# A novel hierarchy of differential–integral equations and their generalized bi-Hamiltonian structures*

Zhai Yun-Yun(翟云云)[a)†], Geng Xian-Guo(耿献国)[a)], and He Guo-Liang(何国亮)[b)]

[a)] School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, China
[b)] School of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou 450002, China

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With the aid of the zero-curvature equation, a novel integrable hierarchy of nonlinear evolution equations associated with a $3\times3$ matrix spectral problem is proposed. By using the trace identity, the bi-Hamiltonian structures of the hierarchy are established with two skew-symmetric operators. Based on two linear spectral problems, we obtain the infinite many conservation laws of the first member in the hierarchy.

**Keywords:** spectral problem, nonlinear evolution equations, bi-Hamiltonian structure, conservation laws

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## 1. Introduction

The theory of an integrable system has always been one of the most intriguing research fields in theoretical physics and applied mathematics. One of the key features of an integrable system is that it possesses a Lax representation (i.e., a compatibility condition of two linear spectral problems), which enables one to solve it by using the inverse scattering transformation,[1,2] the algebra-geometric method,[3–6] the Darboux transformation,[7–10] and others.[11–16] However, besides some well-known models, such as KdV, AKNS, KP, Camassa–Holm, Boussinesq, Toda lattice, Ablowitz–Ladik equations, and others, the integrability of many nonlinear evolution equations is still an open problem in the study of integrable systems. Therefore, it is interesting to search for new integrable systems and to establish their Hamiltonian structures for further investigation.[17–20]

In this paper, we propose a novel hierarchy of nonlinear evolution differential–integral equations associated with a $3\times3$ matrix spectral problem, as described in Section 2. The first nontrivial member can be reduced to the following nonlinear evolution equation

$$\begin{aligned} &u_t=-u_{xx}+uv+u^2w,\\ &v_t=-(uw)_t+2(uw_x)_x,\\ &w_{xt}=w_{xxx}-vw_x-uww_x. \end{aligned} \tag{1}$$

In Section 3, the generalized bi-Hamiltonian structures for the nonlinear evolution hierarchy are established by utilizing the trace identity. Especially, the two skew-symmetric operators and the Hamiltonian functions are explicitly given. In the last section, we obtain the infinite many conservation laws of the first nontrivial equation in the hierarchy by considering the two linear spectral problems associated with it.

## 2. The nonlinear evolution equations

In this section, we first introduce a $3\times3$ matrix spectral problem

$$\psi_x=U\psi,\quad \psi=\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix},\quad U=\begin{pmatrix}0&1&0\\ v-\lambda&0&u\\ 0&w&0\end{pmatrix}, \tag{2}$$

where $u$, $v$, $w$ are three potentials, and $\lambda$ is a constant spectral parameter. Then, we derive a hierarchy of the nonlinear evolution equations associated with the matrix spectral problem (2). In order to derive the corresponding nonlinear evolution hierarchy, we define two sets of Lenard recursion equations

$$\begin{aligned} &Ks_j=Js_{j+1},\ \ j\geq0,\ \ s_j\,|_{(u,v,w)=0}=0,\ \ j\geq1,\\ &K\hat{s}_j=J\hat{s}_{j+1},\ \ j\geq0,\ \ \hat{s}_j\,|_{(u,v,w)=0}=0,\ \ j\geq1, \end{aligned} \tag{3}$$

with the starting points $s_0=(1,0,0,w)^{\mathrm{T}}$, $\hat{s}_0=(0,1,0,0)^{\mathrm{T}}$, and the two operators given by ($\partial\partial^{-1}=\partial^{-1}\partial=1$)

$$K=\begin{pmatrix}\partial u+2u\partial & 3u & \partial^2-v+2uw & 0\\ \partial v+v\partial-u\partial^{-1}vw & \partial^2-u\partial^{-1}w\partial & vw & u\partial^{-1}v\\ -2w\partial+\partial^{-1}vw & -3w+\partial^{-1}w\partial & -2w^2 & \partial-\partial^{-1}v\\ \partial^2+uw & 2\partial & \partial w+w\partial & -u\end{pmatrix}, \tag{4}$$

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†Corresponding author. E-mail: zhai1226@163.com
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$$J=\begin{pmatrix} 0 & 0 & -1 & 0 \\ 2\partial-u\partial^{-1}w & 0 & w & u\partial^{-1} \\ \partial^{-1}w & 0 & 0 & -\partial^{-1} \\ \partial^2+uw & 2\partial & \partial w+w\partial & -u \end{pmatrix}. \tag{5}$$

Thus we can uniquely determine the sequences $s_j$ and $\hat{s}_j$, for example

$$s_1=\begin{pmatrix} \frac{1}{2}v+\frac{1}{2}uw \\ -\frac{1}{4}v_x-\frac{3}{4}uw_x+\frac{3}{4}u_xw \\ -u_x \\ \frac{1}{2}vw-w_{xx}+\frac{1}{2}uw^2 \end{pmatrix}, \quad \hat{s}_1=\begin{pmatrix} 0 \\ 3uw \\ -3u \\ 3w_x \end{pmatrix}, \tag{6}$$

$$s_2=\begin{pmatrix} -\frac{1}{8}v_{xx}+\frac{3}{8}v^2+\frac{3}{4}uvw+\frac{3}{8}u^2w^2-\frac{1}{8}u_{xx}w-\frac{7}{8}uw_{xx}+\frac{1}{2}u_xw_x \\ s_2^{(2)} \\ u_{xxx}-\frac{3}{2}u_xv-\frac{3}{4}uv_x-\frac{9}{4}uu_xw+\frac{3}{4}u^2w_x \\ s_2^{(4)} \end{pmatrix}, \tag{7}$$

$$\hat{s}_2=\begin{pmatrix} 3uw_x \\ -3u_{xx}w-3uw_{xx}+3uvw+3u^2w^2 \\ 3u_{xx}-3uv-3u^2w \\ -3w_{xxx}+3vw_x+6uww_x \end{pmatrix}, \tag{8}$$

where

$$\begin{aligned} s_2^{(2)} &= \frac{1}{16}v_{xx}-\frac{3}{8}vv_x-\frac{15}{16}u_{xxx}w+\frac{15}{16}uw_{xxx}+\frac{5}{16}u_{xx}w_x \\ &\quad -\frac{5}{16}u_xw_{xx}+\frac{9}{8}u_xvw+\frac{3}{8}uv_xw \\ &\quad -\frac{9}{8}uvw_x+\frac{15}{8}uu_xw^2-\frac{15}{8}u^2ww_x, \\ s_2^{(4)} &= w_{xxxx}-\frac{1}{8}v_{xx}w-\frac{3}{4}v_xw_x-\frac{3}{2}vw_{xx}+\frac{3}{8}v^2w \\ &\quad -\frac{1}{8}u_{xx}w^2-\frac{1}{4}u_xww_x-\frac{19}{8}uww_{xx} \\ &\quad -\frac{9}{4}uw_x^2+\frac{3}{4}uvw^2+\frac{3}{8}u^2w^3. \end{aligned}$$

Now, we are in a position to solve the stationary zero-curvature equation

$$V_x-[U,V]=0, \quad V=\begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}, \tag{9}$$

which is equivalent to

$$\begin{aligned} &V_{11,x}+(v-\lambda)V_{12}-V_{21}=0, \\ &V_{12,x}+V_{11}-V_{22}+wV_{13}=0, \\ &V_{13,x}+uV_{12}-V_{23}=0, \\ &V_{21,x}+v(V_{22}-V_{11})-uV_{31}-\lambda(V_{22}-V_{11})=0, \\ &V_{22,x}+V_{21}+wV_{23}-vV_{12}-uV_{32}+\lambda V_{12}=0, \\ &V_{23,x}+u(V_{22}-V_{33})-(v-\lambda)V_{13}=0, \\ &V_{31,x}+vV_{32}-wV_{21}-\lambda V_{32}=0, \\ &V_{32,x}+w(V_{33}-V_{22})+V_{31}=0, \\ &V_{33,x}+uV_{32}-wV_{23}=0. \end{aligned} \tag{10}$$

Let

$$\begin{aligned} &V_{11}=b, \quad V_{12}=a, \quad V_{13}=c, \quad V_{32}=d, \\ &V_{21}=b_x+(v-\lambda)a, \quad V_{22}=a_x+b+wc, \quad V_{23}=c_x+ua, \\ &V_{31}=\partial^{-1}wb_x-\partial^{-1}v(d-wa)+\lambda\partial^{-1}(d-wa), \\ &V_{33}=-a_x-2b-wc, \end{aligned} \tag{11}$$

where

$$(a,b,c,d)=\sum_{j=0}^{\infty}(a_j,b_j,c_j,d_j)\lambda^{-j}.$$

The substitution of Eq. (11) into Eq. (10) yields

$$KS_j=JS_{j+1}, \quad j\geq 0, \quad JS_0=0, \tag{12}$$

where $S_j=(a_j,b_j,c_j,d_j)^{\mathrm{T}}$. Since equation $JS_0=0$ has the solution

$$S_0=\alpha_0s_0+\hat{\alpha}_0\hat{s}_0, \tag{13}$$

then the functions $S_j$ given by

$$S_j=\alpha_0s_j+\hat{\alpha}_0\hat{s}_j, \tag{14}$$

satisfy the recursion equation (12), where $\alpha_0$ and $\hat{\alpha}_0$ are arbitrary constants.

Let $\psi$ satisfy the spectral problem (2) and the auxiliary problem

$$\psi_{t_n}=V^{(n)}\psi, \quad V^{(n)}=\begin{pmatrix} V_{11}^{(n)} & V_{12}^{(n)} & V_{13}^{(n)} \\ V_{21}^{(n)} & V_{22}^{(n)} & V_{23}^{(n)} \\ V_{31}^{(n)} & V_{32}^{(n)} & V_{33}^{(n)} \end{pmatrix}, \tag{15}$$

where $V_{ij}^{(n)}=(\lambda^nV_{ij})_+$. Then, the compatibility condition of Eqs. (2) and (15) yields the zero-curvature equation,

$$U_{t_n}-V_x^{(n)}+[U,V^{(n)}]=0,$$

which is equivalent to a hierarchy of nonlinear evolution equations

$$(u_{t_n},v_{t_n},w_{t_n})^{\mathrm{T}}=P(KS_n)=P(JS_{n+1}), \quad n\geq 0, \tag{16}$$

where $P$ is the projective map

$$P(\gamma^1,\gamma^2,\gamma^3,\gamma^4)^{\mathrm{T}} \to (\gamma^1,\gamma^2,\gamma^3)^{\mathrm{T}}.$$

The first nontrivial member in the hierarchy (16) is as follows:

$$\begin{aligned}
u_{t_1} &= \frac{1}{4}\alpha_0(-4u_{xxx}+6u_xv+3uv_x+9uu_xw-3u^2w_x)\\
&\quad+3\hat{\alpha}_0(-u_{xx}+uv+u^2w),\\
v_{t_1} &= \frac{1}{4}\alpha_0[-v_{xxx}+6vv_x+6uv_xw+3(u_{xx}w)_x-3(uw_{xx})_x\\
&\quad-3u\partial^{-1}v_{xx}w+3u\partial^{-1}w(uw_x)_x-3u\partial^{-1}w(u_xw)_x]\\
&\quad+3\hat{\alpha}_0[(uw)_{xx}-uvw+u\partial^{-1}vw_x-u\partial^{-1}w(uw)_x],\\
w_{t_1} &= \frac{1}{4}\alpha_0[-4w_{xxx}-3v_xw+6vw_x-3u_xw^2+9uww_x\\
&\quad+3\partial^{-1}v_{xx}w-3\partial^{-1}w(uw_x)_x+3\partial^{-1}w(u_xw)_x]\\
&\quad+3\hat{\alpha}_0[w_{xx}-uw^2-\partial^{-1}vw_x+\partial^{-1}w(uw)_x]. \qquad (17)
\end{aligned}$$

As $\alpha_0=0$, $\hat{\alpha}_0=1/3$, $t_1=t$, equation (17) is reduced to

$$\begin{aligned}
u_t &= -u_{xx}+uv+u^2w,\\
v_t &= (uw)_{xx}-uvw+u\partial^{-1}vw_x-u\partial^{-1}w(uw)_x,\\
w_t &= w_{xx}-uw^2-\partial^{-1}vw_x+\partial^{-1}w(uw)_x, \qquad (18)
\end{aligned}$$

which is equivalent to Eq. (1). For $\alpha_0=4$, $\hat{\alpha}_0=0$, $t_1=t$, equation (17) turns into

$$\begin{aligned}
u_t &= -4u_{xxx}+6u_xv+3uv_x+9uu_xw-3u^2w_x,\\
v_t &= -(uw)_t+(-v_{xx}+3v^2-u_{xx}w+4u_xw_x-7uw_{xx}\\
&\quad+6uvw+3u^2w^2)_x,\\
w_{xt} &= -4w_{xxxx}+3v_xw_x+6vw_{xx}\\
&\quad+6u(ww_x)_x+3w_x(uw)_x. \qquad (19)
\end{aligned}$$

## 3. The generalized bi-Hamiltonian structures

This section aims to derive the generalized bi-Hamiltonian structures of Eq. (16) by trace identity[21]

$$\begin{aligned}
&\left(\frac{\delta}{\delta u},\frac{\delta}{\delta v},\frac{\delta}{\delta w}\right)\int\left(V\frac{\partial U}{\partial\lambda}\right)\mathrm{d}x\\
&=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\left[\mathrm{tr}\left(\frac{\partial U}{\partial u}V\right),\mathrm{tr}\left(\frac{\partial U}{\partial v}V\right),\mathrm{tr}\left(\frac{\partial U}{\partial w}V\right)\right]. \qquad (20)
\end{aligned}$$

To this end, we need to calculate the following quantities:

$$\begin{aligned}
&\mathrm{tr}\left(V\frac{\partial U}{\partial\lambda}\right)=-V_{12},\ \ \mathrm{tr}\left(\frac{\partial U}{\partial u}V\right)=V_{32},\\
&\mathrm{tr}\left(\frac{\partial U}{\partial v}V\right)=V_{12},\ \ \mathrm{tr}\left(\frac{\partial U}{\partial w}V\right)=V_{23}. \qquad (21)
\end{aligned}$$

Substituting Eq. (21) into Eq. (20) for $\hat{\alpha}_0=0$ and $\alpha_0=0$

$$\begin{aligned}
&\left(\frac{\delta}{\delta u},\frac{\delta}{\delta v},\frac{\delta}{\delta w}\right)\int(-V_{12}\mid_{\hat{\alpha}_0=0})\mathrm{d}x\\
&=\lambda^{-\gamma_1}\frac{\partial}{\partial\lambda}\lambda^{\gamma_1}(V_{32},V_{12},V_{23})\mid_{\hat{\alpha}_0=0}, \qquad (22)\\
&\left(\frac{\delta}{\delta u},\frac{\delta}{\delta v},\frac{\delta}{\delta w}\right)\int(-V_{12}\mid_{\alpha_0=0})\mathrm{d}x\\
&=\lambda^{-\gamma_2}\frac{\partial}{\partial\lambda}\lambda^{\gamma_2}(V_{32},V_{12},V_{23})\mid_{\alpha_0=0}, \qquad (23)
\end{aligned}$$

where $\gamma_1$ and $\gamma_2$ are two constants to be fixed, we obtain

$$\begin{aligned}
&\left(\frac{\delta}{\delta u},\frac{\delta}{\delta v},\frac{\delta}{\delta w}\right)\int\left(-V_{12,j+1}\mid_{\hat{\alpha}_0=0}\right)\mathrm{d}x\\
&=(\gamma_1-j)\left(V_{32,j},V_{12,j},V_{23,j}\right)\mid_{\hat{\alpha}_0=0},\ \ j\geq 0, \qquad (24)\\
&\left(\frac{\delta}{\delta u},\frac{\delta}{\delta v},\frac{\delta}{\delta w}\right)\int\left(-V_{12,j+1}\mid_{\alpha_0=0}\right)\mathrm{d}x\\
&=(\gamma_2-j)\left(V_{32,j},V_{12,j},V_{23,j}\right)\mid_{\alpha_0=0},\ \ j\geq 0, \qquad (25)
\end{aligned}$$

by, respectively, comparing the coefficients of Eqs. (22) and (23) with respect to the same power for $\lambda$, where

$$V_{ij,l}=\frac{\partial^{n-l}}{(n-l)!\partial\lambda^{n-l}}V_{ij}^{(n)}\mid_{\lambda=0}.$$

Thus we can easily get that $\gamma_1=-1/2$ and $\gamma_2=0$. Therefore, we arrive at

$$\begin{aligned}
&\left(\frac{\delta}{\delta u},\frac{\delta}{\delta v},\frac{\delta}{\delta w}\right)H_j=(V_{32,j},V_{12,j},V_{23,j}),\\
&H_j=\frac{2\alpha_0}{2j+1}\int s_{j+1}^{(1)}\mathrm{d}x+\frac{\hat{\alpha}_0}{j}\int\hat{s}_{j+1}^{(1)}\mathrm{d}x, \qquad (26)
\end{aligned}$$

from which we obtain the generalized bi-Hamiltonian structures of Eq. (16)

$$\begin{aligned}
\begin{pmatrix}u_{t_n}\\v_{t_n}\\w_{t_n}\end{pmatrix}&=\mathscr{K}\begin{pmatrix}\delta/\delta u\\\delta/\delta v\\\delta/\delta w\end{pmatrix}H_n\\
&=\mathscr{J}\begin{pmatrix}\delta/\delta u\\\delta/\delta v\\\delta/\delta w\end{pmatrix}H_{n+1},\ n\geq 1, \qquad (27)
\end{aligned}$$

where $\mathscr{K}$ and $\mathscr{J}$ are two skew-symmetric operators defined by

$$\mathscr{K}=\begin{pmatrix}\frac{3}{2}u\partial^{-1}u & \frac{1}{2}u\partial+v\partial^{-1}u-\frac{1}{2}uw\partial^{-1}u & \mathscr{K}_{13}\\ \frac{1}{2}\partial u+u\partial^{-1}v-\frac{1}{2}u\partial^{-1}uw & \mathscr{K}_{22} & \mathscr{K}_{23}\\ \partial-\partial^{-1}v+\frac{1}{2}\partial^{-1}uw-\frac{3}{2}w\partial^{-1}u & \mathscr{K}_{32} & \mathscr{K}_{33}\end{pmatrix}, \qquad (28)$$

$$\mathscr{J} = \begin{pmatrix} 0 & \partial^{-1}u & -\partial^{-1} \\ u\partial^{-1} & 2\partial - u\partial^{-1}w - w\partial^{-1}u & w\partial^{-1} \\ -\partial^{-1} & \partial^{-1}w & 0 \end{pmatrix}, \tag{29}$$

with

$$\begin{aligned}
\mathscr{K}_{13} &= \partial - v\partial^{-1} + \frac{1}{2}uw\partial^{-1} - \frac{3}{2}u\partial^{-1}w, \\
\mathscr{K}_{22} &= -\frac{1}{2}\partial^3 + \partial v + v\partial - u\partial^{-1}vw - vw\partial^{-1}u \\
&\quad + \frac{1}{2}\partial^2 w\partial^{-1}u + \frac{1}{2}u\partial^{-1}w\partial^2 - \frac{1}{2}u\partial^{-1}w\partial w\partial^{-1}u, \\
\mathscr{K}_{23} &= -\frac{1}{2}\partial w + vw\partial^{-1} + \frac{1}{2}u\partial^{-1}w^2 - \frac{1}{2}\partial^2 w\partial^{-1} \\
&\quad + \frac{1}{2}u\partial^{-1}w\partial w\partial^{-1}, \\
\mathscr{K}_{32} &= -\frac{1}{2}w\partial + \partial^{-1}vw + \frac{1}{2}w^2\partial^{-1}u - \frac{1}{2}\partial^{-1}w\partial^2 \\
&\quad + \frac{1}{2}\partial^{-1}w\partial w\partial^{-1}u, \\
\mathscr{K}_{33} &= -\frac{1}{2}w^2\partial^{-1} + \frac{3}{2}w\partial^{-1}w - \frac{1}{2}\partial^{-1}w\partial w\partial^{-1} - \frac{1}{2}\partial^{-1}w^2.
\end{aligned}$$

For example, the first nontrivial Hamiltonian function reads as

$$\begin{aligned}
H_1 &= \alpha_0 \int \left( \frac{1}{4}v^2 + \frac{1}{2}uvw + \frac{1}{4}u^2w^2 - \frac{1}{12}u_{xx}w \right. \\
&\quad \left. - \frac{7}{12}uw_{xx} + \frac{1}{3}u_x w_x \right) \mathrm{d}x + 3\hat{\alpha}_0 \int uw_x \mathrm{d}x.
\end{aligned}$$

## 4. Infinite many conservation laws

In this section, we shall derive infinite many conservation laws for Eq. (17). Defining $\phi = \psi_{1,x}/\psi_1$, from the spectral problem (2), we arrive at the Riccati equation

$$\begin{aligned}
&\phi_{xx} + 3\phi\phi_x + \phi^3 - \frac{u_x}{u}(\phi_x + \phi^2) \\
&\quad - (v + uw - \lambda)\phi = v_x - \frac{u_x v}{u} + \lambda\frac{u_x}{u}.
\end{aligned} \tag{30}$$

Let $\lambda = -\eta^2$. By inserting the ansatz

$$\phi = \sum_{j=-1}^{\infty} \phi_j \eta^{-j} \tag{31}$$

into Eq. (30) and comparing the coefficients of $\lambda$ with the same power, we obtain the expressions of $\phi_j$, which can be listed as follows:

$$\begin{aligned}
&\phi_{-1} = 1, \quad \phi_0 = 0, \quad \phi_1 = \frac{1}{2}(v + uw), \\
&\phi_2 = -\frac{1}{4}v_x - \frac{1}{4}u_x w - \frac{3}{4}uw_x, \\
&\phi_3 = \frac{1}{8}v_{xx} + \frac{1}{8}u_{xx}w + \frac{1}{2}u_x w_x + \frac{7}{8}uw_{xx} - \frac{1}{8}(v + uw)^2, \\
&2\phi_j = -\phi_{j-2,xx} - 3\sum_{l=-1}^{j-1} \phi_l \phi_{j-2-l,x} - \sum_{\substack{l+m+n=j-2 \\ -1 \le l,m,n < j}} \phi_l\phi_m\phi_n \\
&\quad + \frac{u_x}{u}\left( \phi_{j-2,x} + \sum_{l=-1}^{j-1} \phi_l \phi_{j-2-l} \right) \\
&\quad + (v + uw)\phi_{j-2}, \ j \ge 3.
\end{aligned} \tag{32}$$

On the other hand, from the auxiliary problem (15), we have

$$\frac{\psi_{1,t}}{\psi_1} = V_{11}^{(n)} + V_{12}^{(n)}\phi + V_{13}^{(n)}\left[ \frac{1}{u}(\phi_x + \phi^2) - \frac{v - \lambda}{u} \right]. \tag{33}$$

Denoting

$$\theta = V_{11}^{(n)} + V_{12}^{(n)}\phi + V_{13}^{(n)}\left[ \frac{1}{u}(\phi_x + \phi^2) - \frac{v - \lambda}{u} \right],$$

since

$$\frac{\partial}{\partial_t}\frac{\psi_{1,x}}{\psi_1} = \frac{\partial}{\partial_x}\frac{\psi_{1,t}}{\psi_1},$$

we obtain

$$\phi_t = \theta_x. \tag{34}$$

Noting that $n = 1$,

$$\begin{aligned}
V_{11}^{(1)} &= \alpha_0\left( -\frac{1}{4}v_x - \frac{3}{4}uw_x + \frac{3}{4}u_x w \right) + 3\hat{\alpha}_0 uw + \hat{\alpha}_0\lambda, \\
V_{12}^{(1)} &= \frac{1}{2}\alpha_0(v + uw) + \alpha_0\lambda, \\
V_{13}^{(1)} &= -\alpha_0 u_x - 3\hat{\alpha}_0 u,
\end{aligned} \tag{35}$$

and expanding $\theta$ into the following series of $\lambda$

$$\theta = -\alpha_0\eta^3 - \hat{\alpha}_0\eta^2 + \sum_{j=1}^{\infty} \theta_j \eta^{-j}, \tag{36}$$

we can obtain the expressions of $\theta_j$ immediately from Eq. (34)

$$\begin{aligned}
\theta_1 &= \alpha_0\left[ \frac{3}{8}(v + uw)^2 - \frac{1}{8}v_{xx} - \frac{1}{8}u_{xx}w + \frac{1}{2}u_x w_x - \frac{7}{8}uw_{xx} \right] \\
&\quad + 3\hat{\alpha}_0 uw_x, \\
\theta_j &= \frac{1}{2}\alpha_0(v + uw)\phi_j - \alpha_0\phi_{j+2} - \left( \alpha_0\frac{u_x}{u} + 3\hat{\alpha}_0 \right) \\
&\quad \times \left( \phi_{j,x} + \sum_{l=-1}^{j+1} \phi_l \phi_{j-l} \right), \quad j \ge 1.
\end{aligned} \tag{37}$$

The coefficients $\phi_j$ and $\theta_j$ in the expansion series of $\phi$ and $\theta$ are called conservation densities and currents respectively. The first conservation law for Eq. (17) is

$$\begin{aligned}
\frac{1}{2}(v + uw)_t &= \alpha_0\left[ \frac{3}{8}(v + uw)^2 - \frac{1}{8}v_{xx} - \frac{1}{8}u_{xx}w \right. \\
&\quad \left. + \frac{1}{2}u_x w_x - \frac{7}{8}uw_{xx} \right]_x + 3\hat{\alpha}_0(uw_x)_x.
\end{aligned} \tag{38}$$

Furthermore, we can conclude that the Hamiltonian function $H_1$ is a conserved quantity included in the conserved quantities $\int \phi_j \mathrm{d}x$, $j \ge 1$, since

$$H_1 = -2\alpha_0 \int \phi_3 \mathrm{d}x - 6\hat{\alpha}_0 \int \phi_2 \mathrm{d}x. \tag{39}$$

## 5. Conclusion

In this paper, we have introduced a $3 \times 3$ matrix spectral problem with three potentials, by which a novel hierarchy of differential–integral equations is proposed. Based on trace identity and the Riccati equation of the eigenfunctions, we have established the generalized bi-Hamiltonian structures of this hierarchy and derive infinite many conservation laws. The first nontrivial equation (1) in the hierarchy is in itself interesting and is of potential applied value. In respect that the hierarchy of nonlinear evolution equations are less known, it remains open to question whether they have soliton solutions, Bäculund transformations, Darboux transformation, or other properties. Any further study involved in the equations can help us to explore new integrable systems and investigate the integrable systems that are already known.

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