Noether symmetry and conserved quantity for a Hamilton system with time delay*

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In this paper, the Noether symmetries and the conserved quantities for a Hamilton system with time delay are discussed. Firstly, the variational principles with time delay for the Hamilton system are given, and the Hamilton canonical equations with time delay are established. Secondly, according to the invariance of the function under the infinitesimal transformations of the group, the basic formulas for the variational of the Hamilton action with time delay are discussed, the definitions and the criteria of the Noether symmetric transformations and quasi-symmetric transformations with time delay are obtained, and the relationship between the Noether symmetry and the conserved quantity with time delay is studied. In addition, examples are given to illustrate the application of the results.

Keywords: time delay, Hamilton system, Noether symmetry, conserved quantity

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1. Introduction

Dynamical systems with a time delay are commonly found in nature and engineering practice. Great progress has been made in many fields with time delay, such as mechanics, mechanical engineering, aerospace, and control theory.[1–3] The study of variational problems with time delay can be traced back to El′sgol′c's work.[4] In 1968, the variational and optimal control problems with time delay were discussed by Hughes,[5] who presented the variational necessary and sufficient conditions with time delay for an unconstrained problem. The conjugate point conditions of variational problems with time delay were studied by Palm and Schmitendor,[6] and two conjugate point conditions of variational problems with time delay were derived. In 1988, the differential variational problems with time delay were studied by Rosenblueth,[7,8] the necessary and sufficient conditions in terms of the first and the second variations were derived, and he presented steps to solve constrained variational problems with time delay. Later, Chan and Yung[9] and Lee and Yung[10] presented the first-order and the second-order sufficient conditions for variational and optimal control problems with time delay, respectively.

However, the study of symmetries and conserved quantities with time delay has only just begun. In 2012, Fredrico and Torres[11] first discussed Noether's symmetry theorem for variational and optimal control problems with time delay. The influence of the symmetry of a mechanical system on the dynamical behavior and its basic nature are very important. The extreme importance of symmetry has been shown, from the basic theory to specific applications.[12–30] Therefore, the study of mechanical systems with time delay is of great significance for in-depth understanding of the complex dynamics of mechanical systems and their inherent physical nature. In this paper, we further study the Noether symmetries and the conserved quantities for a Hamilton system with time delay. The structure of this paper is as follows. In Section 2, the Hamilton canonical equations with time delay are presented. In Section 3, the Hamilton action with time delay of phase space is discussed. In Section 4, the definitions and criteria of the Noether symmetric transformations and the Noether quasi-symmetric transformations with time delay are obtained. In Section 5, the relationship between the Noether symmetry and the conserved quantity with time delay is given. In Section 6, two examples are given to illustrate an application of the results.

2. Hamilton canonical equations with time delay

Assume that the configuration of a mechanical system is determined by n generalized coordinates $q_s\,(s=1,\ldots,n)$. The Lagrangian of the system is

$$L=L\left(t,q_s(t),\dot{q}_s(t),q_s(t-\tau),\dot{q}_s(t-\tau)\right). \tag{1}$$

The generalized momentum and the Hamiltonian with time delay are defined as follows:

$$p_s(t)=\frac{\partial L}{\partial \dot{q}_s(t)},\quad p_s(t-\tau)=\frac{\partial L}{\partial \dot{q}_s(t-\tau)},\quad (s=1,2,\ldots,n), \tag{2}$$

$$H=p_s(t)\dot{q}_s(t)+p_s(t-\tau)\dot{q}_s(t-\tau)-L, \tag{3}$$

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where

$$H = H(t, q_s, p_s, q_s(t-\tau), p_s(t-\tau)) = H(t, q_s, p_s, q_{s\tau}, p_{s\tau}).$$

The Hamilton action with time delay is

$$\begin{aligned} S &= \int_{t_1}^{t_2} [p_s(t)\dot{q}_s(t) + p_s(t-\tau)\dot{q}_s(t-\tau) - H]\,\mathrm{d}t \\ &= \int_{t_1}^{t_2} (p_s\dot{q}_s + p_{s\tau}\dot{q}_{s\tau} - H)\,\mathrm{d}t, \end{aligned} \tag{4}$$

with the given boundary conditions

$$q_s(t) = \delta_s(t), \quad t_1 - \tau \le t \le t_1, \quad (s = 1, 2, \ldots, n), \tag{5}$$

$$q_s(t) = q_s(t_2), \quad t = t_2, \quad (s = 1, 2, \ldots, n), \tag{6}$$

where $q_s(\cdot)$ is assumed to be C^2-smooth, $\delta_s(t)$ is a given piecewise smooth function in $t_1 - \tau \le t \le t_1$, and τ is a given positive real number such that $\tau < t_2 - t_1$.

Making the variation of function (4), we have

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \delta(p_s\dot{q}_s + p_{s\tau}\dot{q}_{s\tau} - H)\,\mathrm{d}t \\ &= \int_{t_1}^{t_2} (\delta p_s\dot{q}_s + \delta\dot{q}_s p_s + \delta p_{s\tau}\dot{q}_{s\tau} + \delta\dot{q}_{s\tau}p_{s\tau})\,\mathrm{d}t \\ &\quad - \int_{t_1}^{t_2} \left(\frac{\partial H}{\partial p_s}\delta p_s + \frac{\partial H}{\partial q_s}\delta q_s + \frac{\partial H}{\partial p_{s\tau}}\delta p_{s\tau} \right. \\ &\quad \left. + \frac{\partial H}{\partial q_{s\tau}}\delta q_{s\tau}\right)\mathrm{d}t = 0. \end{aligned} \tag{7}$$

Making a linear change of variable $t = \theta + \tau$ and considering the boundary condition (5), we have

$$\begin{aligned} &\int_{t_1}^{t_2} \left(\delta p_{s\tau}\dot{q}_{s\tau} + \delta\dot{q}_{s\tau}p_{s\tau} - \frac{\partial H}{\partial p_{s\tau}}\delta p_{s\tau} - \frac{\partial H}{\partial q_{s\tau}}\delta q_{s\tau}\right)\mathrm{d}t \\ &= \int_{t_1}^{t_2-\tau} \left[\delta p_s(\theta)\dot{q}_{s\tau}(\theta+\tau) + \delta\dot{q}_s(\theta)p_{s\tau}(\theta+\tau) \right. \\ &\quad \left. - \frac{\partial H}{\partial p_{s\tau}}(\theta+\tau)\delta p_s(\theta) - \frac{\partial H}{\partial q_{s\tau}}(\theta+\tau)\delta q_s(\theta)\right]\mathrm{d}\theta. \end{aligned} \tag{8}$$

By substituting Eq. (8) into Eq. (7), we obtain

$$\begin{aligned} &\int_{t_1}^{t_2-\tau} \left\{ [\dot{q}_s(t) + \dot{q}_{s\tau}(t+\tau)]\,\delta p_s + [p_s(t) + p_{s\tau}(t+\tau)]\,\delta\dot{q}_s \right. \\ &- \left[\frac{\partial H}{\partial p_s}(t) + \frac{\partial H}{\partial p_{s\tau}}(t+\tau)\right]\delta p_s \\ &\left. - \left[\frac{\partial H}{\partial q_s}(t) + \frac{\partial H}{\partial q_{s\tau}}(t+\tau)\right]\delta q_s \right\}\mathrm{d}t \\ &+ \int_{t_2-\tau}^{t_2} \left[\delta p_s\dot{q}_s(t) + \delta\dot{q}_s p_s(t) \right. \\ &\left. - \frac{\partial H}{\partial p_s}(t)\delta p_s - \frac{\partial H}{\partial q_s}(t)\delta q_s\right]\mathrm{d}t = 0. \end{aligned} \tag{9}$$

By using the integration by parts and considering the boundary conditions (5) and (6), we have

$$\begin{aligned} &\int_{t_1}^{t_2-\tau} \left\{\left[\frac{\partial H}{\partial q_s}(t) + \frac{\partial H}{\partial q_{s\tau}}(t+\tau)\right]\delta q_s\right\}\mathrm{d}t \\ &= \left\{\delta q_s \int_{t_2-\tau}^{t} \left[\frac{\partial H}{\partial q_s}(\mu) + \frac{\partial H}{\partial q_{s\tau}}(\mu+\tau)\right]\mathrm{d}\mu\right\}\Bigg|_{t_1}^{t_2-\tau} \\ &\quad - \int_{t_1}^{t_2-\tau} \delta\dot{q}_s \left\{\int_{t_2-\tau}^{t} \left[\frac{\partial H}{\partial q_s}(\mu) + \frac{\partial H}{\partial q_{s\tau}}(\mu+\tau)\right]\mathrm{d}\mu\right\}\mathrm{d}t \\ &= -\int_{t_1}^{t_2-\tau} \delta\dot{q}_s \left\{\int_{t_2-\tau}^{t} \left[\frac{\partial H}{\partial q_s}(\mu) + \frac{\partial H}{\partial q_{s\tau}}(\mu+\tau)\right]\mathrm{d}\mu\right\}\mathrm{d}t, \end{aligned} \tag{10}$$

$$\begin{aligned} &\int_{t_2-\tau}^{t_2} \frac{\partial H}{\partial q_s}\delta q_s\,\mathrm{d}t \\ &= \left[\delta q_s \int_{t_2-\tau}^{t} \frac{\partial H}{\partial q_s}(\mu)\,\mathrm{d}\mu\right]\Bigg|_{t_2-\tau}^{t_2} \\ &\quad - \int_{t_2-\tau}^{t_2} \delta\dot{q}_s \left[\int_{t_2-\tau}^{t} \frac{\partial H}{\partial q_s}(\mu)\,\mathrm{d}\mu\right]\mathrm{d}t \\ &= -\int_{t_2-\tau}^{t_2} \delta\dot{q}_s \left[\int_{t_2-\tau}^{t} \frac{\partial H}{\partial q_s}(\mu)\,\mathrm{d}\mu\right]\mathrm{d}t. \end{aligned} \tag{11}$$

By substituting Eqs. (10) and (11) into Eq. (9), we obtain

$$\begin{aligned} &\int_{t_1}^{t_2-\tau} \left\{ \left[\dot{q}_s(t) + \dot{q}_{s\tau}(t+\tau) - \frac{\partial H}{\partial p_s}(t) - \frac{\partial H}{\partial p_{s\tau}}(t+\tau)\right]\delta p_s \right. \\ &+ \left[p_s(t) + p_{s\tau}(t+\tau) \right. \\ &\left. \left. + \int_{t_2-\tau}^{t} \left(\frac{\partial H}{\partial q_s}(\mu) + \frac{\partial H}{\partial q_{s\tau}}(\mu+\tau)\right)\mathrm{d}\mu\right]\delta\dot{q}_s \right\}\mathrm{d}t \\ &+ \int_{t_2-\tau}^{t_2} \left\{ \left[\dot{q}_s(t) - \frac{\partial H}{\partial p_s}(t)\right]\delta p_s \right. \\ &\left. + \left[p_s(t) + \int_{t_2-\tau}^{t} \frac{\partial H}{\partial q_s}(\mu)\,\mathrm{d}\mu\right]\delta\dot{q}_s \right\}\mathrm{d}t = 0. \end{aligned} \tag{12}$$

Taking the partial derivative of formula (3) with respect to the generalized momentum, we have

$$\begin{aligned} &\dot{q}_s(t) + \dot{q}_{s\tau}(t+\tau) - \frac{\partial H}{\partial p_s}(t) - \frac{\partial H}{\partial p_{s\tau}}(t+\tau) = 0, \\ &\quad t_1 \le t \le t_2 - \tau, \\ &\dot{q}_s(t) - \frac{\partial H}{\partial p_s}(t) = 0, \quad t_2 - \tau < t \le t_2. \end{aligned} \tag{13}$$

By substituting Eq. (13) into Eq. (12), and considering the arbitrariness of the integral interval $[t_1, t_2]$ and the independence of $\delta\dot{q}_s$ $(s = 1, 2, \ldots, n)$, we have

$$\begin{aligned} &p_s(t) + p_{s\tau}(t+\tau) + \int_{t_2-\tau}^{t} \left[\frac{\partial H}{\partial q_s}(\mu) + \frac{\partial H}{\partial q_{s\tau}}(\mu+\tau)\right]\mathrm{d}\mu = 0, \\ &\quad t_1 \le t \le t_2 - \tau, \\ &p_s(t) + \int_{t_2-\tau}^{t} \frac{\partial H}{\partial q_s}(\mu)\,\mathrm{d}\mu = 0, \\ &\quad t_2 - \tau < t \le t_2, \quad (s = 1, 2, \ldots, n). \end{aligned} \tag{14}$$

Taking the derivative of Eq. (14) with respect to t, we have

$$\begin{aligned} &\dot{p}_s(t) + \dot{p}_{s\tau}(t+\tau) + \frac{\partial H}{\partial q_s}(t) + \frac{\partial H}{\partial q_{s\tau}}(t+\tau) = 0, \\ &\quad t_1 \le t \le t_2 - \tau, \\ &\dot{p}_s(t) + \frac{\partial H}{\partial q_s}(t) = 0, \\ &\quad t_2 - \tau < t \le t_2, \quad (s = 1, 2, \ldots, n). \end{aligned} \tag{15}$$

By combining Eq. (13) with Eq. (15), we obtain the Hamilton canonical equations with time delay. If $\tau = 0$, then equations (13) and (15) reduce to the standard Hamilton canonical equations.

3. Variation of Hamilton action with time delay in phase space

If we introduce the infinitesimal transformations of r-parameter finite transformations group[17]

$$\begin{aligned} &\bar{t} = t + \Delta t, \quad \bar{q}_s(\bar{t}) = q_s(t) + \Delta q_s, \\ &\bar{p}_s(\bar{t}) = p_s(t) + \Delta p_s, \end{aligned} \tag{16}$$

then their expansion formulas are

$$\begin{aligned} &\bar{t} = t + \varepsilon_\sigma \xi_0^\sigma(t, q_k, p_k), \quad \bar{q}_s(\bar{t}) = q_s(t) + \varepsilon_\sigma \xi_s^\sigma(t, q_k, p_k), \\ &\bar{p}_s(\bar{t}) = p_s(t) + \varepsilon_\sigma \eta_s^\sigma(t, q_k, p_k), \quad (s = 1, 2, \ldots, n), \end{aligned} \tag{17}$$

where ε_σ $(\sigma = 1, 2, \ldots, r)$ are the infinitesimal parameters, and ξ_0^σ, ξ_s^σ, η_s^σ are the generators or generating functions of the infinitesimal transformations. Under the infinitesimal transformations (16), a curve γ will be transformed to a neighbor curve $\bar{\gamma}$, and the integral functional $S(\gamma)$ will be transformed to $S(\bar{\gamma})$ correspondingly, then we have

$$\begin{aligned} S(\bar{\gamma}) = \int_{\bar{t}_1}^{\bar{t}_2} \Big[&\bar{p}_s(\bar{t}) \dot{\bar{q}}_s(\bar{t}) + \bar{p}_s(\bar{t} - \tau) \dot{\bar{q}}_s(\bar{t} - \tau) \\ &- H(\bar{t}, \bar{q}_s(\bar{t}), \bar{p}_s(\bar{t}), \bar{q}_s(\bar{t} - \tau), \bar{p}_s(\bar{t} - \tau)) \Big] \mathrm{d}\bar{t}. \end{aligned} \tag{18}$$

The main linear part relative to ε in the difference $S(\bar{\gamma}) - S(\gamma)$ is

$$\begin{aligned} \Delta S = \int_{t_1}^{t_2} \Big[&\Delta p_s \dot{q}_s + \Delta \dot{q}_s p_s + \Delta p_{s\tau} \dot{q}_{s\tau} + \Delta \dot{q}_{s\tau} p_{s\tau} \\ &- \frac{\partial H}{\partial t} \Delta t - \frac{\partial H}{\partial q_s} \Delta q_s - \frac{\partial H}{\partial p_s} \Delta p_s \\ &- \frac{\partial H}{\partial q_{s\tau}} \Delta q_{s\tau} - \frac{\partial H}{\partial p_{s\tau}} \Delta p_{s\tau} \\ &+ (p_s \dot{q}_s + p_{s\tau} \dot{q}_{s\tau} - H) \frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) \Big] \mathrm{d}t. \end{aligned} \tag{19}$$

By making a linear change of variable $t = \theta + \tau$ and considering the boundary conditions (5) and (6), we have

$$\begin{aligned} &\int_{t_1}^{t_2} \Big[\Delta p_{s\tau} \dot{q}_{s\tau} + \Delta \dot{q}_{s\tau} p_{s\tau} \\ &\quad - \frac{\partial H}{\partial q_{s\tau}} \Delta q_{s\tau} - \frac{\partial H}{\partial p_{s\tau}} \Delta p_{s\tau} + p_{s\tau} \dot{q}_{s\tau} \frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) \Big] \mathrm{d}t \\ = &\int_{t_1}^{t_2 - \tau} \Big[\Delta p_s \dot{q}_{s\tau}(\theta + \tau) + \Delta \dot{q}_s p_{s\tau}(\theta + \tau) \\ &\quad - \frac{\partial H}{\partial q_{s\tau}}(\theta + \tau) \Delta q_s - \frac{\partial H}{\partial p_{s\tau}}(\theta + \tau) \Delta p_s \\ &\quad + p_{s\tau}(\theta + \tau) \dot{q}_{s\tau}(\theta + \tau) \frac{\mathrm{d}}{\mathrm{d}\theta}(\Delta \theta) \Big] \mathrm{d}\theta. \end{aligned} \tag{20}$$

By substituting Eq. (20) into Eq. (19), we obtain

$$\begin{aligned} \Delta S = &\int_{t_1}^{t_2 - \tau} \Big\{ \Delta p_s [\dot{q}_s(t) + \dot{q}_{s\tau}(t + \tau)] \\ &+ \Delta \dot{q}_s [p_s(t) + p_{s\tau}(t + \tau)] - \frac{\partial H}{\partial t}(t) \Delta t \\ &- \Delta p_s \left[\frac{\partial H}{\partial p_s}(t) + \frac{\partial H}{\partial p_{s\tau}}(t + \tau) \right] \\ &- \Delta q_s \left[\frac{\partial H}{\partial q_s}(t) + \frac{\partial H}{\partial q_{s\tau}}(t + \tau) \right] \\ &+ [p_s(t) \dot{q}_s(t) + p_{s\tau}(t + \tau) \dot{q}_{s\tau}(t + \tau) - H] \frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) \Big\} \mathrm{d}t \\ &+ \int_{t_2 - \tau}^{t_2} \{ \Delta p_s \dot{q}_s(t) + \Delta \dot{q}_s p_s(t) \\ &- \frac{\partial H}{\partial t}(t) \Delta t - \Delta p_s \frac{\partial H}{\partial p_s}(t) - \Delta q_s \frac{\partial H}{\partial q_s}(t) \\ &+ [p_s(t) \dot{q}_s(t) - H] \frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) \Big\} \mathrm{d}t. \end{aligned} \tag{21}$$

Taking notice that

$$\begin{aligned} &\delta q_s = \Delta q_s - \dot{q}_s \Delta t, \quad \delta p_s = \Delta p_s - \dot{p}_s \Delta t, \\ &\delta \dot{q}_s = \frac{\mathrm{d}}{\mathrm{d}t} \Delta q_s - \dot{q}_s \frac{\mathrm{d}}{\mathrm{d}t} \Delta t, \end{aligned} \tag{22}$$

and using transformations (17), we can express formula (21) as follows:

$$\begin{aligned} \Delta S = &\int_{t_1}^{t_2 - \tau} \Big\{ \frac{\mathrm{d}}{\mathrm{d}t} \Big[(p_s(t) + p_{s\tau}(t + \tau)) \delta q_s \\ &+ (p_s(t) \dot{q}_s(t) + p_{s\tau}(t + \tau) \dot{q}_{s\tau}(t + \tau) - H) \Delta t \Big] \\ &+ \delta q_s \left[-\dot{p}_s(t) - \dot{p}_{s\tau}(t + \tau) - \frac{\partial H}{\partial q_s}(t) - \frac{\partial H}{\partial q_{s\tau}}(t + \tau) \right] \\ &+ \delta p_s \left[\dot{q}_s(t) + \dot{q}_{s\tau}(t + \tau) - \frac{\partial H}{\partial p_s}(t) - \frac{\partial H}{\partial p_{s\tau}}(t + \tau) \right] \Big\} \mathrm{d}t \\ &+ \int_{t_2 - \tau}^{t_2} \Big\{ \frac{\mathrm{d}}{\mathrm{d}t} [\delta q_s p_s(t) + (p_s(t) \dot{q}_s(t) - H) \Delta t] \\ &+ \delta q_s \left[-\dot{p}_s(t) - \frac{\partial H}{\partial q_s}(t) \right] + \delta p_s \left[\dot{q}_s(t) - \frac{\partial H}{\partial p_s}(t) \right] \Big\} \mathrm{d}t \\ = &\int_{t_1}^{t_2 - \tau} \varepsilon_\sigma \Big\{ \frac{\mathrm{d}}{\mathrm{d}t} \Big[(p_s(t) + p_{s\tau}(t + \tau)) \bar{\xi}_s^\sigma \\ &+ (p_s(t) \dot{q}_s(t) + p_{s\tau}(t + \tau) \dot{q}_{s\tau}(t + \tau) - H) \xi_0^\sigma \Big] \\ &+ \bar{\xi}_s^\sigma \left[-\dot{p}_s(t) - \dot{p}_{s\tau}(t + \tau) - \frac{\partial H}{\partial q_s}(t) - \frac{\partial H}{\partial q_{s\tau}}(t + \tau) \right] \\ &+ \bar{\eta}_s^\sigma \left[\dot{q}_s(t) + \dot{q}_{s\tau}(t + \tau) - \frac{\partial H}{\partial p_s}(t) - \frac{\partial H}{\partial p_{s\tau}}(t + \tau) \right] \Big\} \mathrm{d}t \\ &+ \int_{t_2 - \tau}^{t_2} \varepsilon_\sigma \Big\{ \frac{\mathrm{d}}{\mathrm{d}t} [p_s(t) \bar{\xi}_s^\sigma + (p_s(t) \dot{q}_s(t) - H) \xi_0^\sigma] \\ &+ \bar{\xi}_s^\sigma \left[-\dot{p}_s(t) - \frac{\partial H}{\partial q_s}(t) \right] \\ &+ \bar{\eta}_s^\sigma \left[\dot{q}_s(t) - \frac{\partial H}{\partial p_s}(t) \right] \Big\} \mathrm{d}t, \end{aligned} \tag{23}$$

where

$$\bar{\xi}_s^\sigma = \xi_s^\sigma - \dot{q}_s\xi_0^\sigma, \quad \bar{\eta}_s^\sigma = \eta_s^\sigma - \dot{p}_s\xi_0^\sigma. \tag{24}$$

Formulas (21) and (23) are the basic formulas for the variation of the Hamilton action with time delay in phase space.

4. Noether symmetric transformations with time delay in phase space

Firstly, we study the definitions and the criteria of Noether symmetric transformations with time delay.

Definition 1 If the Hamilton action (4) is an invariant of the infinitesimal transformations of group (16), that is, for each of the infinitesimal transformations, the formula

$$\Delta S = 0 \tag{25}$$

holds, then the infinitesimal transformations are called the Noether symmetric transformations with time delay.

From Definition 1 and formulas (21) and (23), we obtain the following criteria.

Criterion 1 For the infinitesimal transformations of group (16), if the condition

$$\begin{aligned}&\Delta p_s\left[\dot{q}_s(t)+\dot{q}_{s\tau}(t+\tau)\right]+\Delta\dot{q}_s\left[p_s(t)+p_{s\tau}(t+\tau)\right]\\&-\frac{\partial H}{\partial t}(t)\Delta t-\Delta p_s\left[\frac{\partial H}{\partial p_s}(t)+\frac{\partial H}{\partial p_{s\tau}}(t+\tau)\right]\\&-\Delta q_s\left[\frac{\partial H}{\partial q_s}(t)+\frac{\partial H}{\partial q_{s\tau}}(t+\tau)\right]\\&+\left[p_s(t)\dot{q}_s(t)+p_{s\tau}(t+\tau)\dot{q}_{s\tau}(t+\tau)-H\right]\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)=0\end{aligned} \tag{26}$$

is satisfied when $t_1 \le t \le t_2-\tau$, and the condition

$$\begin{aligned}&\Delta p_s\dot{q}_s(t)+\Delta\dot{q}_s p_s(t)-\frac{\partial H}{\partial t}(t)\Delta t-\Delta p_s\frac{\partial H}{\partial p_s}(t)\\&-\Delta q_s\frac{\partial H}{\partial q_s}(t)+\left[p_s(t)\dot{q}_s(t)-H\right]\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)=0\end{aligned} \tag{27}$$

is satisfied when $t_2-\tau < t \le t_2$, then the infinitesimal transformations (16) are Noether symmetric transformations with time delay.

Criterion 2 For the infinitesimal transformations of group (17), if the r equations

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(p_s(t)+p_{s\tau}(t+\tau)\right)\bar{\xi}_s^\sigma\right.\\&\left.+\left(p_s(t)\dot{q}_s(t)+p_{s\tau}(t+\tau)\dot{q}_{s\tau}(t+\tau)-H\right)\xi_0^\sigma\right]\\&+\bar{\xi}_s^\sigma\left[-\dot{p}_s(t)-\dot{p}_{s\tau}(t+\tau)-\frac{\partial H}{\partial q_s}(t)-\frac{\partial H}{\partial q_{s\tau}}(t+\tau)\right]\\&+\bar{\eta}_s^\sigma\left[\dot{q}_s(t)+\dot{q}_{s\tau}(t+\tau)-\frac{\partial H}{\partial p_s}(t)-\frac{\partial H}{\partial p_{s\tau}}(t+\tau)\right]=0,\\&\quad(\sigma=1,2,\ldots,r)\end{aligned} \tag{28}$$

are satisfied when $t_1 \le t \le t_2-\tau$, and the r equations

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[p_s(t)\bar{\xi}_s^\sigma+\left(p_s(t)\dot{q}_s(t)-H\right)\xi_0^\sigma\right]\\&+\bar{\xi}_s^\sigma\left[-\dot{p}_s(t)-\frac{\partial H}{\partial q_s}(t)\right]+\bar{\eta}_s^\sigma\left[\dot{q}_s(t)-\frac{\partial H}{\partial p_s}(t)\right]=0,\\&\quad(\sigma=1,2,\ldots,r)\end{aligned} \tag{29}$$

are satisfied when $t_2-\tau < t \le t_2$, then the infinitesimal transformations (17) are the Noether symmetric transformations with time delay.

According to formula (17), taking use of the relationships (22) and (13), and considering the independence of ε_σ, we have the following results. From Eq. (26), when $t_1 \le t \le t_2-\tau$, we have

$$\begin{aligned}&\left[p_s(t)+p_{s\tau}(t+\tau)\right]\dot{\xi}_s^\sigma-\frac{\partial H}{\partial t}(t)\xi_0^\sigma\\&-\left[\frac{\partial H}{\partial q_s}(t)+\frac{\partial H}{\partial q_{s\tau}}(t+\tau)\right]\xi_s^\sigma-H\dot{\xi}_0^\sigma=0,\\&\quad(\sigma=1,2,\ldots,r).\end{aligned} \tag{30}$$

From Eq. (27), when $t_2-\tau < t \le t_2$, we have

$$\begin{aligned}&p_s(t)\dot{\xi}_s^\sigma-\frac{\partial H}{\partial t}(t)\xi_0^\sigma-\frac{\partial H}{\partial q_s}(t)\xi_s^\sigma-H\dot{\xi}_0^\sigma=0,\\&\quad(\sigma=1,2,\ldots,r).\end{aligned} \tag{31}$$

When $r=1$, equations (30) and (31) are called the Noether identity with time delay.

Secondly, we discuss the Noether quasi-symmetric transformations with time delay in phase space.

Suppose that H_1 is another Hamiltonian with time delay, if the transformations (16) accurate to the first order infinitesimal satisfy the condition

$$\begin{aligned}&\int_{t_1}^{t_2}\left[p_s(t)\dot{q}_s(t)+p_s(t-\tau)\dot{q}_s(t-\tau)\right.\\&\left.-H(t,q_s(t),p_s(t),q_s(t-\tau),p_s(t-\tau))\right]\mathrm{d}t\\=&\int_{\bar{t}_1}^{\bar{t}_2}\left[\bar{p}_s(\bar{t})\dot{\bar{q}}_s(\bar{t})+\bar{p}_s(\bar{t}-\tau)\dot{\bar{q}}_s(\bar{t}-\tau)\right.\\&\left.-H_1(\bar{t},\bar{q}_s(\bar{t}),\bar{p}_s(\bar{t}),\bar{q}_s(\bar{t}-\tau),\bar{p}_s(\bar{t}-\tau))\right]\mathrm{d}\bar{t},\end{aligned} \tag{32}$$

then this invariance is called the quasi-invariance of Hamilton action (4) under the infinitesimal transformations of group (16). The Hamiltonians H_1 and H determined by formula (32) satisfy the same differential equations. Then, the transformations (16) are called the Noether quasi-symmetric transformations with time delay.

Definition 2 If Hamilton action (4) is an quasi-invariant of the infinitesimal transformations of group (16), that is, for each of the infinitesimal transformations, the formula

$$\Delta S = -\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G)\,\mathrm{d}t \tag{33}$$

holds, where $G = G(t,q_s(t),p_s(t),q_s(t-\tau),p_s(t-\tau),q_s(t+\tau),p_s(t+\tau))$, then the infinitesimal transformations (16) are

called the Noether quasi-symmetric transformations with time delay.

From Definition 2 and formulas (21) and (23), we obtain the following criteria.

Criterion 3 For the infinitesimal transformations of group (16), if the condition

$$\begin{aligned}&\Delta p_s[\dot{q}_s(t)+\dot{q}_{s\tau}(t+\tau)]+\Delta\dot{q}_s[p_s(t)+p_{s\tau}(t+\tau)]\\&-\frac{\partial H}{\partial t}(t)\Delta t-\Delta p_s\left[\frac{\partial H}{\partial p_s}(t)+\frac{\partial H}{\partial p_{s\tau}}(t+\tau)\right]\\&-\Delta q_s\left[\frac{\partial H}{\partial q_s}(t)+\frac{\partial H}{\partial q_{s\tau}}(t+\tau)\right]\\&+[p_s(t)\dot{q}_s(t)+p_{s\tau}(t+\tau)\dot{q}_{s\tau}(t+\tau)-H]\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\\=&-\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G)\end{aligned}\tag{34}$$

is satisfied when $t_1\le t\le t_2-\tau$, and the condition

$$\begin{aligned}&\Delta p_s\dot{q}_s(t)+\Delta\dot{q}_s p_s(t)-\frac{\partial H}{\partial t}(t)\Delta t-\Delta p_s\frac{\partial H}{\partial p_s}(t)\\&-\Delta q_s\frac{\partial H}{\partial q_s}(t)+[p_s(t)\dot{q}_s(t)-H]\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\\=&-\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G)\end{aligned}\tag{35}$$

is satisfied when $t_2-\tau<t\le t_2$, then the infinitesimal transformations (16) are the Noether quasi-symmetric transformations with time delay.

Criterion 4 For the infinitesimal transformations of group (17), if the r equations

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}[(p_s(t)+p_{s\tau}(t+\tau))\bar{\xi}_s^\sigma+(p_s(t)\dot{q}_s(t)\\&+p_{s\tau}(t+\tau)\dot{q}_{s\tau}(t+\tau)-H)\xi_0^\sigma+G^\sigma]\\&+\bar{\xi}_s^\sigma\left[-\dot{p}_s(t)-\dot{p}_{s\tau}(t+\tau)-\frac{\partial H}{\partial q_s}(t)-\frac{\partial H}{\partial q_{s\tau}}(t+\tau)\right]\\&+\bar{\eta}_s^\sigma\left[\dot{q}_s(t)+\dot{q}_{s\tau}(t+\tau)-\frac{\partial H}{\partial p_s}(t)-\frac{\partial H}{\partial p_{s\tau}}(t+\tau)\right]=0,\\&(\sigma=1,2,\cdots,r)\end{aligned}\tag{36}$$

are satisfied when $t_1\le t\le t_2-\tau$, and the r equations

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[p_s(t)\bar{\xi}_s^\sigma+(p_s(t)\dot{q}_s(t)-H)\xi_0^\sigma+G^\sigma\right]\\&+\bar{\xi}_s^\sigma\left[-\dot{p}_s(t)-\frac{\partial H}{\partial q_s}(t)\right]+\bar{\eta}_s^\sigma\left[\dot{q}_s(t)-\frac{\partial H}{\partial p_s}(t)\right]=0,\\&(\sigma=1,2,\ldots,r)\end{aligned}\tag{37}$$

are satisfied when $t_2-\tau<t\le t_2$, where $\Delta G=\varepsilon_\sigma G^\sigma$, then the infinitesimal transformations (17) are the Noether quasi-symmetric transformations with time delay.

Conditions (34) and (35) can also be expressed as follows: when $t_1\le t\le t_2-\tau$,

$$\begin{aligned}&[p_s(t)+p_{s\tau}(t+\tau)]\dot{\xi}_s^\sigma-\frac{\partial H}{\partial t}(t)\xi_0^\sigma\\&-\left[\frac{\partial H}{\partial q_s}(t)+\frac{\partial H}{\partial q_{s\tau}}(t+\tau)\right]\xi_s^\sigma-H\dot{\xi}_0^\sigma=-\dot{G}^\sigma,\\&(\sigma=1,2,\ldots,r);\end{aligned}\tag{38}$$

and when $t_2-\tau<t\le t_2$,

$$\begin{aligned}&p_s(t)\dot{\xi}_s^\sigma-\frac{\partial H}{\partial t}(t)\xi_0^\sigma-\frac{\partial H}{\partial q_s}(t)\xi_s^\sigma-H\dot{\xi}_0^\sigma=-\dot{G}^\sigma,\\&(\sigma=1,2,\ldots,r).\end{aligned}\tag{39}$$

When $r=1$, equations (38) and (39) are called the Noether identity with time delay.

By using Criterion 1 or 2, we can verify the Noether symmetry with time delay, By using Criterion 3 or 4, we can verify the Noether quasi-symmetry with time delay. If $\tau=0$, then criteria 1–4 are expressed as the criteria of the standard Noether symmetry.

5. Noether theorem with time delay of Hamilton system

In this section, we will discuss the Noether theorem for a Hamilton system with time delay.

Firstly, we give the definition of the conserved quantity for the Hamilton system with time delay.

Definition 3 A function $I(t,t+\tau,q_s(t),p_s(t),q_s(t-\tau),p_s(t-\tau),q_s(t+\tau),p_s(t+\tau))$ is said to be a conserved quantity of the Hamilton system under study if, and only if,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}I(t,t+\tau,q_s(t),p_s(t),q_s(t-\tau),p_s(t-\tau),\\&q_s(t+\tau),p_s(t+\tau))=0\end{aligned}\tag{40}$$

holds along all the solution curves of the Hamilton canonical equations (11) with time delay.

For the Hamilton system with time delay, if the Noether symmetric or quasi-symmetric transformations with time delay can be found, then the conserved quantities corresponding to these symmetries can be obtained by using the following Noether theorems.

Theorem 1 For the Hamilton system (13) and (15), if the infinitesimal transformations of group (16) are the Noether symmetric transformations under Definition 1, then the system exists with the following r linear independent conserved quantities with time delay: when $t_1\le t\le t_2-\tau$, there are

$$\begin{aligned}&I^\sigma=[p_s(t)+p_{s\tau}(t+\tau)]\xi_s^\sigma-H\xi_0^\sigma=\text{const.},\\&(\sigma=1,2,\ldots,n),\end{aligned}\tag{41}$$

and when $t_2-\tau<t\le t_2$, there are

$$I^\sigma=p_s(t)\xi_s^\sigma-H\xi_0^\sigma=\text{const.},\quad(\sigma=1,2,\ldots,n).\tag{42}$$

Proof According to Definition 1 and Criterion 2, and substituting Eqs. (13) and (15) into Eqs. (28) and (29), when $t_1\le t\le t_2-\tau$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[(p_s(t)+p_{s\tau}(t+\tau))\bar{\xi}_s^\sigma+(p_s(t)\dot{q}_s(t)$$

$$+p_{s\tau}(t+\tau)\dot{q}_{s\tau}(t+\tau)-H)\xi_0^\sigma]$$
$$=\frac{\mathrm{d}}{\mathrm{d}t}[(p_s(t)+p_{s\tau}(t+\tau))\xi_s^\sigma-H\xi_0^\sigma]=0,$$
$$(\sigma=1,2,\ldots,n), \tag{43}$$

and when $t_2-\tau<t\le t_2$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[p_s(t)\bar{\xi}_s^\sigma+(p_s(t)\dot{q}_s(t)-H)\xi_0^\sigma\right]$$
$$=\frac{\mathrm{d}}{\mathrm{d}t}[p_s(t)\xi_s^\sigma-H\xi_0^\sigma]=0,$$
$$(\sigma=1,2,\ldots,n). \tag{44}$$

Integrating Eqs. (43) and (44), we obtain the theorem.

Theorem 2 For the Hamilton system (13) and (15), if the infinitesimal transformations of group (16) are the Noether quasi-symmetric transformations under Definition 2, then the system exists with the following r linear independent conserved quantities with time delay: when $t_1\le t\le t_2-\tau$, there are

$$I^\sigma=[p_s(t)+p_{s\tau}(t+\tau)]\,\xi_s^\sigma-H\xi_0^\sigma+G^\sigma=\text{const.},$$
$$(\sigma=1,2,\ldots,n), \tag{45}$$

and when $t_2-\tau<t\le t_2$, there are

$$I^\sigma=p_s(t)\xi_s^\sigma-H\xi_0^\sigma+G^\sigma=\text{const.},\quad(\sigma=1,2,\ldots,n). \tag{46}$$

Proof According to Definition 2 and Criterion 4, and substituting Eqs. (13) and (15) into Eqs. (36) and (37), when $t_1\le t\le t_2-\tau$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[(p_s(t)+p_{s\tau}(t+\tau))\bar{\xi}_s^\sigma+(p_s(t)\dot{q}_s(t)$$
$$+p_{s\tau}(t+\tau)\dot{q}_{s\tau}(t+\tau)-H)\xi_0^\sigma+G^\sigma]$$
$$=\frac{\mathrm{d}}{\mathrm{d}t}[(p_s(t)+p_{s\tau}(t+\tau))\xi_s^\sigma-H\xi_0^\sigma+G^\sigma]=0,$$
$$(\sigma=1,2,\ldots,n), \tag{47}$$

and when $t_2-\tau<t\le t_2$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[p_s(t)\bar{\xi}_s^\sigma+(p_s(t)\dot{q}_s(t)-H)\xi_0^\sigma+G^\sigma\right]$$
$$=\frac{\mathrm{d}}{\mathrm{d}t}[p_s(t)\xi_s^\sigma-H\xi_0^\sigma+G^\sigma]=0,$$
$$(\sigma=1,2,\ldots,n). \tag{48}$$

By integrating Eqs. (47) and (48), we obtain the theorem.

Theorems 1 and 2 are called the Noether theorem for the Hamilton system with time delay. Using Theorem 1 or Theorem 2, we can obtain a conserved quantity from a Noether symmetry for the Hamilton system with time delay. If $\tau=0$, then the Noether theorem with time delay is reduced to the standard Noether theorem.

6. Examples

Example 1 Let us study a mechanical system with time delay whose Lagrangian is

$$L=\frac{1}{2}[\dot{q}(t)+\dot{q}(t-\tau)]^2, \tag{49}$$

where $t\in[t_1,t_2]$, and $\tau<t_2-t_1$ is a given real number. The following boundary conditions are satisfied: when $t\in[t_1-\tau,t_1]$, $q(t)=\delta(t)$, where $\delta(t)$ is a given piecewise smooth function in $[t_1-\tau,t_1]$; when $t=t_2$, $q(t)=q(t_2)$, where $q(t_2)$ is a certain value.

From formulas (2) and (3), we have

$$p(t)=\frac{\partial L}{\partial\dot{q}}=\dot{q}(t)+\dot{q}(t-\tau),$$
$$p(t-\tau)=\frac{\partial L}{\partial\dot{q}(t-\tau)}=\dot{q}(t)+\dot{q}(t-\tau),$$
$$H=\frac{1}{8}[p(t)+p(t-\tau)]^2. \tag{50}$$

From the differential equations (13) and (15) of the system, we have

$$\dot{q}(t)+\dot{q}_\tau(t+\tau)-\frac{1}{4}[2p(t)+p(t-\tau)+p(t+\tau)]=0,$$
$$\dot{p}(t)+\dot{p}_\tau(t+\tau)=0,\quad t\in[t_1,t_2-\tau],$$
$$\dot{q}(t)-\frac{1}{4}[p(t)+p(t-\tau)]=0,$$
$$\dot{p}(t)=0,\quad t\in(t_2-\tau,t_2]. \tag{51}$$

Then, equations (51) are the Hamilton canonical equations of the system with time delay.

Form Criterion 4, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(p(t)+p_\tau(t+\tau))\bar{\xi}+\left(p(t)\dot{q}(t)+p_\tau(t+\tau)\dot{q}_\tau(t+\tau)\right.\right.$$
$$\left.\left.-\frac{1}{8}(p(t)+p(t-\tau))^2\right)\xi_0\right]$$
$$+\left[\dot{q}(t)+\dot{q}_\tau(t+\tau)-\frac{1}{4}(2p(t)+p(t-\tau)+p(t+\tau))\right]\bar{\eta}$$
$$-[\dot{p}(t)+\dot{p}_\tau(t+\tau)]\bar{\xi}=-\dot{G},\quad t\in[t_1,t_2-\tau],$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[p(t)\bar{\xi}+\left(p(t)\dot{q}(t)-\frac{1}{8}(p(t)+p(t-\tau))^2\right)\xi_0\right]-\dot{p}(t)\bar{\xi}$$
$$+\left[\dot{q}(t)-\frac{1}{4}(p(t)+p(t-\tau))\right]\bar{\eta}=-\dot{G},$$
$$t\in(t_2-\tau,t_2]. \tag{52}$$

Equation (52) has a solution

$$\xi=1,\quad \xi_0=\eta=0. \tag{53}$$

By substituting Eq. (53) into Eq. (52), we have

$$G=0. \tag{54}$$

The generator (53) corresponds to a Noether symmetry. From Theorem 1, the system has a conserved quantity corresponding to symmetry (53), that is

$$I=p(t)+p_\tau(t+\tau)=\text{const.},\quad t\in[t_1,t_2-\tau],$$

$$I = p(t) = \text{const.}, \quad t \in (t_2 - \tau, t_2]. \tag{55}$$

Let

$$\xi_0 = 0, \quad \xi = \eta = 1. \tag{56}$$

By substituting Eq. (56) into Eq. (52), we have

$$\begin{aligned} &G = -q(t) + \frac{3}{4}q(t-\tau) + \frac{1}{4}q(t+\tau), \quad t \in [t_1, t_2 - \tau], \\ &G = -\frac{1}{2}q(t) + \frac{1}{4}q(t-\tau), \quad t \in (t_2 - \tau, t_2]. \end{aligned} \tag{57}$$

The generator (56) corresponds to a Noether quasi-symmetry. From Theorem 2, the system has a conserved quantity corresponding to symmetry (56), that is

$$\begin{aligned} &I = p(t) + p_\tau(t+\tau) - q(t) + \frac{3}{4}q(t-\tau) + \frac{1}{4}q(t+\tau) = \text{const.}, \\ &\quad t \in [t_1, t_2 - \tau], \\ &I = p(t) - \frac{1}{2}q(t) + \frac{1}{4}q(t-\tau) = \text{const.}, \quad t \in (t_2 - \tau, t_2]. \end{aligned} \tag{58}$$

Example 2 Let us study a mechanical system with two degrees of freedom, the Lagrangian of the system is

$$\begin{aligned} L = {} & \frac{1}{2}m\left\{[(\dot{q}_1(t) + \dot{q}_1(t-\tau)]^2 + [(\dot{q}_2(t) + \dot{q}_2(t-\tau)]^2\right\} \\ & - \frac{1}{2}k\left\{[q_1(t) + q_1(t-\tau)]^2 + [q_2(t) + q_2(t-\tau)]^2\right\}. \end{aligned} \tag{59}$$

The generalized momentum and the Hamiltonian are

$$\begin{aligned} &p_1(t) = m[\dot{q}_1(t) + \dot{q}_1(t-\tau)], \\ &p_2(t) = m[\dot{q}_2(t) + \dot{q}_2(t-\tau))], \\ &p_1(t-\tau) = p_{1\tau} = m[\dot{q}_1(t) + \dot{q}_1(t-\tau)], \\ &p_2(t-\tau) = p_{2\tau} = m[\dot{q}_2(t) + \dot{q}_2(t-\tau)], \\ &H = \frac{1}{8m}\left\{[p_1(t) + p_1(t-\tau)]^2 + [p_2(t) + p_2(t-\tau)]^2\right\} \\ &\quad + \frac{1}{2}k\left\{[q_1(t) + q_1(t-\tau)]^2 + [q_2(t) + q_2(t-\tau)]^2\right\}. \end{aligned} \tag{60}$$

The Hamilton canonical equations with time delay of the system are

$$\begin{aligned} &\dot{q}_1(t) + \dot{q}_{1\tau}(t+\tau) - \frac{1}{4m}[2p_1(t) + p_1(t-\tau) + p_1(t+\tau)] = 0, \\ &\dot{p}_1(t) + \dot{p}_{1\tau}(t+\tau) - k[2q_1(t) + q_1(t-\tau) + q_1(t+\tau)] = 0, \\ &\dot{q}_2(t) + \dot{q}_{2\tau}(t+\tau) - \frac{1}{4m}[2p_2(t) + p_2(t-\tau) + p_2(t+\tau)] = 0, \\ &\dot{p}_2(t) + \dot{p}_{2\tau}(t+\tau) - k[2q_2(t) + q_2(t-\tau) + q_2(t+\tau)] = 0, \\ &\quad t \in [t_1, t_2 - \tau], \\ &\dot{q}_1(t) - \frac{1}{4m}[p_1(t) + p_1(t-\tau)] = 0, \\ &\dot{p}_1(t) + k[q_1(t) + q_1(t-\tau)] = 0, \\ &\dot{q}_2(t)) - \frac{1}{4m}[p_2(t) + p_2(t-\tau)] = 0, \\ &\dot{p}_2(t) + k[q_2(t) + q_2(t-\tau)] = 0, \quad t \in (t_2 - \tau, t_2]. \end{aligned} \tag{61}$$

The Noether identities (38) and (39) give

$$\begin{aligned} &[p_1(t) + p_{1\tau}(t+\tau)]\dot{\xi}_1 + [p_2(t) + p_{2\tau}(t+\tau)]\dot{\xi}_2 \\ &- k[2q_1(t) + q_1(t-\tau) + q_1(t+\tau)]\xi_1 \\ &- k[2q_2(t) + q_2(t-\tau) + q_2(t+\tau)]\xi_2 - H\dot{\xi}_0 = -\dot{G}, \\ &\quad t \in [t_1, t_2 - \tau], \end{aligned} \tag{62}$$

$$\begin{aligned} &p_1(t)\dot{\xi}_1 + p_2(t)\dot{\xi}_2 - k[q_1(t) + q_1(t-\tau)]\xi_1 \\ &- k[q_2(t) + q_2(t-\tau)]\xi_2 - H\dot{\xi}_0 = -\dot{G}, \quad t \in (t_2 - \tau, t_2]. \end{aligned} \tag{63}$$

Equation (62) has a solution

$$\begin{aligned} &\xi_0 = 0, \quad \xi_1 = \frac{1}{m}[p_1(t) + p_{1\tau}(t+\tau)], \\ &\xi_2 = \frac{1}{m}[p_2(t) + p_{2\tau}(t+\tau)], \\ &G = -\frac{1}{2m}\left\{[p_1(t) + p_{1\tau}(t+\tau)]^2 + [p_2(t) + p_{2\tau}(t+\tau)]^2\right\} \\ &\quad + \frac{k}{2}[2q_1(t) + q_1(t-\tau) + q_1(t+\tau)]^2 \\ &\quad + \frac{k}{2}[2q_2(t) + q_2(t-\tau) + q_2(t+\tau)]^2, \\ &\quad t \in [t_1, t_2 - \tau]. \end{aligned} \tag{64}$$

Equation (63) has a solution

$$\begin{aligned} &\xi_0 = 0, \quad \xi_1 = \frac{1}{m}p_1(t), \quad \xi_2 = \frac{1}{m}p_2(t), \\ &G = -\frac{1}{2m}\left[p_1^2(t) + p_2^2(t)\right] + \frac{k}{2}\{[q_1(t) + q_1(t-\tau)] \\ &\quad + [q_2(t) + q_2(t-\tau)]\}, \quad t \in (t_2 - \tau, t_2]. \end{aligned} \tag{65}$$

The generators (64) and (65) correspond to the Noether quasi-symmetry of the system. From Theorem 2, we have

$$\begin{aligned} &\frac{1}{2m}\left\{[p_1(t) + p_{1\tau}(t+\tau)]^2 + [p_2(t) + p_{2\tau}(t+\tau)]^2\right\} \\ &+ \frac{k}{2}[2q_1(t) + q_1(t-\tau) + q_1(t+\tau)]^2 \\ &+ \frac{k}{2}[2q_2(t) + q_2(t-\tau) + q_2(t+\tau)]^2 = \text{const.}, \\ &\quad t \in [t_1, t_2 - \tau], \end{aligned} \tag{66}$$

$$\begin{aligned} &\frac{1}{2m}\left[p_1^2(t) + p_2^2(t)\right] + \frac{k}{2}\{[q_1(t) + q_1(t-\tau)]^2 \\ &+ [q_2(t) + q_2(t-\tau)]^2\} = \text{const.}, \quad t \in (t_2 - \tau, t_2]. \end{aligned} \tag{67}$$

Formulas (66) and (67) give the conserved quantities corresponding to the Noether quasi-symmetries (64) and (65).

7. Conclusion

In this manuscript, the Noether symmetries and the conserved quantities are studied for the Hamilton system with time delay. The Hamilton canonical equations with time delay are established. The Hamilton action with time delay is obtained. According to the Hamilton action with time delay, the definitions and the criteria of the Noether symmetric transformations and quasi-symmetric transformations are given. The relationship between a Noether symmetry and a conserved quantity is discussed. The results of this paper are of universal meaning. The theory of the Noether symmetries

and the conserved quantities for a standard Hamilton system is a special case of our results. If we take $\tau = 0$, then Theorems 1 and 2 reduce to the Noether theorem for a standard Hamilton system. The approach of this paper can be further generalized to various types of constrained mechanical systems with time delay, optimal control systems with time delay, Birkhoffian systems with time delay, and so on.

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