensional systems have advantages in fault-tolerant quantum computation and quantum error correction.[40] Motivated by these facts, in this paper we study the entanglement evolution and the non-Markovianity of the spin-S systems embedded in dephasing environments. Special attention is paid to the dependence of entanglement evolution on the initial-state parameter and bath structure, and the dependence of non-Markovianity on the system dimension.

This paper is organized as follows. In Section 2, we provide the microscopic model for a spin-S system coupled to a dephasing environment and present the exact solution. In Section 3, we study the evolution of entanglement for the three-dimensional Werner-like state and the d-dimensional GHZ state. In Section 4, we study the dependence of the non-Markovianity on the dimension of the system. Finally, a conclusion is given in Section 5.

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11275064 and 11075050), the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20124306110003), the Program for Changjiang Scholars and Innovative Research Team in Universities of China (Grant No. IRT0964), and the Construct Program of the National Key Discipline.

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2. Interaction model

Consider a spin-S system interacting with a thermal reservoir, which is modeled by an infinite chain of quantum harmonic oscillators, with ω_k, b_k, and $b_k^\dagger$ being, respectively, the frequency, annihilation, and creation operators for the k-th oscillator. The total Hamiltonian in the Schrödinger picture is given by

$$H = \omega_0 S_z + \sum_k \omega_k b_k^\dagger b_k + \sum_k S_z(g_k b_k^\dagger + g_k^* b_k), \tag{1}$$

where ω_0 is the transition frequency between any neighboring energy states of the spin, and S_z is the z component of spin operator, which in the eigen-basis $\{|j\rangle | j = s, \ldots, -s\}$ can be denoted by a diagonal matrix $S_z = \mathrm{diag}[s, s-1, \ldots, -s]$. The spin couples to its environment via dephasing interaction with coupling strength g_k. This kind of dephasing model is reasonable. It is a natural generalization of dephasing interactions from spin-1/2 systems to higher dimensional spin systems, and in the limit of infinite dimensions it describes the dephasing of harmonic oscillators.[41] It can also be used to describe the dephasing of N spin-1/2 particles in a common environment with $S_z = \sum_{i=1}^N \sigma_{iz}$.[1,41] By introducing Eq. (1) into the interaction picture, one has,

$$H_\mathrm{I} = \sum_k S_z(g_k b_k^\dagger \mathrm{e}^{\mathrm{i}\omega_k t} + g_k^* b_k \mathrm{e}^{-\mathrm{i}\omega_k t}), \tag{2}$$

which, up to an overall phase factor, produces the unitary propagator,

$$V(t) = \exp\left[\frac{1}{2} S_z \sum_k (\alpha_k b_k^\dagger - \alpha_k^* b_k)\right], \tag{3}$$

with $\alpha_k = 2g_k(1 - \mathrm{e}^{\mathrm{i}\omega_k t})/\omega_k$. This is a displacement operator acting on the bath modes, with the displacement parameters conditional on the spin states.

Suppose that the initial state of the spin-bath system is in a product state $\rho_\mathrm{T}(0) = \rho(0) \otimes \rho_\mathrm{B}$, where $\rho(0)$ is the initial state of the spin, and

$$\rho_\mathrm{B} = \frac{1}{Z_\mathrm{B}} \mathrm{e}^{-\beta \Sigma_k \omega_k b_k^\dagger b_k}$$

is the thermal equilibrium state of the bath with partition function Z_B and inverse temperature $\beta = 1/k_\mathrm{B}T$. The dynamical density-matrix elements of the spin system can thus be calculated as

$$\begin{aligned} \rho_{nm}(t) &= \langle n | \mathrm{tr}_\mathrm{B}[V(t)\rho_\mathrm{T}(0)V^{-1}(t)] | m \rangle \\ &= \rho_{nm}(0) \langle \exp[\sum_k (\beta_k b_k^\dagger - \beta_k^* b_k)] \rangle, \end{aligned} \tag{4}$$

where $n, m = s, \ldots, -s$ and $\beta_k = (n-m)\alpha_k/2$. The angular brackets denote the expectation value with respect to the thermal state ρ_B. Following the process in Ref. [1], the expectation can be easily evaluated and, finally, equation (4) may be expressed as

$$\rho_{nm}(t) = \rho_{nm}(0) \exp[-(n-m)^2 \Gamma(t)], \tag{5}$$

where, in the continuum-mode limit, the decoherence function is defined by

$$\Gamma(t) = \int_0^\infty J(\omega) \coth(\omega/2k_\mathrm{B}T) \frac{1 - \cos(\omega t)}{\omega^2}, \tag{6}$$

with spectral density $J(\omega) = \sum_k |g_k|^2 \delta(\omega - \omega_k)$. It is shown that the decoherence function determines the evolution of the off-diagonals of spin density matrix. All the off-diagonals decay exponentially with $\Gamma(t)$, with the rates proportional to the distances $(n-m)^2$ of the elements ρ_{nm} to the diagonals ρ_{nn} or ρ_{mm}. If $\Gamma(t)$ is a monotonous increasing function of time, then the off-diagonals (or equivalently the coherence) decay monotonously with time. If there exist some time intervals in which $\Gamma(t)$ decreases, then the off-diagonals increase in these time intervals; that is, produces the phenomenon of recoherence. The behavior of $\Gamma(t)$ is closely related to the characteristics of environments. In this paper, we mainly involve the Ohmic-like reservoirs with spectral density

$$J(\omega) = \frac{\omega^r}{\omega_\mathrm{c}^{r-1}} \mathrm{e}^{-\omega/\omega_\mathrm{c}}, \tag{7}$$

where ω_c is the bath cutoff frequency. By changing the parameter r, one can go from sub-Ohmic reservoir ($r < 1$) to Ohmic ($r = 1$) and super-Ohmic ($r > 1$) reservoirs, respectively. Such engineering of the Ohmic-like spectrum is possible when simulating the dephasing model in trapped ultracold atoms.[42] In Fig. 1, we present the evolution of the decoherence function $\Gamma(\tau)$ versus the dimensionless time $\tau = tk_\mathrm{B}T$ for Ohmic-like reservoirs, where we set the dimensionless cutoff frequency $\tilde{\omega}_\mathrm{c} = \omega_\mathrm{c}/k_\mathrm{B}T = 20$. This shows that $\Gamma(\tau)$ increases monotonically with time for sub-Ohmic or Ohmic reservoirs, but appears as an oscillation for the super-Ohmic reservoir. In fact it was shown[43] that the monotonic increasing property of $\Gamma(\tau)$

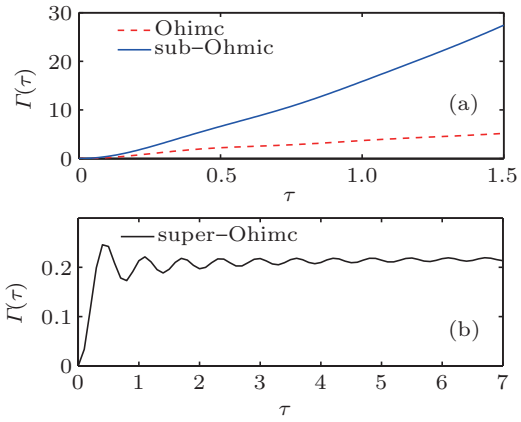


Fig. 1. (color online) Evolution of $\Gamma(\tau)$ versus dimensionless time τ for (a) sub-Ohmic reservoir with $r = 1/2$ (solid line) and Ohmic reservoir (dash line), (b) super-Ohmic reservoir with $r = 2.5$.

could be broken only for the super-Ohmic spectrum with $r \geq 2$. In addition, we see from the figure that at given times, $\Gamma(\tau)$ for super-Ohmic reservoir is apparently much smaller than that for Ohmic or sub-Ohmic reservoirs, implying that the decoherence induced by the super-Ohmic reservoir is relatively much weaker.

3. Entanglement evolution

Entanglement is a unique quantum mechanical phenomenon that has no counterpart in the classical world. So far, much attention has been poured into its research and many effective measures have been presented. Here, we will use the notion of negativity[33,34] to describe the entanglement of the considered system. Given the state ρ^{AB} of a bipartite system, the negativity is defined by

$$\mathcal{N}(\rho^{AB}) = \sum_k \left|\lambda_k^{\mathrm{T}_A(-)}\right| = \frac{\Sigma_k|\lambda_k^{\mathrm{T}_A}|-1}{2}, \tag{8}$$

where $\lambda_k^{\mathrm{T}_A(-)}$, $\lambda_k^{\mathrm{T}_A}$ are, respectively, the negative and all eigenvalues of the partial transpose of ρ^{AB} with respect to subsystem A.

Now, we consider two spins A and B, which are embedded in two equal but independent dephasing environments. According to the method presented by Bellomo *et al.*,[44] the dynamics of the two-spin system can be obtained from one of the individuals. After a tedious but straightforward deduction, we arrive at the dynamical density-matrix elements of the two-spin system

$$\begin{aligned}&\langle n_A n_{\mathrm{B}}|\rho^{AB}(t)|m_A m_{\mathrm{B}}\rangle\\ =\ &\langle n_A n_{\mathrm{B}}|\rho^{AB}(0)|m_A m_{\mathrm{B}}\rangle\\ &\times \exp\left\{-[(n_A-m_A)^2+(n_{\mathrm{B}}-m_{\mathrm{B}})^2]\Gamma(t)\right\},\end{aligned} \tag{9}$$

with $n_A, n_{\mathrm{B}}, m_A, m_{\mathrm{B}} = s, \ldots, -s$.

Let us firstly study the entanglement evolution of the Werner-like state,[45] which is composed by a maximally mixed state and a maximally entangled state. For a two-qutrit system (i.e., $s=1$), the Werner-like state may be written as

$$\rho_\varepsilon = \frac{(1-\varepsilon)}{9} I + \varepsilon|\Psi^{AB}\rangle\langle\Psi^{AB}|, \tag{10}$$

where I denotes the three-dimensional identity matrix, and $|\Psi^{AB}\rangle = (|1^A1^B\rangle + |0^A0^B\rangle + |-1^A-1^B\rangle)/\sqrt{3}$ is a two-qutrit GHZ state. This state is separable for $0 \leq \varepsilon \leq 1/4$, and entangled for $1/4 < \varepsilon \leq 1$. By defining the tensor basis by

$$\begin{aligned}&|1^A1^B\rangle = |1\rangle,\ \ |1^A0^B\rangle = |2\rangle,\ \ |1^A-1^B\rangle = |3\rangle,\\ &|0^A1^B\rangle = |4\rangle,\ \ |0^A0^B\rangle = |5\rangle,\ \ |0^A-1^B\rangle = |6\rangle,\\ &|-1^A1^B\rangle = |7\rangle,\ \ |-1^A0^B\rangle = |8\rangle,\\ &|-1^A-1^B\rangle = |9\rangle,\end{aligned} \tag{11}$$

the dynamical density matrix of the Werner-like state in the local dephasing baths has only the following nonzero elements,

$$\begin{aligned}\rho_{22} &= \rho_{33} = \rho_{44} = \rho_{66} = \rho_{77} = \rho_{88} = \frac{1-\varepsilon}{9},\\ \rho_{11} &= \rho_{55} = \rho_{99} = \frac{1+2\varepsilon}{9},\\ \rho_{15} &= \rho_{19} = \rho_{59} = \frac{\varepsilon}{3}\mathrm{e}^{-8\Gamma(t)}.\end{aligned} \tag{12}$$

After some straightforward calculations, we get its negativity,

$$\begin{aligned}\mathcal{N}(\rho) =\ & -\frac{1}{6}(1-\varepsilon) + \frac{\varepsilon}{6}\left[2\mathrm{e}^{-2\Gamma(t)} + \mathrm{e}^{-8\Gamma(t)}\right]\\ &+ \left|\frac{1-\varepsilon}{9} - \frac{\varepsilon}{3}\mathrm{e}^{-2\Gamma(t)}\right|\\ &+ \frac{1}{2}\left|\frac{1-\varepsilon}{9} - \frac{\varepsilon}{3}\mathrm{e}^{-8\Gamma(t)}\right|.\end{aligned} \tag{13}$$

It is clear that $\mathcal{N}(\rho)$ is a function of $\Gamma(t)$. In order to more clearly find the dependent relation of $\mathcal{N}(\rho)$ on $\Gamma(t)$, we distinguish the problem into three cases according to the values of $\Gamma(t)$. Firstly, for small $\Gamma(t)$ satisfying $\mathrm{e}^{-8\Gamma(t)} > (1-\varepsilon)/3\varepsilon$, both of the two expressions in the inner of $|\cdot|$ of the third and fourth terms in Eq. (13) are negative, thus equation (13) becomes

$$\mathcal{N}_1 = -\frac{1}{3}(1-\varepsilon) + \frac{2\varepsilon}{3}\mathrm{e}^{-2\Gamma(t)} + \frac{\varepsilon}{3}\mathrm{e}^{-8\Gamma(t)}. \tag{14}$$

Secondly, for intermediate $\Gamma(t)$ satisfying $\mathrm{e}^{-8\Gamma(t)} \leq (1-\varepsilon)/3\varepsilon \leq \mathrm{e}^{-2\Gamma(t)}$, equation (13) becomes,

$$\mathcal{N}_2 = -\frac{2}{9}(1-\varepsilon) + \frac{2\varepsilon}{3}\mathrm{e}^{-2\Gamma(t)}. \tag{15}$$

Finally, for large $\Gamma(t)$ satisfying $\mathrm{e}^{-2\Gamma(t)} < (1-\varepsilon)/3\varepsilon$, equation (13) becomes

$$\mathcal{N}_3 = 0. \tag{16}$$

Note that equations (14)–(16) form a continuous function of $\Gamma(t)$, that is, at the crossovers $\mathrm{e}^{-8\Gamma(t)} = (1-\varepsilon)/3\varepsilon$ and $\mathrm{e}^{-2\Gamma(t)} = (1-\varepsilon)/3\varepsilon$, one has $\mathcal{N}_1 = \mathcal{N}_2$ and $\mathcal{N}_2 = \mathcal{N}_3$, respectively. Since both $\mathcal{N}_1$ and $\mathcal{N}_2$ are the monotonic decreasing functions of $\Gamma(t)$, we thus conclude that $\mathcal{N}(\rho)$ is a monotonous decreasing function of $\Gamma(t)$.

For sub-Ohmic or Ohmic reservoirs, since $\Gamma(t)$ is the monotonous increasing function of time, the negativity is thus the monotonic decreasing function of time. The condition and time for negativity sudden death is determined by

$$\Gamma(t) = -\frac{1}{2}\ln\frac{1-\varepsilon}{3\varepsilon}.$$

For the super-Ohmic reservoir, since the monotonicity of $\Gamma(t)$ can be broken, the negativity may appear in the phenomenon of revivals. In Figs. 2, 3, and 4, we plot the evolution of negativity versus time and the initial-state parameter ε for sub-Ohmic, Ohmic, and super-Ohmic reservoirs, respectively. For clarity, we make the areas where the negativity is zero in blue.

It is clear that the initial entanglement is nonzero only for $1/4 < \varepsilon \leq 1$. From these figures, we can sum up two features. Firstly, the entanglement for the sub-Ohmic or Ohmic environments decays monotonically, revealing the feature of Markovian dynamics. While for the super-Ohmic environment it decays oscillatorily, revealing a non-Markovian dynamical feature. Secondly, the speed of the decay of entanglement from sub-Ohmic reservoir to Ohmic and super-Ohmic reservoirs decreases in turn (also see Fig. 5), implying that super-Ohmic reservoir is more favorable for entanglement preservation.

Apart from qutrits, the entanglement evolution of more higher dimensional systems may be studied in a similar way, but the process will become more and more cumbersome. After an incomplete induction, we find when the initial state of the two spins is in the maximally entangled GHZ state,

$$|\psi^{AB}\rangle = \frac{1}{\sqrt{2s+1}} \sum_{k=-s}^{s} |k^A k^B\rangle, \tag{17}$$

then the corresponding negativity of the evolved state seems to have a universal expression

$$\mathcal{N}(\psi) = \sum_{n=1}^{d-1} \frac{d-n}{d} \mathrm{e}^{-2n^2\Gamma(t)}, \tag{18}$$

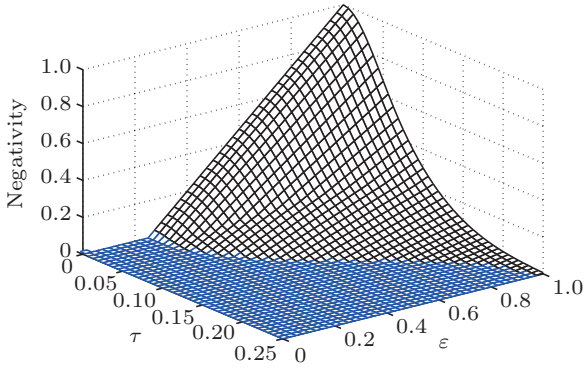


Fig. 2. (color online) Negativity as a function of the dimensionless time τ and the initial state parameter ε for sub-Ohmic spectrum reservoir with $r = 1/2$ and $\tilde{\omega}_c = 20$.

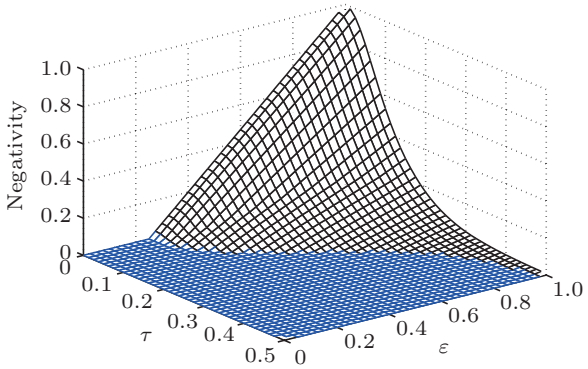


Fig. 3. (color online) Negativity as a function of the dimensionless time τ and the initial state parameter ε for Ohmic spectrum reservoir with $\tilde{\omega}_c = 20$.

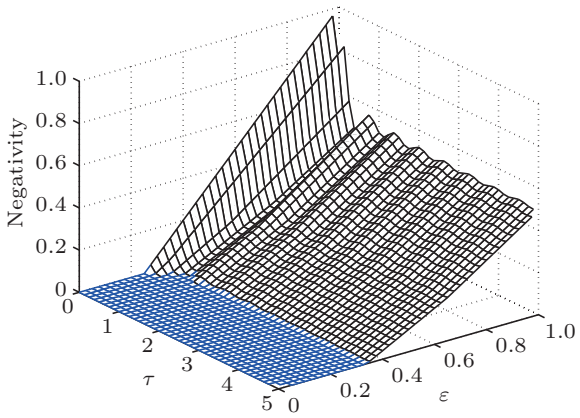


Fig. 4. (color online) Negativity as a function of the dimensionless time τ and the initial state parameter ε for super-Ohmic spectrum reservoir with $r = 2.5$ and $\tilde{\omega}_c = 20$.

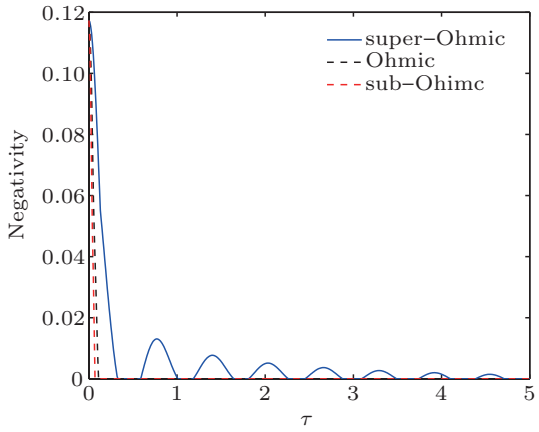


Fig. 5. (color online) Comparison of entanglement evolutions for three Ohmic-like reservoirs: Red, black, and blue lines indicate respectively sub-Ohmic ($r = 1/2$), Ohmic, and super-Ohmic ($r = 2.5$) reservoirs, where $\varepsilon = 0.338$ and $\tilde{\omega}_c = 20$.

with $d = 2s+1$ being the dimension of the spins. This equation is verified to be correct for systems with a spin quantum number up to $s = 5/2$. Equation (18), which is a sum of $(d-1)$ positive functions, is clearly a decreasing function of $\Gamma(t)$. Thus, for sub-Ohmic or Ohmic reservoirs, $\mathcal{N}(\psi)$ monotonously decreases in time, while for some super-Ohmic reservoirs with non-monotonic $\Gamma(t)$, $\mathcal{N}(\psi)$ appears non-monotonicity.

It is worthwhile to point out that Haikka *et al.*[43] have recently studied the non-Markovianity of qubit dynamics in Ohmic-like dephasing environments and find that only a super-Ohmic bath can give rise to the increase of entanglement. Here, from the discussion of entanglement evolution for two-qutrit Werner-like state and high-spin GHZ state, we seem to also conclude the same results. However, from the deduction processes we see clearly that the generalization from qubit to high-spin systems is nontrivial. Moreover, due to the complexity of deductions, we consider only two special kinds of initial states. For other types of high-dimensional initial states, further verifications are needed.

In Fig. 6, we give the time evolution of negativity for different spin quantum numbers and in the super-Ohmic reservoir with $r = 2.5$, $\tilde{\omega}_c = 20$. This shows that larger spin quantum numbers have a larger initial entanglement and the entanglement decays faster in the beginning stage. This implies that the entanglement of high-dimensional systems seems to lose more easily.

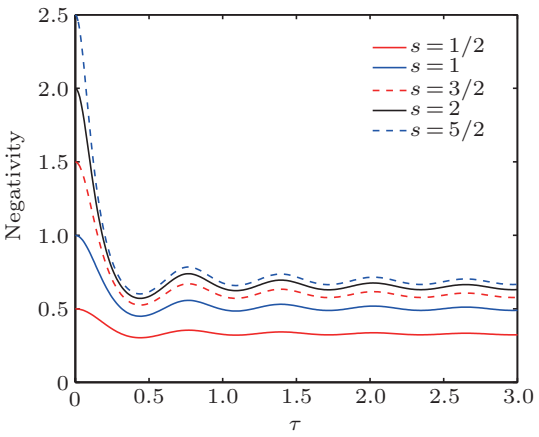


Fig. 6. (color online) Time evolution of negativity for different spin quantum numbers and in the super-Ohmic reservoir ($r = 2.5$, $\tilde{\omega}_c = 20$), where the initial state is given by Eq. (17).

4. Non-Markovianity

From the above section, we see that the entanglement for sub-Ohmic and Ohmic reservoirs decreases monotonously, revealing Markovian behavior. While for super-Ohmic reservoir with $r > 2$ it can increase in some time intervals, revealing non-Markovian behavior. In this section, we proceed to study the non-Markovianity of the dynamical process of the open spin-S system. We will use the non-Markovian measure[27] based on the entanglement with an ancillary particle. Suppose that a system of interest is initially prepared in a maximally entangled state with an ancillary particle, where only the system is influenced by a noise environment and the ancillary particle is noise-free. Then, the quantity for measuring the non-Markovianity of the quantum process is defined as[27]

$$\mathscr{I} = \int_{\dot{E}(t)>0} \dot{E}(t)\,\mathrm{d}t, \tag{19}$$

where $\dot{E}(t)$ denotes the time derivative of some kind of entanglement between the system and the ancillary particle, and the integration is extended over all time intervals in which $\dot{E}(t)$ is positive. For two-level systems, one usually measures the entanglement $E(t)$ by concurrence, while for higher dimensional systems the entanglement negativity may be a suitable choice. In the following, we will use the time evolution of negativity to signify the dynamical non-Markovianity of spin-S systems.

To return to our problem, we suppose that two spins A and B are initially prepared in the maximally entangled GHZ state of Eq. (17), where only spin A is influenced by the dephasing bath and spin B keeps noise-free. In terms of the incomplete induction as given above, we obtain the dynamical negativity between the two high-dimensional spins,

$$\mathscr{N}(t) = \sum_{k=1}^{d-1} \frac{d-k}{d}\,\mathrm{e}^{-k^2\Gamma(t)}, \tag{20}$$

with dimension d defined as before. This equation is also verified for systems with up to six dimensions. By replacing the entanglement $E(t)$ in Eq. (19) by the negativity $\mathscr{N}(t)$, we obtain the non-Markovianity of d-dimensional system,

$$\mathscr{I}_d = -\sum_{k=1}^{d-1} \frac{k^2(d-k)}{d} \int_{\dot{\Gamma}(t)<0} \exp[-k^2\Gamma(t)]\dot{\Gamma}(t)\,\mathrm{d}t. \tag{21}$$

It is clear that $\mathscr{I}_d > 0$ if and only if $\dot{\Gamma}(t) < 0$ in some time intervals. Thus, the non-Markovianity can only possibly appear in the super-Ohmic reservoir, while sub-Ohmic or Ohmic reservoirs cannot induce non-Markovian dynamics. Note that for two-dimensional systems, $\dot{\Gamma}(t)$ is the dephasing rate.[46] This is also true for high-dimensional spin systems because the dynamical state of Eq. (5) satisfies the time-dependent master equation,

$$\dot{\rho} = \dot{\Gamma}(t)[2S_z\rho S_z - S_z^2\rho - \rho S_z^2]. \tag{22}$$

Thus, for the high-dimensional systems in the dephasing environments, we conclude a consistent result with the case of qubit systems that negative decay rates lead to non-Markovianity. It is easy to find that the difference of the non-Markovianity measures for d-dimensional and $(d-1)$-dimensional spin systems is

$$\mathscr{I}_d - \mathscr{I}_{d-1} = -\sum_{k=1}^{d-1} \frac{k^3}{d(d-1)} \int_{\dot{\Gamma}(t)<0} \times \exp[-k^2\Gamma(t)]\dot{\Gamma}(t)\,\mathrm{d}t. \tag{23}$$

Obviously, $\mathscr{I}_d - \mathscr{I}_{d-1} > 0$ provided $\dot{\Gamma}(t) < 0$ in some intervals. Therefore, non-Markovianity increases with the system dimension. We have stressed that this result is correct for any structured dephasing bath. In Fig. 7, we plot the relation of non-Markovianity versus the system dimension in a super-Ohmic environment.

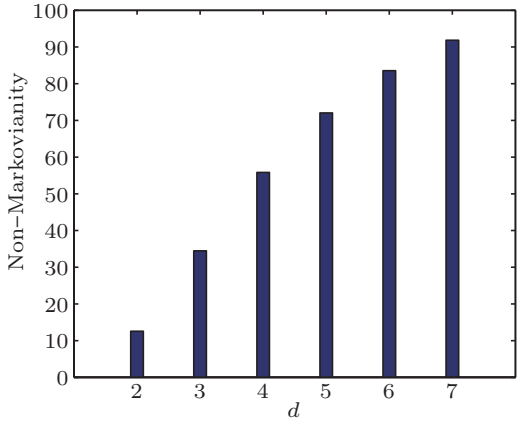


Fig. 7. (color online) Schematic diagram of non-Markovianity versus system dimension d for the super-Ohmic reservoir with $r = 2.5$ and $\tilde{\omega}_c = 20$.

Intuitively, one can imagine that, as the system's dimension increases, the degrees of freedom coupling to the environment also increase, so that the exchange ability of the energy or information enhances, leading to a stronger non-Markovianity. However, recent research[47] has showed that non-Markovianity can increase or decrease with the number of qubits, depending on whether they are embedded in independent or common environments. Therefore, the relation between non-Markovianity and the system size is subtle and nontrivial.

5. Conclusion

We have studied the dynamics of a spin-S system embedded in the dephasing environment and found that the decoherence function plays an important role in the evolution of coherence and entanglement, as well as in the non-Markovianity of the dynamical process. First, all the off-diagonal elements of spin density matrix decay exponentially with respect to the decoherence function, with the decay rates proportional to the distances of the elements to the diagonals. Next, we have found that the entanglement evolution, whether for the three-dimensional Werner-like state or for d-dimensional GHZ state in the local dephasing environment, is a monotonous decreasing function of the decoherence function. The condition of entanglement sudden death for three-dimensional Werner-like state is also given by the decoherence function. Finally, we have quantified the non-Markovianity of d-dimensional systems in terms of the measure based on entanglement, which shows that the condition for the emergence of non-Markovianity is given by $\dot{\Gamma}(t) < 0$. We have also analytically demonstrated that non-Markovianity increases with the system dimension.

These results are in principle independent of the structure of environment; that is, independent of the spectral density. As an exemplary example, we have numerically simulated the evolution of entanglement and the non-Markovianity of the system dynamical process for Ohmic-like environments. We have found that, for sub-Ohmic and Ohmic reservoirs, the coherence (i.e., off-diagonals of the system density matrix) and the entanglement decay monotonously in time, and the dynamics is Markovian. However, for some super-Ohmic environments with non-monotonic $\Gamma(t)$, the coherence and entanglement appear the phenomenon of revivals, and the dynamics become non-Markovian.

Our main contributions are the exact solution of the dephasing model for high-dimensional spin systems, and the subsequent analytical results for the entanglement evolution and the non-Markovianity of the dynamical process. We find that, on the one hand, some similar phenomena observed in qubit systems, such as ESD[48] and the relation between non-Markovianity and negative decay rates,[46] can also exist in high-dimensional systems. On the other hand, some new results or properties, such as the dependency of entanglement and non-Markovianity on the system dimensions, have been found. We believe that these results are useful in the study of non-Markovianity of high-dimensional systems and in quantum information processing.

Appendix A: Derivation for Eqs. (18) and (20)

The derivation of Eq. (18) is straightforward but tedious. Here we give some outlines of the derivation. Starting from the initial state Eq. (17) and after each spin influenced by the local dephasing environments, the dynamical state at time t is governed by Eq. (9). For spin-1/2 system and in the tensor basis

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = |1\rangle, \quad \left|\frac{1}{2},-\frac{1}{2}\right\rangle = |2\rangle,$$
$$\left|-\frac{1}{2},\frac{1}{2}\right\rangle = |3\rangle, \quad \left|-\frac{1}{2},-\frac{1}{2}\right\rangle = |4\rangle,$$

the dynamical state at time t can be written as

$$\rho_{AB}(t) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2}\mathrm{e}^{-2\Gamma(t)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2}\mathrm{e}^{-2\Gamma(t)} & 0 & 0 & \frac{1}{2} \end{pmatrix}. \tag{A1}$$

The partial transpose matrix with respect to subsystem A is

$$\rho_{AB}^{\mathrm{T}_A}(t) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\mathrm{e}^{-2\Gamma(t)} & 0 \\ 0 & \frac{1}{2}\mathrm{e}^{-2\Gamma(t)} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \tag{A2}$$

which has eigenvalues

$$\lambda_0 = \frac{1}{2}, \qquad \text{2-fold};$$
$$\lambda_{\pm 1} = \pm\frac{1}{2}\mathrm{e}^{-2\Gamma(t)}, \ \text{singlet},$$

that is, λ_0 is two-fold-degenerate and $\lambda_{\pm 1}$ are not degenerate. Thus, the negativity is $\mathcal{N}_{(d=2)} = \mathrm{e}^{-2\Gamma(t)}/2$. Following this process, we can in principle obtain the negativity of more higher dimensional systems. For example when $s = 1$, the corresponding partial transpose matrix has 9 eigenvalues:

$$\lambda_0 = \frac{1}{3}, \qquad \text{3-fold};$$
$$\lambda_{\pm 1} = \pm\frac{1}{3}\mathrm{e}^{-2\Gamma(t)}, \ \text{2-fold};$$
$$\lambda_{\pm 2} = \pm\frac{1}{3}\mathrm{e}^{-8\Gamma(t)}, \ \text{singlet}.$$

Thus the negativity is

$$\mathcal{N}_{(d=3)} = \frac{2}{3}\mathrm{e}^{-2\Gamma(t)} + \frac{1}{3}\mathrm{e}^{-8\Gamma(t)}.$$

For $s = 3/2$, the corresponding partial transpose matrix has 16 eigenvalues:

$$\lambda_0 = \frac{1}{4}, \qquad \text{4-fold};$$

$$\lambda_{\pm 1} = \pm\frac{1}{4}\mathrm{e}^{-2\Gamma(t)}, \quad \text{3-fold};$$

$$\lambda_{\pm 2} = \pm\frac{1}{4}\mathrm{e}^{-8\Gamma(t)}, \quad \text{2-fold};$$

$$\lambda_{\pm 3} = \pm\frac{1}{4}\mathrm{e}^{-18\Gamma(t)}, \quad \text{singlet}.$$

Therefore, the negativity is

$$\mathcal{N}_{(d=4)} = \frac{3}{4}\mathrm{e}^{-2\Gamma(t)} + \frac{2}{4}\mathrm{e}^{-8\Gamma(t)} + \frac{1}{4}\mathrm{e}^{-18\Gamma(t)}.$$

Through the above derivation, we can summarize the following rules. For spin-S or equivalently $d = 2s+1$ dimensional systems, the corresponding partial transpose matrix has total d^2 eigenvalues. Apart from the d-fold degenerate λ_0, the other eigenvalues are paired in positive–negative forms. Thus, there are total sd negative eigenvalues. We can compactly express these negative eigenvalues as

$$\lambda_{-n} = -\frac{1}{d}\mathrm{e}^{-2n^2\Gamma(t)}, \tag{A3}$$

with $n = 1, 2, \ldots, (d-1)$. The degeneracy for λ_{-n} is $(d-n)$. Therefore, the negativity for this d-dimensional system is

$$\mathcal{N}_d = \sum_{n=1}^{d-1} |\lambda_{-n}|(d-n) = \sum_{n=1}^{d-1} \frac{d-n}{d}\mathrm{e}^{-2n^2\Gamma(t)},$$

in agreement with Eq. (18). We have justified this rule for dimensions up to $d = 6$.

The derivation for Eq. (20) can be done in the same way as above, but note that only spin A is influenced by the dephasing environment. In fact, we find that the corresponding partial transpose matrices have a similar distribution of eigenvalues as mentioned above, but they only require to halve the exponentials in all the above eigenvalues. For example, all the negative eigenvalues for d-dimensional systems can now be expressed as

$$\lambda_{-n} = -\frac{1}{d}\mathrm{e}^{-n^2\Gamma(t)}, \tag{A4}$$

where $n = 1, 2, \ldots, (d-1)$ and the degeneracy for λ_{-n} is still $(d-n)$. Thus, the negativity in this case becomes

$$\mathcal{N}_d = \sum_{n=1}^{d-1} |\lambda_{-n}|(d-n) = \sum_{n=1}^{d-1} \frac{d-n}{d}\mathrm{e}^{-n^2\Gamma(t)},$$

which is in agreement with Eq. (20).

References

[1] Breuer H P and Petruccione F 2002 *The Theory of Open Quantum Systems* (Oxford: Oxford University Press)
[2] Liu B H, Li L, Huang Y F, Li C F, Guo G C, Laine E M, Breuer H P and Piilo J 2011 *Nat. Phys.* **7** 931
[3] Tang J S, Li C F, Li Y L, Zou X B, Guo G C, Breuer H P, Laine E M and Piilo J 2012 *Europhys. Lett.* **97** 10002
[4] Ji Y H and Hu J J 2010 *Chin. Phys. B* **19** 060304
[5] Hoeppe U, Wolff C, Küchenmeister J, Niegemann J, Drescher M, Benner H and Busch K 2012 *Phys. Rev. Lett.* **108** 043603
[6] Shao J 2004 *J. Chem. Phys.* **120** 5053
[7] Pomyalov A and Tannor D J 2005 *J. Chem. Phys.* **123** 204111
[8] Chin A W, Datta A, Caruso F, Huelga S F and Plenio M B 2010 *New J. Phys.* **12** 065002
[9] Guérin T, Bénichou O and Voituriez R 2012 *Nat. Chem.* **4** 568
[10] Bellomo B, Lo Franco R, Maniscalco S and Compagno G 2008 *Phys. Rev. A* **78** 060302
[11] Zhang Y J, Yang X Q, Han W and Xia Y J 2013 *Chin. Phys. B* **22** 090307
[12] Wang X Y, Ding B F and Zhao H P 2013 *Chin. Phys. B* **22** 020309
[13] Wang X Y, Ding B F and Zhao H P 2013 *Chin. Phys. B* **22** 040308
[14] Zeng H S, Zheng Y P, Tang N and Wang G Y 2013 *Quantum Inf. Process.* **12** 1637
[15] Wang Z, Guo Y and Zhou D L 2012 arXiv: 1207.2036
[16] Chin A W, Huelga S F and Plenio M B 2012 *Phys. Rev. Lett.* **109** 233601
[17] Xiang G Y and Guo G C 2013 *Chin. Phys. B* **22** 110601
[18] Vasile R, Olivares S, Paris M G A and Maniscalco S 2011 *Phys. Rev. A* **83** 042321
[19] Laine E M, Breuer H P and Piilo J 2012 arXiv: 1210.8266
[20] Tang N, Xu T T and Zeng H S 2013 *Chin. Phys. B* **22** 030304
[21] Zou H M, Fang M F and Yang B Y 2013 *Chin. Phys. B* **22** 120303
[22] Tang N, Fan Z L and Zeng H S 2013 arXiv: 1306.0676
[23] Bylicka B, Chruściński D and Maniscalco S 2013 arXiv: 1301.2585
[24] Li Y L and Fang M F 2011 *Chin. Phys. B* **20** 100312
[25] Breuer H P, Laine E M and Piilo J 2009 *Phys. Rev. Lett.* **103** 210401
[26] Usha Devi A R, Rajagopal A K and Sudha 2011 *Phys. Rev. A* **83** 022109
[27] Rivas A, Huelga S F and Plenio M B 2010 *Phys. Rev. Lett.* **105** 050403
[28] Luo S, Fu S and Song H 2012 *Phys. Rev. A* **86** 044101
[29] Wolf M M, Eisert J, Cubitt T S and Cirac J I 2008 *Phys. Rev. Lett.* **101** 150402
[30] Chruściński D, Kossakowski A and Rivas Á 2011 *Phys. Rev. A* **83** 052128
[31] Haikka P, Cresser J D and Maniscalco S 2011 *Phys. Rev. A* **83** 012112
[32] Zeng H S, Tang N, Zheng Y P and Xu T T 2012 *Eur. Phys. J. D* **66** 255
[33] Peres A 1996 *Phys. Rev. Lett.* **77** 1413
[34] Horodečki P 1997 *Phys. Lett. A* **232** 333
[35] Yang J, Wang Y, Wang Z, Rong X, Duan C K, Su J H and Du J 2012 *Phys. Rev. Lett.* **108** 230501
[36] Thomale R, Rachel S, Schmitteckert P and Greiter M 2012 *Phys. Rev. B* **85** 195149
[37] Schlottmann P and Zvyagin A A 2012 *Phys. Rev. B* **85** 205129
[38] Bruß D and Macchiavello C 2002 *Phys. Rev. Lett.* **88** 127901
[39] Cerf N J, Bourennane M, Karlsson A and Gisin N 2002 *Phys. Rev. Lett.* **88** 127902
[40] Knill E 2004 quant-ph/0402171
[41] Walls D F and Milburn G J 1985 *Phys. Rev. A* **31** 2403
[42] Haikka P, McEndoo S, De Chiara D, Palma G M and Maniscalco S 2011 *Phys. Rev. A* **84** 031602
[43] Haikka P, Johnson T H and Maniscalco S 2013 *Phys. Rev. A* **87** 010103
[44] Bellomo B, Lo Franco R and Compagno G 2007 *Phys. Rev. Lett.* **99** 160502
[45] Werner R F 1989 *Phys. Rev. A* **40** 4277
[46] Zeng H S, Tang N, Zheng Y P and Wang G Y 2011 *Phys. Rev. A* **84** 032118
[47] Fanchini F F, Karpat G, Castelano L K and Rossatto D Z 2013 *Phys. Rev. A* **88** 012105
[48] Yu T and Eberly J H 2004 *Phys. Rev. Lett.* **93** 140404