New generating function formulae of even- and odd-Hermite polynomials obtained and applied in the context of quantum optics*

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By combining the operator Hermite polynomial method and the technique of integration within an ordered product of operators, for the first time we derive the generating function of even- and odd-Hermite polynomials which will be useful in constructing new optical field states. We then show that the squeezed state and photon-added squeezed state can be expressed by even- and odd-Hermite polynomials.

Keywords: generating function, even- and odd-Hermite polynomials, Hermite polynomial method, technique of integral within an ordered product of operators

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1. Introduction

It is common knowledge that Hermite polynomials $\mathrm{H}_n(x)$ are widely used in quantum mechanics, mathematical physics, nucleon physics, and quantum optics; for example, $\mathrm{H}_n(x)$ are the wave function of quantum harmonic oscillator and the eigenfunction of fractional Fourier transformation. Mathematically, Hermite polynomials can be defined from their generating function

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_n(x)=\mathrm{e}^{2tx-t^2}. \tag{1}$$

One can see from some mathematical handbooks that

$$\sum_{n=0}^{\infty}\frac{1}{(2n)!}\mathrm{H}_{2n}(x)=\frac{1}{\mathrm{e}}\cosh 2x,$$
$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\mathrm{H}_{2n+1}(x)=\frac{1}{\mathrm{e}}\sinh 2x. \tag{2}$$

Then, an interesting question naturally arises: what are the summation formulae for the even-Hermite polynomials (since $\mathrm{H}_n(-x)=(-1)^n\mathrm{H}_n(x)$)

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{2n}(x)=? \tag{3}$$

and odd-Hermite polynomials

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{2n+1}(x)=? \tag{4}$$

And, what is

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{2n+l}(x)=?$$

Are there some physical applications of these summation formulae? To our knowledge, these formulas have not been reported in the literature before, so in this paper we tackle them by combining the operator-Hermite-polynomial method (OHPM) and the technique of integration within an ordered product (IWOP) of operators.[1–3] In Section 2 we briefly review the OHPM. In Sections 3 and 4 we derive the generating function of even $\mathrm{H}_{2n}(x)$ and odd $\mathrm{H}_{2n+1}(x)$, respectively. In Section 5 we deduce the generating function of $\mathrm{H}_{2n+l}(x)$, where l is a positive integer. In Section 6 we derive new expansion of the squeezed state and the photon-added squeezed state in terms of $\mathrm{H}_{2n+l}(X)$, where X is understood as the optical field's quadrature in quantum optics theory. This exhibits that the quantum mechanical operator-ordering method can solve some difficult problems in mathematical physics.

2. Brief review the operator Hermite-polynomial method

The so-called quantum mechanical operator-Hermite-polynomial method replaces the ordinary Hermite polynomial $\mathrm{H}_n(x)$ by $\mathrm{H}_n(X)$, X is the coordinate operator, $X=(a+a^\dagger)/\sqrt{2}$, $[a,a^\dagger]=1$, $[X,P]=\mathrm{i}\hbar$, $P=(a-a^\dagger)/\sqrt{2}\mathrm{i}$. Since the normally ordered form of $\mathrm{H}_n(X)$ has been given by[4–6]

$$\mathrm{H}_n(X)=:(2X)^n:=:\sqrt{2^n}(a+a^\dagger)^n:, \tag{5}$$

which is a power function within the normal ordering symbol : :, within : : the creation operator $a^\dagger$ is commutable with the annihilation operator a, we have much convenience in handling $:X^n:$ than tackling with Hermite-polynomials. In Ref. [7] we have also derived the inverse relation of Eq. (5); that is,

$$X^n=(2\mathrm{i})^{-n}:\mathrm{H}_n(\mathrm{i}X):. \tag{6}$$

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Equations (5) and (6) are the core formulae of the OHPM, and are also useful when studying the properties of function $\mathrm{H}_n(x)$. For instance, using Eq. (5) we have $\mathrm{H}_n(-X) =: (-2X)^n := (-1)^n \mathrm{H}_n(X)$, and using Baker–Hausdorff formula we can proceed

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{(2n)!} \mathrm{H}_{2n}(X) \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n)!} :(2X)^{2n}: \\
&= :\cosh(2X): := \frac{1}{2}(:\mathrm{e}^{\sqrt{2}(a+a^{\dagger})}: + :\mathrm{e}^{-\sqrt{2}(a+a^{\dagger})}:) \\
&= \frac{1}{2}(\mathrm{e}^{\sqrt{2}(a+a^{\dagger})}/\mathrm{e} + \mathrm{e}^{-\sqrt{2}(a+a^{\dagger})}/\mathrm{e}) \\
&= \frac{1}{\mathrm{e}} \cosh 2X, \qquad (7)
\end{aligned}$$

so equation (2) is proved. There is another formula that we will use shortly

$$\mathrm{H}_n(fX) = (\sqrt{1-f^2})^n : \mathrm{H}_n\left(\frac{fX}{\sqrt{1-f^2}}\right) :, \quad f \neq 1, \qquad (8)$$

which can be derived as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H}_n(fX) \\
&= \mathrm{e}^{2tfX - t^2} \\
&= \mathrm{e}^{\sqrt{2}tf(a+a^{\dagger}) - t^2} \\
&= :\mathrm{e}^{-t^2(1-f^2) + \sqrt{2}tf(a+a^{\dagger})}: \\
&= :\exp\left[-t^2(1-f^2) + 2t\sqrt{(1-f^2}\frac{fX}{\sqrt{1-f^2}}\right]: \\
&= :\sum_{n=0}^{\infty} \frac{(t\sqrt{1-f^2})^n}{n!} \mathrm{H}_n\left(\frac{fX}{\sqrt{1-f^2}}\right): \qquad (9)
\end{aligned}$$

by comparing the two sides of this equation we can obtain Eq. (8).

3. The generating function of even $\mathrm{H}_{2n}(x)$

According to OHPM and using Eq. (5) we consider

$$\sum_{=0}^{\infty} \frac{t^n}{n!} \mathrm{H}_{2n}(X) = \sum_{n=0}^{\infty} \frac{t^n}{n!} 2^{2n} :X^{2n}: =: \mathrm{e}^{4tX^2}:, \qquad (10)$$

then we use the IWOP form of the coordinate representation

$$\int_{-\infty}^{\infty} \mathrm{d}x \left|x\right\rangle \left\langle x\right| =: \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{\pi}} \mathrm{e}^{-(x-X)^2} := 1, \qquad (11)$$

where

$$\left|x\right\rangle = \pi^{-1/4} \exp\left[-\frac{x^2}{2} + \sqrt{2}xa^{\dagger} - \frac{1}{2}a^{\dagger 2}\right] \left|0\right\rangle, \qquad (12)$$

we can deduce

$$\begin{aligned}
\mathrm{e}^{\lambda X^2} &= \int_{-\infty}^{\infty} \mathrm{d}x \mathrm{e}^{\lambda X^2} \left|x\right\rangle \left\langle x\right| \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{\pi}} :\exp\left[-(1-\lambda)x^2 + \sqrt{2}x(a+a^{\dagger})\right. \\
&\quad \left. - \frac{1}{2}(a+a^{\dagger})^2\right]: \\
&= \frac{1}{\sqrt{1-\lambda}} \exp\left[\frac{\lambda a^{\dagger 2}}{2(1-\lambda)}\right] \exp\left[-a^{\dagger} a \ln(1-\lambda)\right] \\
&\quad \times \exp\left[\frac{\lambda a^2}{2(1-\lambda)}\right] \\
&= \frac{1}{\sqrt{1-\lambda}} :\exp\left[\frac{\lambda}{1-\lambda} X^2\right]:, \qquad (13)
\end{aligned}$$

so

$$:\mathrm{e}^{4tX^2}: := \frac{1}{\sqrt{1+4t}} \exp\left(\frac{4t}{1+4t} X^2\right), \qquad (14)$$

then

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H}_{2n}(X) = \frac{1}{\sqrt{1+4t}} \exp\left(\frac{4t}{1+4t} X^2\right). \qquad (15)$$

Changing X back to x we have the generating function of even $\mathrm{H}_{2n}(x)$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H}_{2n}(x) = \frac{1}{\sqrt{1+4t}} \exp\left(\frac{4t}{1+4t} x^2\right). \qquad (16)$$

One can see that the above approach is concise and neat.

4. The generating function of odd $\mathrm{H}_{2n+1}(x)$

According to OHPM we replace $\mathrm{H}_{2n+1}(x)$ by $\mathrm{H}_{2n+1}(X)$, using Eqs. (5) and (13) we consider

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H}_{2n+1}(X) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} 2^{2n+1} :X^{2n+1}: \\
&= :2X\mathrm{e}^{4tX^2}: \\
&= :\sqrt{2}(a+a^{\dagger})\mathrm{e}^{4tX^2}: \\
&= \sqrt{2}a^{\dagger} :\mathrm{e}^{4tX^2}: + \sqrt{2} :\mathrm{e}^{4tX^2}: a \\
&= \frac{\sqrt{2}}{\sqrt{1+4t}} \left(a^{\dagger} \exp\left(\frac{4t}{1+4t} X^2\right)\right. \\
&\quad \left. + \exp\left(\frac{4t}{1+4t} X^2\right) a\right). \qquad (17)
\end{aligned}$$

As a result of Baker–Hausdorff formula

$$\begin{aligned}
\mathrm{e}^{gX^2} a \mathrm{e}^{-gX^2} &= \mathrm{e}^{g(a+a^{\dagger})^2/2} a \mathrm{e}^{-g(a+a^{\dagger})^2/2} \\
&= a + \frac{g}{2}\left[a^{\dagger 2} + 2a^{\dagger}a, a\right] + \cdots \\
&= a - ga^{\dagger} - ga + \cdots \\
&= a - \sqrt{2}gX. \qquad (18)
\end{aligned}$$

It then follows that

$$\begin{aligned}
&\exp\left(\frac{4t}{1+4t} X^2\right) a \\
&= \exp\left(\frac{4t}{1+4t} X^2\right) a \exp\left(-\frac{4t}{1+4t} X^2\right) \exp\left(\frac{4t}{1+4t} X^2\right)
\end{aligned}$$

$$= \left[a - \sqrt{2}\frac{4t}{1+4t}X\right] \exp\left[\frac{4t}{1+4t}X^2\right]. \quad (19)$$

Substituting Eq. (19) into Eq. (17) yields

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{2n+1}(X)\\ &= \frac{\sqrt{2}}{\sqrt{1+4t}}\left(a^\dagger + a - \sqrt{2}\frac{4t}{1+4t}X\right)\exp\left[\frac{4t}{1+4t}X^2\right]\\ &= \frac{2}{(1+4t)^{3/2}}X\exp\left(\frac{4t}{1+4t}X^2\right). \quad (20)\end{aligned}$$

Changing X back to x we have the generating function of odd $\mathrm{H}_{2n+1}(x)$

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{2n+1}(x) = \frac{2x}{(1+4t)^{3/2}}\exp\left(\frac{4t}{1+4t}x^2\right), \quad (21)$$

we think that equations (16) and (21) should appear in later mathematical handbooks because they are fundamental and useful.

5. The generating function of $\mathrm{H}_{2n+l}(x)$

Now, we hope to tackle $\sum_{n=0}^{\infty}t^n\mathrm{H}_{2n+l}(x)/n!$ to unite Eqs. (16) and (21), $l = 0,1,2,3\ldots$, still we consider the operator

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{2n+l}(X) &= \sum_{n=0}^{\infty}\frac{t^n}{n!}:2^{2n+l}X^{2n+l}:= 2^l:X^l\,\mathrm{e}^{4tX^2}:\\ &= \sqrt{2^l}:(a+a^\dagger)^l\,\mathrm{e}^{4tX^2}:\\ &= \sqrt{2^l}\sum_{k=0}^{l}\binom{l}{k}a^{\dagger k}:\mathrm{e}^{4tX^2}:a^{l-k}. \quad (22)\end{aligned}$$

Then, using Eqs. (13) and (19) we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{2n+l}(X)\\ &= \frac{\sqrt{2^l}}{\sqrt{1+4t}}\sum_{k=0}^{l}\binom{l}{k}a^{\dagger k}\exp\left(\frac{4t}{1+4t}X^2\right)a^{l-k}\\ &\quad\times\exp\left(\frac{-4t}{1+4t}X^2\right)\exp\left(\frac{4t}{1+4t}X^2\right)\\ &= \frac{\sqrt{2^l}}{\sqrt{1+4t}}\sum_{k=0}^{l}\binom{l}{k}a^{\dagger k}\left(a-\frac{4\sqrt{2}t}{1+4t}X\right)^{l-k}\\ &\quad\times\exp\left(\frac{4t}{1+4t}X^2\right). \quad (23)\end{aligned}$$

Due to the operator identity

$$\begin{aligned}&(\mu a+\upsilon a^\dagger)^n\\ &= \left(-\mathrm{i}\sqrt{\frac{\mu\upsilon}{2}}\right)^n:\mathrm{H}_n\left(\mathrm{i}\sqrt{\frac{\mu}{2\upsilon}}a+\mathrm{i}\sqrt{\frac{\upsilon}{2\mu}}a^\dagger\right):, \quad (24)\end{aligned}$$

we can express $(a-(4\sqrt{2}t)/(1+4t)X)^{l-k}$ as

$$\begin{aligned}\left(a-\frac{4\sqrt{2}t}{1+4t}X\right)^{l-k} &= \left(\frac{1}{1+4t}a-\frac{4t}{1+4t}a^\dagger\right)^{l-k}\\ &= \left(\frac{-\mathrm{i}\sqrt{-2t}}{1+4t}\right)^{l-k}:\mathrm{H}_{l-k}\\ &\quad\times\left(\mathrm{i}\sqrt{\frac{1}{-8t}}a+\mathrm{i}\sqrt{-2t}a^\dagger\right):. \quad (25)\end{aligned}$$

Substituting Eq. (25) into Eq. (23) leads to

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{2n+l}(X)\\ &= \frac{\sqrt{2^l}}{\sqrt{1+4t}}\sum_{k=0}^{l}\binom{l}{k}\left(\frac{-\mathrm{i}\sqrt{-2t}}{1+4t}\right)^{l-k}\\ &\quad\times:a^{\dagger k}\mathrm{H}_{l-k}\left(\mathrm{i}\sqrt{\frac{1}{-8t}}a+\mathrm{i}\sqrt{-2t}a^\dagger\right)\\ &\quad\times:\exp\left(\frac{4t}{1+4t}X^2\right). \quad (26)\end{aligned}$$

To proceed, we must have a new binomial theorem regarding the Hermite polynomial, we shall make summation $\sum_{l=0}^{m}\binom{m}{l}y^l q^{m-l}\mathrm{H}_{m-l}(x)$, for this aim, we replace $\mathrm{H}_{m-l}(x)$ by $\mathrm{H}_{m-l}(X)$ and consider

$$\begin{aligned}&\sum_{l=0}^{m}\binom{m}{l}y^l q^{m-l}\mathrm{H}_{m-l}(X)\\ &= \sum_{l=0}^{m}\binom{m}{l}y^l(2q)^{m-l}:X^{m-l}:=:(2qX+y)^m:. \quad (27)\end{aligned}$$

By multiplying both sides $\sum_m^{\infty}\lambda^m/m!$ from the left and making the summation we have

$$\begin{aligned}&\sum_{m=0}^{\infty}\frac{\lambda^m}{m!}\sum_{l=0}^{\infty}\binom{m}{l}y^l q^{m-l}\mathrm{H}_{m-l}(X)\\ &= \sum_{m=0}^{\infty}\frac{\lambda^m}{m!}:(2qX+y)^m:=:\mathrm{e}^{\lambda(2qX+y)}:. \quad (28)\end{aligned}$$

Using Baker–Hausdorff formula and Eq. (1) we see that

$$\begin{aligned}:\mathrm{e}^{\lambda(2qX+y)}: &= \exp\left[2\lambda q(X+\frac{y}{2q})-\lambda^2q^2\right]\\ &= \sum_{m=0}^{\infty}\frac{(\lambda q)^m}{m!}\mathrm{H}_m\left(X+\frac{y}{2q}\right). \quad (29)\end{aligned}$$

Comparing Eqs. (28) and (29) we obtain a new operator identity

$$\sum_{l=0}^{m}\binom{m}{l}y^{m-l}q^l\mathrm{H}_l(X) = q^m\mathrm{H}_m\left(X+\frac{y}{2q}\right). \quad (30)$$

Returning $X\to x$, we deduce a new binomial theorem about $\mathrm{H}_l(x)$

$$\sum_{l=0}^{\infty}\binom{m}{l}y^{m-l}q^l\mathrm{H}_l(x) = q^m\mathrm{H}_m\left(x+\frac{y}{2q}\right). \quad (31)$$

Now, we continue to discuss Eq. (26), using Eq. (31) it becomes

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{2n+l}(X)$$

$$
\begin{aligned}
&= \frac{\sqrt{2^l}}{\sqrt{1+4t}} : \left(\frac{-\mathrm{i}\sqrt{-2t}}{1+4t}\right)^l \\
&\quad \times \mathrm{H}_l\left(\mathrm{i}\sqrt{\frac{1}{-8t}}a + \frac{a^\dagger}{-\mathrm{i}2\sqrt{-2t}}\right) : \exp\left(\frac{4t}{1+4t}X^2\right) \\
&= \frac{\sqrt{2^l}}{\sqrt{1+4t}}\left(\frac{-\mathrm{i}\sqrt{-2t}}{1+4t}\right)^l : \mathrm{H}_l\left(\frac{\mathrm{i}X}{2\sqrt{-t}}\right) \\
&\quad \times : \exp\left(\frac{4t}{1+4t}X^2\right), \qquad (32)
\end{aligned}
$$

where

$$
: \mathrm{H}_l\left(\frac{\mathrm{i}X}{2\sqrt{-t}}\right) := \left(\frac{4t+1}{4t}\right)^{l/2} \mathrm{H}_l\left(\sqrt{\frac{1}{4t+1}}X\right), \qquad (33)
$$

which is the result of using Eq. (8). Substituting Eq. (33) into Eq. (32) leads to

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{2n+l}(X) \\
&= \frac{\sqrt{2^l}}{\sqrt{1+4t}}\left(\frac{\sqrt{2t}}{1+4t}\right)^l\left(\frac{4t+1}{4t}\right)^{l/2} \\
&\quad \times \mathrm{H}_l\left(\sqrt{\frac{1}{4t+1}}X\right)\exp\left(\frac{4t}{1+4t}X^2\right) \\
&= \frac{1}{(\sqrt{1+4t})^{l+1}}\mathrm{H}_l\left(\sqrt{\frac{1}{4t+1}}X\right)\exp\left(\frac{4t}{1+4t}X^2\right). \qquad (34)
\end{aligned}
$$

Replacing $X \to x$, we obtain the formula about the generating function of $\mathrm{H}_{2n+l}(x)$

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{2n+l}(x) \\
&= \frac{1}{(\sqrt{1+4t})^{l+1}}\mathrm{H}_l\left(\sqrt{\frac{1}{4t+1}}x\right)\exp\left(\frac{4t}{1+4t}x^2\right). \qquad (35)
\end{aligned}
$$

In particular, for $l=0$, and $l=1$, since $\mathrm{H}_0(x)=1$, $\mathrm{H}_1(x)=2x$, equation (34) respectively reduces to Eqs. (16) and (21). For $l=2$, and $l=3$,

$$
\mathrm{H}_2(x) = 4x^2-2, \;\; \mathrm{H}_3(x) = 8x^3-12x, \qquad (36)
$$

from Eq. (35) we have

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{2n+2}(x) &= \frac{4x^2-8t-2}{(4t+1)^{5/2}}\exp\left(\frac{4t}{1+4t}x^2\right), \\
\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{2n+3}(x) &= \frac{8x^3-12x-48tx}{(1+4t)^{7/2}}\exp\left(\frac{4t}{1+4t}x^2\right). \qquad (37)
\end{aligned}
$$

6. Physical applications

Theoretically, we introduce a new optical state in the Fock space

$$
|\,\rangle_t = \sqrt{1-16t^2}\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{2n}(X)|0\rangle, \qquad (38)
$$

where X is now understood as optical field's quadrature. Using Eq. (10) we know

$$
|\,\rangle_t = \sqrt{1-16t^2} : \mathrm{e}^{4tX^2} : |0\rangle = \sqrt{1-16t^2}\,\mathrm{e}^{2ta^{\dagger 2}}|0\rangle, \qquad (39)
$$

which is a normalized squeezed state, since

$$
{}_t\langle|\,\rangle_t = (1-16t^2)\langle 0|\,\mathrm{e}^{2ta^2}\,\mathrm{e}^{2ta^{\dagger 2}}|0\rangle = 1, \qquad (40)
$$

where we have used the operator identity

$$
\begin{aligned}
&\mathrm{e}^{fa^2}\,\mathrm{e}^{ga^{\dagger 2}} \\
&= \frac{1}{\sqrt{1-4fg}}\exp\left[\frac{ga^{\dagger 2}}{1-4fg}\right] \\
&\quad \times \exp[-a^\dagger a\ln(1-4fg)]\exp\left[\frac{fa^2}{1-4fg}\right]. \qquad (41)
\end{aligned}
$$

Combining Eqs. (39) and (41) we see

$$
\sqrt{1-16t^2}\,\mathrm{e}^{2ta^{\dagger 2}}|0\rangle = \sqrt{1-16t^2}\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{2n}(X)|0\rangle. \qquad (42)
$$

Let $t=(1/4)\tanh\lambda$, equation (42) becomes

$$
\begin{aligned}
&\mathrm{sech}^{1/2}\lambda\,\mathrm{e}^{a^{\dagger 2}\tanh\lambda/2}|0\rangle \\
&= \mathrm{sech}^{1/2}\lambda\sum_{n=0}^{\infty}\frac{(\tanh\lambda)^n}{4^n n!}\mathrm{H}_{2n}(X)|0\rangle, \qquad (43)
\end{aligned}
$$

this shows that a squeezed state can also be expanded as the sum of $\mathrm{H}_{2n}(X)|0\rangle$, the Hermite-polynomial $\mathrm{H}_{2n}(X)$ excitation of the vacuum state. One may compare this with the ordinary expansion in terms of the even-number Fock state

$$
\begin{aligned}
&\mathrm{sech}^{1/2}\lambda\,\mathrm{e}^{a^{\dagger 2}\tanh\lambda/2}|0\rangle \\
&= \mathrm{sech}^{1/2}\lambda\sum_{n=0}^{\infty}\frac{(\tanh\lambda)^n\sqrt{(2n)!}}{2^n n!}|2n\rangle. \qquad (44)
\end{aligned}
$$

Using Eq. (42) we can directly obtain the wave function

$$
\begin{aligned}
&\mathrm{sech}^{1/2}\lambda\,\langle x|\,\mathrm{e}^{a^{\dagger 2}\tanh\lambda/2}|0\rangle \\
&= \mathrm{sech}^{1/2}\lambda\sum_{n=0}^{\infty}\frac{(\tanh\lambda)^n}{4^n n!}\mathrm{H}_{2n}(x)\,\mathrm{e}^{-x^2/2} \\
&= \mathrm{e}^{-\lambda/2}\exp\left[\frac{\tanh\lambda-1}{2(1+\tanh\lambda)}x^2\right]. \qquad (45)
\end{aligned}
$$

Similarly, we construct the state

$$
\begin{aligned}
|\psi\rangle &\equiv \sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{2n+l}(X)|0\rangle \\
&= \sqrt{2^l}\sum_{k=0}^{l}\binom{l}{k}a^{\dagger k} : \mathrm{e}^{4tX^2} : a^{l-k}|0\rangle \\
&= \sqrt{2^l}a^{\dagger l}\,\mathrm{e}^{2ta^{\dagger 2}}|0\rangle, \qquad (46)
\end{aligned}
$$

which is the photon-added squeezed state. We directly know the wave function of $|\psi\rangle$

$$
\psi(x) = \sqrt{2^l}\,\langle x|\,a^{\dagger l}\,\mathrm{e}^{2ta^{\dagger 2}}|0\rangle
$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H}_{2n+l}(x)$$
$$= \frac{1}{(\sqrt{1+4t})^{l+1}} \mathrm{H}_l\left(\sqrt{\frac{1}{4t+1}}x\right) \times \exp\left(\frac{4t}{1+4t}x^2\right). \quad (47)$$

In summary, we have derived for the first time the generating function of even- and odd-Hermite polynomials, which will be useful in constructing new optical field states. We believe that these new fundamental formulae will appear in mathematical handbooks in the near future.

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