Properties of surface waves in granular media under gravity*

Zheng He-Peng(郑鹤鹏)†

Department of Mathematics and Physics, Guilin Medical University, Guilin 541004, *China*

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Acoustical waves propagating along the free surface of granular media under gravity are investigated in the framework of elasticity theory. The influence of stress on a surface wave is analyzed. The results have shown that two types of surface waves, namely sagittal and transverse modes exist depending on initial stress states, which may have some influence on the dispersion relations of surface waves, but the influence is not great. Considering that the present experimental accuracy is far from distinguishing this detail, the validity of elasticity theory on the surface waves propagating in granular media can still be maintained.

Keywords: granular matter, stress, surface waves

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1. Introduction

Granular materials refer to any system composed of a large conglomeration of discrete macroscopic particles.[1] They are widely spread in both geophysical and industrial contexts. For dry, non-cohesive granular media under gravity, a solid-fluid transition would be reached at their mechanically free surface.[2] The behavior of granular media in this critical state is a poorly understood phenomenon in physics and it continues to attract the attention of researchers.[3] Obviously, acoustics offer an important technique for investigation of this phenomenon. Some experimental data[4] and natural phenomena[5] have shown that sound waves at the surface of sand propagate at a speed of approximately 40 m/s. They are localized close to the surface, just like Rayleigh waves. However, the very low velocity of such waves remains unexplained, and their physical nature is still controversial.

Recently, two independent theoretical calculations by Bonneau *et al.*[6,7] and Gusev *et al.*[8,9] have shown that surface waves in sand should be ascribed to a gravity induced index gradient, which produces a wave-guide effect and allows for the propagation of an infinite, yet discrete, number of dispersive surface waves. Gusev and collaborators took the sound velocities increasing with depth z in granular media near the surface as $c \sim (\rho g z)^{1/6}$ (where ρ is the density and g is the gravitational acceleration) into Helmholtz equations,[9] and then obtained the dispersion relations of surface waves through numerical solutions. In contrast, Bonneau and collaborators obtained the properties of these waves in the framework of granular elasticity (or GE) established by Jiang and Liu.[10,11] Note that elasticity theory is not only used for analysing acoustic waves but also for mechanic behaviors.[12] From the physical point of view, analyzing acoustic modes based on granular elasticity is more orthodox. Therefore, in this paper, we continue to use this elasticity theory to describe the propagation of surface waves in an inclined granular layer, and we especially take into account the following issues: (i) What is the influence of initial stress and tilt of granular layers on surface waves? (ii) How do we choose the parameters for a GE model? (iii) When does the coupling between sagittal and transverse surface waves occur?

The present work is a development of Bonneau and collaborators' work, and aims to study the influence of initial stress in granular solids on surface waves. Bonneau *et al.* only considered the simplest possible stress state, and found its theoretical calculations were well agreement with experimental data, except for a difference in the model parameter value. Because the sample is not always in this state, it is worth considering properties of surface waves in other stress states. In addition, the discrepancy of the parameters deserves further discussion.

2. Granular elasticity model

According to elasticity theory, the dynamic equation for the displacements u_i reads

$$\rho \ddot{u}_i = -\nabla_j \sigma_{ij} + \rho g_i, \tag{1}$$

where ρ is the mass density, g_i the vector of gravitational acceleration. Neglecting heat effect, the stress tensor σ_{ij} is given by[12]

$$\sigma_{ij} = -\frac{\partial V}{\partial u_{ij}}, \tag{2}$$

where $V(u_{ij})$ is the deformation energy density, strain $u_{ij} = (\nabla_j u_i + \nabla_i u_j)/2$ for small displacements. Given an explicit expression for the elastic energy V and appropriate boundary conditions, one may obtain elastic displacement u_i, elastic strain u_{ij}, and stress σ_{ij}. For isotropic, normal solids,

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†Corresponding author. E-mail: hepengzheng@163.com

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$V \sim (K\Delta^2 + \mu\Delta_2^2)$ (denoting $\Delta \equiv -u_{ll}$, $\Delta_2 \equiv \sqrt{u_{ij}^* u_{ij}^*}$, $u_{ij}^* \equiv u_{ij} + \Delta\delta_{ij}/3$), is a quadratic function of u_{ij} (a summation on the repeating foot index is assumed), where K, μ are two material-dependent constants, which are referred to as the bulk and shear modulus.[12] This expression accounts well for normal solids, but not for granular systems.[13] To obtain a useful energy expression for granular solids, the contact mechanics of two spheres given by Hertz is a good hint. Based on this hint, Jiang and Liu[10,11] chose the energetic expression of noncohesive granular solids as

$$V = B\sqrt{\Delta}\left(\frac{2}{5}\Delta^2 + \frac{1}{\xi}\Delta_2^2\right), \tag{3}$$

where B, ξ are model coefficients. Typically, take $B = 7$ GPa and $\xi = 5/3$ for sand. An interesting result of Eq. (3) is that the energy V is convex only for $\Delta_2/\Delta \le \sqrt{2\xi}$, and no elastic solution is possible beyond this point. This feature is able to well account for a yield transition of granular matter. It is worth noting that (3) is just an assumption, without a rigorous mathematical proof. A practical way to verify its validity is to compare its results with experiments. Fortunately, up to now this formulation of elastic energy is indeed capable of accounting for many qualitative features observed experimentally, including statics forces, yield, small-amplitude stress-strain relation, stress-induced anisotropy or anisotropic propagation of elastic waves.[14–18] In the present paper, we will address the issue of surface waves propagation by this granular elasticity model.

3. Surface waves in a granular layer under gravity

Consider a static granular layer and the coordinate system shown in Fig. 1, of which the stress varies only with the layer depth z : $\sigma_{ij} = \sigma_{ij}(z)$, as does the strain $u_{ij} = u_{ij}(z)$ by the stress-strain relation (2). Since the energetic expression (3) is not a quadratic function of strain, equation (2) will be a nonlinear elastic relation. For simplicity we neglect mass inhomogeneity (taking ρ = constant).

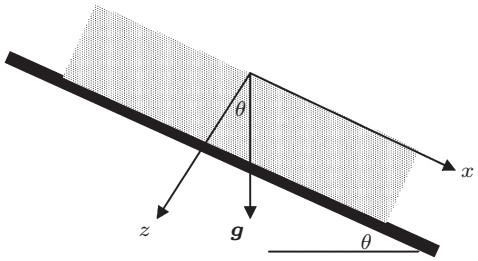


Fig. 1. Granular layer on inclined plane.

Considering the boundary condition that the layer top $z = 0$ is a free surface, on which all stress and strain components are vanishing

$$\sigma_{ij}(z=0) = 0, \tag{4}$$

$$u_{ij}(z=0) = 0. \tag{5}$$

To solve Eqs. (1)–(3) under this boundary condition, one may obtain the static strain tensor

$$u_{ij} = \begin{pmatrix} C_1 z & C_3 z & -\dfrac{z\sin\theta}{2\ell\sqrt{\Delta}} \\ C_3 z & C_2 z & 0 \\ -\dfrac{z\sin\theta}{2\ell\sqrt{\Delta}} & 0 & -\Delta - (C_1 + C_2)z \end{pmatrix}, \tag{6}$$

with C_{1-3} integral constants, $\ell = B/(\xi g\rho)$ is a characteristic length. The dependence of Δ on z is given by (see Ref. [19] for detailed computations)

$$\left[\frac{1}{4}\left(\frac{\sin\theta}{\ell\sqrt{\Delta}}\right)^2 + \left(C_1C_2 + C_1^2 + C_2^2 + C_3^2\right)\right] z^2 + \left[3(C_1 + C_2) - \frac{\cos\theta}{\ell\sqrt{\Delta}}\right] z\Delta + \left(\frac{5}{3} + \xi\right)\Delta^2 = 0. \tag{7}$$

Special solutions with vanishing C_{1-3} were first studied in Ref. [10] to analyze the stability of granular layers, and they were then used to investigate the properties of surface sound waves and the booming instabilities in sand dunes.[6,7] Here, we investigate the properties of surface waves with nonvanishing C_{1-3}. To do this, we take a superposition of modes of the form

$$\tilde{u}_i = A_i(z)\, \mathrm{e}^{\mathrm{i}(kx - \omega t)}, \tag{8}$$

where k is the wave number, ω is the angular frequency, and $A_i(z)$ is the amplitude. The equation of motion (1) can be written as

$$\rho\frac{\partial^2 \tilde{u}_i}{\partial t^2} = -\nabla_j \tilde{\sigma}_{ij}, \tag{9}$$

with $\tilde{\sigma}_{ij}$ is the disturbance to the stress tensor under the following linear approximation: $\tilde{\sigma}_{ij} = \overline{M}_{ijlk}\tilde{u}_{lk}$, while $\overline{M}_{ijkl} = -\partial^2 V/\partial u_{ij}\partial u_{kl}$ is the stiffness tensor. By inserting Eq. (8) into Eq. (9), we get the ordinary differential equation

$$\overline{M}_{izlz}A_l'' + \left[\mathrm{i}k\left(\overline{M}_{ixlz} + \overline{M}_{izlx}\right) + \nabla_z\overline{M}_{izlz}\right]A_l' + \left(\mathrm{i}k\nabla_z\overline{M}_{izlx} - k^2\overline{M}_{ixlx} - \rho\omega^2\delta_{il}\right)A_l = 0. \tag{10}$$

Here, the prime denotes the z derivative. This equation contains three equations about variables A_x, A_y, A_z. Especially when $\overline{M}_{xxxy} = \overline{M}_{xxyz} = \overline{M}_{zzyx} = \overline{M}_{zzyz} = \overline{M}_{yxxx} = \overline{M}_{yzzz} = 0$ (namely $C_3 = 0$), equation (10) expands to

$$\overline{M}_{yzyz}A_y'' + \left(\nabla_z\overline{M}_{yzyz}\right)A_y' - \left(k^2\overline{M}_{yxyx} + \rho\omega^2\right)A_y = 0, \tag{11}$$

$$\begin{aligned} &\overline{M}_{xzxz}A_x'' + \left(\mathrm{i}k\overline{M}_{xxxz} + \mathrm{i}k\overline{M}_{xzxx} + \nabla_z\overline{M}_{xzxz}\right)A_x' \\ &+ \left(\mathrm{i}k\nabla_z\overline{M}_{xzxx} - k^2\overline{M}_{xxxx} - \rho\omega^2\right)A_x \\ &+ \overline{M}_{xzzz}A_z'' + \left(\mathrm{i}k\overline{M}_{xxzz} + \mathrm{i}k\overline{M}_{xzzx} + \nabla_z\overline{M}_{xzzz}\right)A_z' \\ &+ \left(\mathrm{i}k\nabla_z\overline{M}_{xzzx} - k^2\overline{M}_{xxzx}\right)A_z = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} &\overline{M}_{zzxz}A_x'' + \left(\mathrm{i}k\overline{M}_{zxxz} + ik\overline{M}_{zzxx} + \nabla_z\overline{M}_{zzxz}\right)A_x' \\ &+ \left(\mathrm{i}k\nabla_z\overline{M}_{zzxx} - k^2\overline{M}_{zxxx}\right)A_x \end{aligned}$$

$$+\overline{M}_{zzzz}A_z'' + \left(\mathrm{i}k\overline{M}_{zxz} + \mathrm{i}k\overline{M}_{zzzx} + \nabla_z\overline{M}_{zzzz}\right)A_z' + \left(\mathrm{i}k\nabla_z\overline{M}_{zzzx} - k^2\overline{M}_{zxzx} - \rho\omega^2\right)A_z = 0. \quad (13)$$

The component A_y can be solved independently of the components (A_x, A_z), which correspond to two types of surface waves, namely: transverse and sagittal modes, respectively. Conversely, the transverse and sagittal modes will be coupled if the integral constant C_3 is not equal to zero. Equation (11) is the governing equation of transverse surface waves. It is subject to the following conditions:

$$A_y'|_{z=0} = 0, \ \ A_y(H) = 0, \ \ A_y(0) = 1. \quad (14)$$

The first boundary condition in Eq. (14) corresponds to the vanishing shear stress σ_{yz} at the free surface, while the second one ensures localization of the modes, and the third condition represents the choice of normalization for the eigenfunctions. Equations (12) and (13) are the governing equations of sagittal waves, with the conditions

$$\left[\overline{M}_{xzzz}A_z' + \overline{M}_{xzxz}A_x' + \mathrm{i}k\left(\overline{M}_{xzxx}A_x + \overline{M}_{xzxz}A_z\right)\right]|_{z=0} = 0,$$
$$\left[\overline{M}_{zzzz}A_z' + \overline{M}_{zzxz}A_x' + \mathrm{i}k\left(\overline{M}_{zzxx}A_x + \overline{M}_{zzxz}A_z\right)\right]|_{z=0} = 0,$$
$$A_{x,z}(H) = 0, \ \ A_z(0) = 1. \quad (15)$$

Similarly the two first boundary conditions correspond to the absence of the stress $\sigma_{xz} = \sigma_{zz} = 0$ at the surface, the third one is the localization condition, and the last one fixes the normalization of the eigenfunctions.

4. Results and discussion

In this section, some numerical solutions are shown from Eqs. (11)–(15), taking $B = 1.5$ GPa, $\xi = 5/3$, $\rho = 1500$ kg/m^3, $g = 9.8$ m/s^2, $C_1 = C_2 = C$, $C_3 = 0$, $\theta = 0$. The grain size is $d = 150 \pm 25$ μm, and the layer of thickness $H = 0.20$ m. Figure 2 shows the shape and dispersion relation of the first sagittal mode for different integral constants C, and the experimental data come from Ref. [7]. The calculated A_z is a real number, A_x a pure imaginary: $A_x = \mathrm{i}\bar{A}_x$ ($\bar{A}_x$ the real), the phase difference between A_x and A_z is 90°. The value $\bar{A}_x$ becomes negative around 0.04 m depth. The absolute value of A_x, shown in Fig. 2(a), is compared with the measurement. From Fig. 2(b), it is found that different stress states (C) have little effect on the dispersion relation of sagittal mode. In addition, a waveguide cutoff frequency, about 200 Hz, appears at the limit of vanishing kH due to finite depth effect ($H = 0.20$ m) in Fig. 2(b).

Similarly, figure 3 shows the shape and dispersion relation of the first transverse mode for different integral constants C, and the experimental data come from Ref. [7]. The influence of initial stress on transverse mode is similar to the sagittal mode. The range of C, agrees well with the experiments, seems

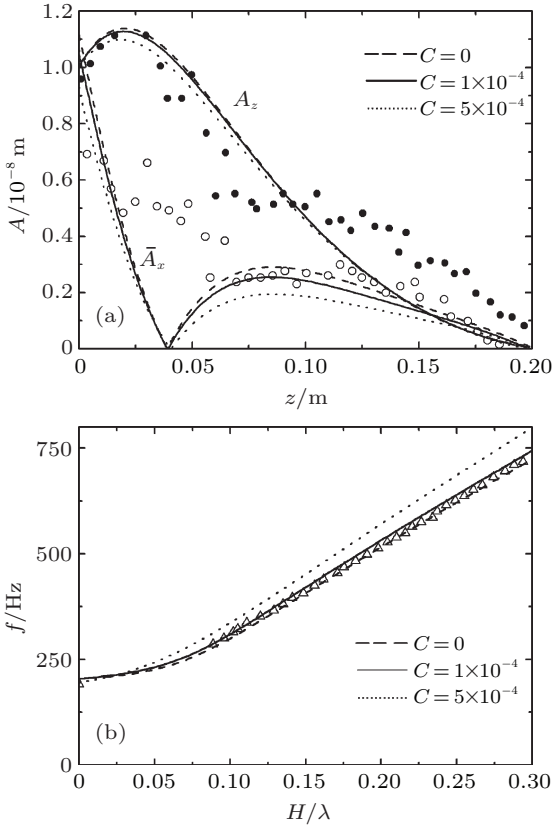


Fig. 2. First sagittal mode under different integral constants. Symbols (spheres, filled spheres, and triangles) represent the experimental data extracted from Ref. [7]. (a) Amplitude of the horizontal (○) and vertical displacement (●) as a function of depth. (b) Dispersion relation.

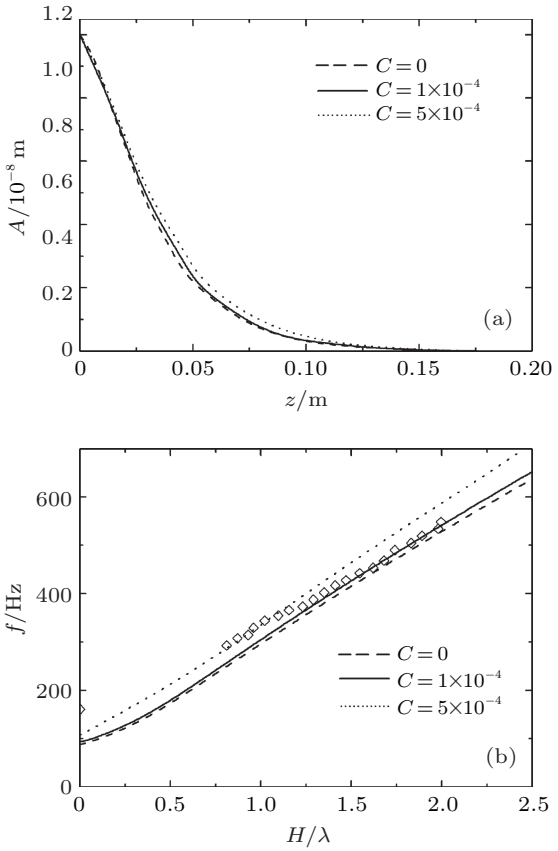


Fig. 3. First transverse mode under different integral constants. (a) Shape of the mode. (b) Dispersion relation.

between 0 and -5×10^{-4}. In addition, the cutoff frequency of transverse mode by theoretical calculation is lower than the measurement value; however, the reason for this is unclear.

It needs to be noted that the special solution ($C=0$) has been studied by Bonneau *et al.*,[6,7] but they took the parameter $\xi=6$ in order to accord with the experimental data. However, the parameter ξ in GE is essentially fixed by the yield angle, and its value is around $\xi\sim1$ for granular systems with yield angle about 30°. Thus, the adjustment range of ξ is small, it would not take $\xi=6$. To satisfy the experimental data, we adjust the other parameter B, while keeping the value $\xi=5/3$ invariant. From Figs. 2 and 3 we can see that theoretical results are also in good agreement with the experiments by taking $B=1.5$ GPa. It is worth noting that the value $B=1.5$ GPa is smaller than the typical value 7 GPa obtained from mechanical testing. The reason for the anomalous decrease of B near the free surface of granular layers remains unclear. Parameter B accounts for overall rigidity in granular elasticity model.[14] It increases with the density ρ, or the packing ratio $\phi\equiv\rho/\rho_{\rm G}$, where $\rho_{\rm G}$ is the bulk density of granular materials. In other words, the value B of dense granular media is larger than that of unconsolidated granular media. Because the confining pressure is very small near the free surface, granular matter is in a loose state. This may be the reason why B decreases.

When the granular layers are inclined ($\theta\neq0$), the transverse mode propagation properties are independent of θ. The shape and dispersion relation of the first sagittal mode in an inclined granular layer with $\theta=26°$ is shown in Figs. 4 and 5. Figure 4 gives the complex amplitudes (A_x, A_z) as a function of the reduced depth kz, where Re and Im denoted the real part and imaginary part of amplitude, respectively. It is seen that the phase difference between A_x and A_z deviates from 90°. Figure 5 presents the dispersion relation of the first sagittal mode with different integral constants: $C=0$ (Fig. 5(a)), and $C=-5\times10^{-4}$ (Fig. 5(b)). The results of the horizontal layers ($\theta=0$) are given simultaneously for comparison. We observe from Fig. 5 that the velocity of surface waves in an inclined granular layer is slightly lower than that in the horizontal layer at a given frequency. These results again confirm that the dispersion relations of surface waves have a weak dependence on the initial stress and tilt of granular layers.

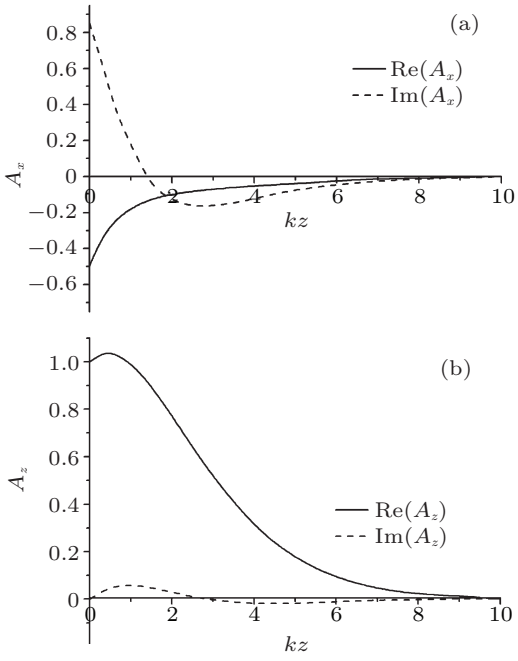


Fig. 4. Complex amplitudes of the first sagittal mode as a function of the reduced depth kz in an inclined layer at $\theta=26°$ and $C=-5\times10^{-4}$. Solid line: the real part; dotted line: the imaginary part of amplitude. (a) A_x and (b) A_z.

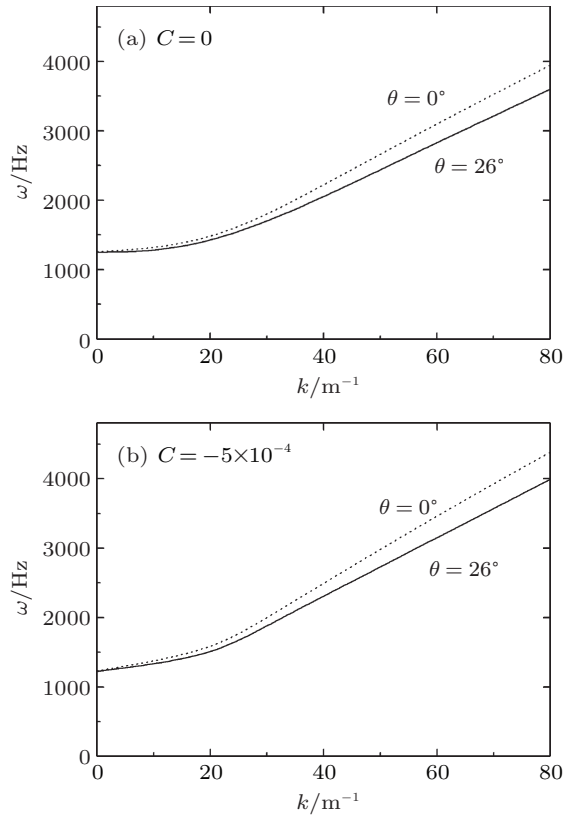


Fig. 5. Dispersion relation of the first sagittal mode with $\theta=26°$ under different integral constants $C=0$ (a) and $C=-5\times10^{-4}$ (b).

5. Conclusion

In summary, we have performed a calculation of the influences of initial stress and tilt of a granular layer under gravity on surface wave propagation properties in the context of granular elasticity model. Our results have shown that the stress and tilt angle have little effect on the properties of surface waves. This means that the elasticity theory mentioned above does not conflict with existing experimental data of surface waves. The acoustic data could be accounted for by the elastic potential model. Nevertheless, how to choose the model parameters is still a notable problem. Mechanical data suggested that the parameters are around $\xi\sim5/3$ and $B\sim7$ GPa, but the speed of surface wave calculated by such parameters is higher than the measurement. Although increasing the value

of ξ (Bonneau *et al.*) or decreasing that of B (present work) could match the measurements better, a divergence of parameter values between describing surface waves and other mechanical data does still exist. Some possible reasons for this divergence include: (i) the granular elasticity model has a defect at the free surface of the layer and needs to be modified; (ii) the density is very small near the free surface leading to the decrease of B; and, (iii) the particles become loose around their positions due to acoustic excitation.

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