Pinning sampled-data synchronization for complex networks with probabilistic coupling delay*

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We deal with the problem of pinning sampled-data synchronization for a complex network with probabilistic time-varying coupling delay. The sampling period considered here is assumed to be less than a given bound. Without using the Kronecker product, a new synchronization error system is constructed by using the property of the random variable and input delay approach. Based on the Lyapunov theory, a delay-dependent pinning sampled-data synchronization criterion is derived in terms of linear matrix inequalities (LMIs) that can be solved effectively by using MATLAB LMI toolbox. Numerical examples are provided to demonstrate the effectiveness of the proposed scheme.

Keywords: complex network, probabilistic time-varying coupling delay, sampled-data synchronization, pinning control

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1. Introduction

Over the past decades, complex networks have received a great deal of attention from many researchers from different areas, for instance, biology, physics, sociology, mathematics, and so on.[1–3] Complex networks can be used to describe the structures of various practical systems including the World Wide Web, food webs, electric power grids, scientific citation networks, and social networks. A complex network is composed of a large number of interconnected nodes, in which each node represents an element with certain dynamical system and edge represents the relationship between them. Synchronization, as a collective behavior, is a significant topic in the study of complex networks. The main reason for this is that network synchronization can not only explain many natural phenomena but also has many potential applications, such as secure communication, synchronous information exchange over the internet, and the synchronous transfer of digital signals in communication networks. Until now, synchronization of large-scale complex network consisting of coupled dynamical nodes has attracted a great deal of interest from scientists and engineers.[4–16]

Since a real-world network usually has a large number of nodes, it is difficult to add controllers to all nodes. To reduce the number of controlled nodes, pinning control, in which the feedback controllers are only applied to a fraction of network nodes, has been introduced in many works.[17–22] In Ref. [17], the authors found that one can pin the linearly coupled networks by introducing fewer locally negative feedback controllers. They also found that the pinning strategy based on the highest connection degree has better performance than randomly pinning. In Ref. [18], Chen *et al.* pinned a complex network to a homogenous solution by a single controller under a large enough coupling strength. By using adaptive pinning control method, Zhou *et al.*[19] investigated local and global pinning synchronizations of complex networks and presented some low-dimensional pinning synchronization criteria. Later, Yu *et al.*[20] showed that the nodes with low degrees should be pinned first when the coupling strength is small, which is different from the traditional results. In Ref. [21], Song and Cao presented some low-dimensional pinning schemes for global synchronization of both directed and undirected complex networks, and proposed specific pinning schemes to select pinned nodes by investigating the relationship among pinning synchronization, network topology and the coupling strength. The authors in Ref. [22] considered the pinning anti-synchronization between two general dynamical networks.

With the rapid development of modern microprocessors, the sampled-data control technology is more efficient and useful than the common control in real-world applications. The sampled-data control system, whose control signals are allowed to change only at discrete sampling instants, can drastically reduce communication traffic and save energy. Therefore, sampled-data control has been discussed extensively by some researchers.[23,24] The input delay approach[23] is popular in the study of sampled control systems, where the system is modeled as a continuous-time system with a time-varying sawtooth delay in the control input induced by sample-and-hold. In Ref. [24], by constructing the time-dependent Lyapunov functional, a refined input delay approach was pro-

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posed. Based on the above two approaches, there are some results on chaos synchronization problem via sampled-data control.[25–27] Recently, in the framework of the input delay approach, the sampled-data synchronization problem has been investigated for a class of general complex network with time-varying coupling delay.[28] Furthermore, an improved sampled-data synchronization criterion has been derived in Ref. [29], and the desired sampled-data feedback controllers have been designed in terms of the solution to certain linear matrix inequalities (LMIs). It is noted that the results in Refs. [28] and [29] were obtained by using the Kronecker product, which needed to verify the given LMIs with large dimensions and had a huge computational burden. How to reduce the computational burden is an open question. In addition, control actions were added to all nodes of the network. It inevitably brings some inconvenience in practice. This motivates us to study the synchronization problem of a complex network under pinning control and sampled-data control with small computational burden.

Time delay is ubiquitous in many physical systems. It may cause undesirable dynamic behavior and poor performance. Therefore, the synchronization of a complex network with coupling delay has been investigated deeply. In general, the coupling delay of the complex network is inevitable and deterministic. However, as pointed out in Ref. [30], coupling delay often exists in a random form, that is to say, some values of the coupling delay are very large, but the probability of the delay taking such large values is very small, and it may lead to a more conservative result if only the information of variation range of the coupling delay is considered. Therefore, it is interesting and meaningful to investigate the synchronization in a network with probabilistic coupling delay. To the best of our knowledge, the synchronization of a complex network with probabilistic time-varying coupling delay and sampled-data information has not been considered in the literature.

Motivated by the above discussion, in this paper, the pinning sampled-data synchronization problem is investigated for a complex network with probabilistic interval time-varying coupling delay. The sampling period considered here is assumed to be time-varying but less than a given bound. Without using the Kronecker product, a new synchronization error dynamics is obtained by using the property of the random variable and input delay approach. Based on the Lyapunov theory and Jensen's inequality, a delay-dependent pinning sampled-data synchronization condition is proposed in terms of LMIs that can be solved effectively by using available software. Finally, numerical examples are given to illustrate the effectiveness of the proposed pinning sampled-data method.

The rest of this paper is organized as follows. In Section 2, the model of complex dynamical network with probabilistic time-varying coupling delay is presented and some preliminaries are also provided. A delay-dependent pinning sampled-data synchronization criterion is discussed in Section 3. Numerical simulations are given in Section 4. Finally, a conclusion is presented in Section 5.

Notation The notations used throughout this paper are fairly standard, R^n and $R^{m\times n}$ denote the n-dimensional Euclidean space and the set of all $m\times n$ real matrix, respectively. $P>0$ (≥ 0) or $P<0$ (≤ 0) means that P is symmetric and positive-definite (semi-positive-definite) or negative definite (semi-negative-definite). The superscript "T" represents the transpose, and "I" and "0" denote the identity and zero matrixes with appropriate dimensions, $\mathrm{diag}\{l_1,l_2,\ldots,l_n\}$ stands for a block diagonal matrix, $\|\cdot\|$ denotes the Euclidean norm of a vector. The symmetric terms in a symmetric matrix are denoted by $*$.

2. Model and preliminaries

Consider a complex dynamical network consisting of N identical coupled nodes as follows:

$$\begin{aligned}\dot{x}_i(t) &= f(x_i(t))+\sum_{j=1}^{N}G_{ij}Dx_j(t)\\ &\quad+\sum_{j=1}^{N}G_{ij}Ax_j(t-\tau(t))+u_i(t),\ i=1,2,\ldots,N,\end{aligned}\tag{1}$$

where $x_i=(x_{i1},x_{i2},\ldots,x_{in})^{\mathrm{T}}\in R^n$ and $u_i(t)\in R^n$ are, respectively, the state variable and the control input of the node i, $f:R^n\to R^n$ is a continuous vector-valued function, $\tau(t)$ denotes the time-varying coupling delay, $D=\mathrm{diag}\{d_1,\ldots,d_n\}$ and $A=\mathrm{diag}\{a_1,\ldots,a_n\}$ are the constant inner-coupling matrices, $G=(g_{ij})\in R^{N\times N}$ is the coupling configuration matrix, where g_{ij} is defined as follows: if there is a connection between node i and node j ($i\neq j$), then $g_{ij}>0$; otherwise, $g_{ij}=0$, and the diagonal elements of matrix G are defined by $g_{ii}=-\sum_{j=1,j\neq i}^{N}G_{ij}$, $i=1,2,\ldots,N$.

Remark 1 The coupling configuration matrix G represents the topological structure of network (1). In this paper, the matrix G is not assumed to be symmetric or irreducible. In Refs. [28] and [29], the coupling configuration matrix is assumed to be symmetric, which is quite restrictive in practice. In this regards, the network model considered here is more general.

Assumption 1 There exist constants h_1 and h_2 satisfying $0\leq h_1\leq h_2$ such that either $\tau(t)\in[0,h_1)$ or $\tau(t)\in[h_1,h_2]$. Furthermore, the probability distribution of $\tau(t)$ taking values in $[0,h_1)$ and $[h_1,h_2]$ is known in advance.

Under assumption 1, the probability distribution of $\tau(t)$ is assumed to be $\mathrm{Prob}\{\tau(t)\in[0,h_1)\}=\beta_0$ and $\mathrm{Prob}\{\tau(t)\in[h_1,h_2]\}=1-\beta_0$, where $0\leq\beta_0\leq 1$. Then, define a Bernoulli

random variable as follows:

$$\beta(t) = \begin{cases} 1, & \tau(t) \in [0, h_1), \\ 0, & \tau(t) \in [h_1, h_2]. \end{cases} \tag{2}$$

Therefore, it can be derived from Eq. (2) that

$$\text{Prob}\{\beta(t) = 1\} = E\{\beta(t)\} = \beta_0, \tag{3}$$

$$\text{Prob}\{\beta(t) = 0\} = 1 - E\{\beta(t)\} = 1 - \beta_0. \tag{4}$$

Furthermore, from the definition of $\beta(t)$, it can be shown that

$$E\{\beta_0 - \beta(t)\} = 0, \quad E\{(\beta_0 - \beta(t))^2\} = \beta_0(1-\beta_0). \tag{5}$$

Now, we introduce two time-varying delays $\tau_1(t)$ and $\tau_2(t)$ such that

$$\tau(t) = \begin{cases} \tau_1(t), & \tau(t) \in [0, h_1), \\ \tau_2(t), & \tau(t) \in [h_1, h_2]. \end{cases}$$

As for $\tau_1(t)$, $\tau_2(t)$, and $f(x)$, we make the following assumptions.

Assumption 2 There are constants μ_1 and μ_2 such that $\dot{\tau}_1(t) \le \mu_1 < \infty$ and $\dot{\tau}_2(t) \le \mu_2 < \infty$.

Assumption 3 There exist some constants F_k^-, F_k^+, $k = 1, 2, \ldots, n$, such that the nonlinear function $f(\cdot)$ is satisfied $F_k^- \le [f_k(x) - f_k(y)]/(x - y) \le F_k^+$ for all $x \ne y$.

By using the new functions $\tau_1(t)$, $\tau_2(t)$, and $\beta(t)$, the system (1) can be written as

$$\begin{aligned} \dot{x}_i \;=\; & f(x_i(t)) + \sum_{j=1}^{N} G_{ij} D x_j(t) + \beta(t) \sum_{j=1}^{N} G_{ij} A x_j(t - \tau_1(t)) \\ & + [1 - \beta(t)] \sum_{j=1}^{N} G_{ij} A x_j(t - \tau_2(t)) + u_i(t), \\ & i = 1, 2, \ldots, N. \end{aligned} \tag{6}$$

Remark 2 In this paper, we assume the time-varying coupling delay has a certain probability distribution. It is worth noting that when $\beta(t) = 1$ or 0, which means $\tau_1(t) = \tau_2(t) = \tau(t)$, system (6) includes the model considered in Refs. [28] and [29] as a special case.

Let $e_i(t) = x_i(t) - s(t)$ be the synchronization error, where $s(t) \in R^n$ is the state trajectory of the unforced isolate node $\dot{s}(t) = f(x(t))$. Then, the error dynamics is given by

$$\begin{aligned} \dot{e}_i \;=\; & g(e_i(t)) + \sum_{j=1}^{N} G_{ij} D e_j(t) + \beta(t) \sum_{j=1}^{N} G_{ij} A e_j(t - \tau_1(t))) \\ & + (1 - \beta(t)) \sum_{j=1}^{N} G_{ij} A e_j(t - \tau_2(t))) + u_i(t), \\ & i = 1, 2, \ldots, N, \end{aligned} \tag{7}$$

where

$$\begin{aligned} & g(e_i(t)) = [f_1(e_{i1}(t)) \;\; f_2(e_{i2}(t)) \;\; \cdots \;\; f_n(e_{in}(t))]^{\mathrm{T}}, \\ & f_j(e_{ij}(t)) = f_j(x_{ij}(t)) - f_j(s_j(t)). \end{aligned}$$

The control signal is assumed to be generalized by using a zero-order-hold (ZOH) function with a sequence of hold times $0 = t_0 < t_1 < \cdots < t_k < \cdots$. Therefore, the state feedback controllers take the following form:

$$u_i = K_i e_i(t_k), \quad t_k \le t < t_{k+1}, \quad i = 1, 2, \ldots, N, \tag{8}$$

where $K_i \ge 0$ reflects the control strength, $e_i(t_k)$ is the discrete measurement of $e_i(t)$ at sampling instant t_k, $\lim_{k \to +\infty} t_k = +\infty$. In this paper, the sampling is not required to be periodic, and the only assumption is that the distance between any two consecutive sampling instants is less than a given bound. It is assumed that $t_{k+1} - t_k = h_k \le p$ for any integer $k \ge 0$, where $p > 0$ represents the largest sampling interval.

Remark 3 While $K_i = 0$, it means that the node i is not be controlled, which implies that pinning control is employed. Thus, in this paper, we combine the sampled-data and pinning control to investigate the problem of synchronization for network. This is very different from Refs. [28] and [29].

By using the input delay approach and substituting Eq. (8) into Eq. (7), we can obtain

$$\begin{aligned} \dot{e}_i \;=\; & g(e_i(t)) + \sum_{j=1}^{N} G_{ij} D e_j(t) + \beta(t) \sum_{j=1}^{N} G_{ij} A e_j(t - \tau_1(t)) \\ & + (1 - \beta(t)) \sum_{j=1}^{N} G_{ij} A e_j(t - \tau_2(t)) - K_i e_i(t - d(t)), \\ & i = 1, 2, \ldots, N, \end{aligned} \tag{9}$$

where $d(t) = t - t_k$.

Remark 4 Similar to Refs. [28] and [29], based on the Kronecker product, we can get the following compact error dynamics:

$$\begin{aligned} \dot{e}(t) \;=\; & \bar{g}(e(t)) + (G \otimes D) e(t) + \beta(t)(G \otimes A) e(t - \tau_1(t)) \\ & + [1 - \beta(t)](G \otimes A) e(t - \tau_2(t)) + K e(t - d(t)), \end{aligned} \tag{10}$$

where

$$\begin{aligned} & e(t) = [e_1^{\mathrm{T}} \;\; e_2^{\mathrm{T}} \;\; \cdots \;\; e_N^{\mathrm{T}}]^{\mathrm{T}}, \\ & \bar{g}(e(t)) = [g^{\mathrm{T}}(e_1(t)) \; g^{\mathrm{T}}(e_2(t)) \;\; \cdots \;\; g^{\mathrm{T}}(e_n(t))]^{\mathrm{T}}, \\ & K = \text{diag}\{\bar{K}_1, \bar{K}_2, \ldots, \bar{K}_N\}_{Nn}, \\ & \bar{K}_i = \text{diag}\{K_i, \ldots, K_i\}_n. \end{aligned}$$

Furthermore, some synchronization stability conditions will be derived in terms of LMIs by considering the following Lyapunov functional:

$$V(t) = V_1(t) + V_2(t), \tag{11}$$

where

$$\begin{aligned} V_1(t) \;=\; & e^{\mathrm{T}}(t) P e(t) + \int_{t-p}^{t} e^{\mathrm{T}}(s) Q_1 e(s) \mathrm{d}s \\ & + \int_{t-h_1}^{t} e^{\mathrm{T}}(s) Q_2 e(s) \mathrm{d}s \\ & + \int_{t-h_2}^{t-h_1} e^{\mathrm{T}}(s) Q_3 e(s) \mathrm{d}s + \int_{t-\tau_1(t)}^{t} e^{\mathrm{T}}(s) Q_4 e(s) \mathrm{d}s \\ & + \int_{t-\tau_2(t)}^{t} e^{\mathrm{T}}(s) Q_5 e(s) \mathrm{d}s, \end{aligned}$$

$$V_2(t) = p\int_{-p}^{0}\int_{t+\theta}^{t}\dot{e}^{\mathrm{T}}(s)Z_1\dot{e}(s)\mathrm{d}s\mathrm{d}\theta + h_1\int_{-h_1}^{0}\int_{t+\theta}^{t}\dot{e}^{\mathrm{T}}(s)Z_2\dot{e}(s)\mathrm{d}s\mathrm{d}\theta + (h_2-h_1)\int_{-h_2}^{-h_1}\int_{t+\theta}^{t}\dot{e}^{\mathrm{T}}(s)Z_3\dot{e}(s)\mathrm{d}s\mathrm{d}\theta.$$

By calculation, the decision variables of the positive matrix P (Z_i, Q_i) is $0.5Nn(Nn+1)$. Obviously, the more decision variables, the proposed LMIs involves, the bigger computational complexity will be. On the other hand, if the number of nodes is big enough, it will lead to verify the given LMIs with large dimension with Kronecker product and have a huge computational burden. Thus, from the practical view, we expect to obtain a synchronization stability condition with less computational burden.

Now, we will rewrite the error system (9) by some new defined error state variables. Inspired by Ref. [32], we define

$$\bar{e}_k(t) = [e_{1k}(t)\ \ e_{2k}(t)\ \ \ldots\ \ e_{Nk}(t)]^{\mathrm{T}},\quad k=1,2,\ldots,n.$$

The new state $\bar{e}_k(t)$ is N-dimensional, which is composed of the k-th element of each error state. It is noted that if $e_i(t)$ satisfies $\lim_{t\to\infty} e_i(t)=0$, $i=1,2,\ldots,N$, then we have $\lim_{t\to\infty}\bar{e}_i(t)=0$, $i=1,2,\ldots,n$. Thus, $\bar{e}_k(t)$ can be treated as the new error state, and the number of new error state becomes n. This can be regarded as rearranging the error states in the network. From Eq. (9), we have the following new error dynamics:

$$\dot{\bar{e}}_k(t) = \phi_k(t) + [\beta(t)-\beta_0]\omega_k(t),\quad k=1,2,\ldots,n, \tag{12}$$

where

$$\begin{aligned}\phi_k(t) &= \bar{g}_k(\bar{e}_k(t)) + d_kG\bar{e}_k(t) + a_k\beta_0G\bar{e}_k(t-\tau_1(t)) + a_k(1-\beta_0)G\bar{e}_k(t-\tau_2(t)) + K\bar{e}_k(t-d(t)),\\ \omega_k(t) &= a_kG\bar{e}_k(t-\tau_1(t)) - a_kG\bar{e}_k(t-\tau_2(t)),\\ K &= \mathrm{diag}\{K_1,K_2,\ldots,K_N\},\\ \bar{g}_k(\bar{e}_k(t)) &= [f_k(x_{1k}(t))-f_k(s_k(t))\ \ f_k(x_{2k}(t)) - f_k(s_k(t))\ \ \ldots\ f_k(x_{Nk}(t))-f_k(s_k(t))]^{\mathrm{T}}.\end{aligned}$$

Remark 5 From Eqs. (9)–(12), the arrangement of $e_{ij}(t)$ is changed, but there is any change in real network structure. Hence, the stability of error states will not be changed, i.e., $\lim_{t\to\infty}\bar{e}_i(t)=0$, $i=1,2,\ldots,n$, implying $\lim_{t\to\infty}e_i(t)=0$, $i=1,2,\ldots,N$. The rearrangement of error states in the network is only benefit to derive a concise synchronization criterion with less computational burden without using the Kronecker product, which will be seen in the next section.

The aim of this paper is to find some sufficient conditions to ensure error system (12) to be stable in mean square under the given pinning sampled-data controllers. To proceed further, the following definition and useful lemmas are needed.

Definition 1 The coupled complex network (6) is said to be globally synchronized in mean square sense if $\lim_{t\to\infty}E\{\|e_i(t)\|^2\}=0$, $i=1,2,\ldots,N$, holds for any initial values.

Lemma 1 (Jensen's inequality[31]) For any positive symmetric constant matrix $Z=Z^{\mathrm{T}}>0$, scalar $\gamma>0$ and vector function $\omega:[0,\gamma]\to R^n$ such that the integrations concerned are well defined, then

$$\int_0^{\gamma}\omega^{\mathrm{T}}(s)\mathrm{d}s Z\int_0^{\gamma}\omega(s)\mathrm{d}s \le \gamma\int_0^{\gamma}\omega^{\mathrm{T}}(s)Z\omega(s)\mathrm{d}s. \tag{13}$$

Lemma 2 (Schur complement[22]) The following linear matrix inequality

$$\begin{bmatrix}Q(x) & S(x)\\ S(x)^{\mathrm{T}} & R(x)\end{bmatrix}<0,$$

where $Q(x)=Q(x)^{\mathrm{T}}$ and $R(x)=R(x)^{\mathrm{T}}$, is equivalent to one of the following conditions:

a) $Q(x)<0$, $R(x)-S(x)^{\mathrm{T}}Q(x)^{-1}S(x)<0$;

b) $R(x)<0$, $Q(x)-S(x)R(x)^{-1}S(x)^{\mathrm{T}}<0$.

3. Main results

In this section, a sufficient delay-dependent condition will be established, which can guarantee the error system (6) to be globally stable in a mean square sense.

Theorem 1 Under assumptions 1–3, the controlled complex network (6) with sampled-data controllers can achieve pinning synchronization in mean square sense, if there exist positive definition matrices P_k, Q_{ik} ($i=1,\ldots,5$), Z_{ik} ($i=1,2,3$), and some scalars $\varepsilon_k>0$, $k=1,2,\ldots,n$, such that

$$\begin{bmatrix}\Sigma_k & \Omega_{1k}^{\mathrm{T}}Z_k & \sqrt{\beta_0(1-\beta_0)}\Omega_{2k}^{\mathrm{T}}Z_k\\ * & -Z_k & 0\\ * & * & -Z_k\end{bmatrix}<0, \tag{14}$$

where

$$\Sigma_k = \begin{bmatrix}\Sigma_{11k} & P_kK+Z_{1k} & 0 & \Sigma_{14k} & \Sigma_{15k} & 0 & 0 & \Sigma_{18k}\\ * & -2Z_{1k} & Z_{1k} & 0 & 0 & 0 & 0 & 0\\ * & * & -Q_{1k}+Z_{1k} & 0 & 0 & 0 & 0 & 0\\ * & * & * & \Sigma_{44k} & 0 & Z_{2k} & 0 & 0\\ * & * & * & * & \Sigma_{55k} & Z_{3k} & Z_{3k} & 0\\ * & * & * & * & * & \Sigma_{66k} & 0 & 0\\ * & * & * & * & * & * & -Q_{3k}-Z_{3k} & 0\\ * & * & * & * & * & * & * & -\varepsilon_kI\end{bmatrix},$$

$$
\begin{aligned}
\Sigma_{11k} &= d_k P_k G + d_k G^{\mathrm{T}} P_k + Q_{1k} + Q_{2k} + Q_{4k} \\
&\quad + Q_{5k} - Z_{1k} - Z_{2k} - \varepsilon_k F_k^- F_k^+ I, \\
\Sigma_{14k} &= \beta_0 a_k P_k G + Z_{2k}, \quad \Sigma_{15k} = (1-\beta_0) a_k P_k G, \\
\Sigma_{18k} &= P_k + \varepsilon_k \frac{F_k^- + F_k^+}{2} I, \quad \Sigma_{44k} = -(1-\mu_1) Q_{4k} - 2Z_{2k}, \\
\Sigma_{55k} &= -(1-\mu_2) Q_{5k} - 2Z_{3k}, \\
\Sigma_{66k} &= -Q_{2k} + Q_{3k} - Z_{2k} - Z_{3k}, \\
\Omega_{1k} &= [d_k G \ \ K \ \ 0 \ \ \beta_0 a_k G \ \ (1-\beta_0) a_k G \ \ 0 \ \ 0 \ \ I], \\
\Omega_{2k} &= [0 \ \ 0 \ \ 0 \ \ a_k G \ \ -a_k G \ \ 0 \ \ 0 \ \ 0], \\
Z_k &= p^2 Z_{1k} + h_1^2 Z_{2k} + (h_2 - h_1)^2 Z_{3k}.
\end{aligned}
$$

Proof Consider the following Lyapunov functional:

$$
V(t) = V_1(t) + V_2(t), \tag{15}
$$

where

$$
\begin{aligned}
V_1(t) &= \sum_{k=1}^{n} [\bar{e}_k^{\mathrm{T}}(t) P_k \bar{e}_k(t) + \int_{t-p}^{t} \bar{e}_k^{\mathrm{T}}(s) Q_{1k} \bar{e}_k(s) \mathrm{d}s \\
&\quad + \int_{t-h_1}^{t} \bar{e}_k^{\mathrm{T}}(s) Q_{2k} \bar{e}_k(s) \mathrm{d}s + \int_{t-h_2}^{t-h_1} \bar{e}_k^{\mathrm{T}}(s) Q_{3k} \bar{e}_k(s) \mathrm{d}s \\
&\quad + \int_{t-\tau_1(t)}^{t} \bar{e}_k^{\mathrm{T}}(s) Q_{4k} \bar{e}_k(s) \mathrm{d}s \\
&\quad + \int_{t-\tau_2(t)}^{t} \bar{e}_k^{\mathrm{T}}(s) Q_{5k} \bar{e}_k(s) \mathrm{d}s], \\
V_2(t) &= \sum_{i=1}^{n} [p \int_{-p}^{0} \int_{t+\theta}^{t} \dot{\bar{e}}_k^{\mathrm{T}}(s) Z_{1k} \dot{\bar{e}}_k(s) \mathrm{d}s \mathrm{d}\theta \\
&\quad + h_1 \int_{-h_1}^{0} \int_{t+\theta}^{t} \dot{\bar{e}}_k^{\mathrm{T}}(s) Z_{2k} \dot{\bar{e}}_k(s) \mathrm{d}s \mathrm{d}\theta \\
&\quad + (h_2 - h_1) \int_{-h_2}^{-h_1} \int_{t+\theta}^{t} \dot{\bar{e}}_k^{\mathrm{T}}(s) Z_{3k} \dot{\bar{e}}_k(s) \mathrm{d}s \mathrm{d}\theta].
\end{aligned}
$$

Let $LV(t) = \lim\limits_{\Delta \to 0+} \Delta^{-1}[E\{V(t+\Delta)|e(t)\} - V(t)]$ be the infinitesimal operator of $V(t)$. It follows from Eq. (15) that

$$
LV(t) = LV_1(t) + LV_2(t). \tag{16}
$$

Taking the derivative of Eq. (16) along the solution of system Eq. (12) yields

$$
\begin{aligned}
LV_1(t) &\le \sum_{k=1}^{n} [2\bar{e}_k^{\mathrm{T}}(t) P_k \varphi(t) + \bar{e}_k^{\mathrm{T}}(t) Q_{1k} \bar{e}_k(t) \\
&\quad - \bar{e}_k^{\mathrm{T}}(t-p) Q_{1k} \bar{e}_k(t-p) + \bar{e}_k^{\mathrm{T}}(t) Q_{2k} \bar{e}_k(t) \\
&\quad - \bar{e}_k^{\mathrm{T}}(t-h_1) Q_{2k} \bar{e}_k(t-h_1) \\
&\quad + \bar{e}_k^{\mathrm{T}}(t-h_1) Q_{3k} \bar{e}_k(t-h_1) - \bar{e}_k^{\mathrm{T}}(t-h_2) Q_{3k} \bar{e}_k(t-h_2) \\
&\quad + \bar{e}_k^{\mathrm{T}}(t) Q_{4k} \bar{e}_k(t) - (1-\mu_1) \bar{e}_k^{\mathrm{T}} \\
&\quad \times (t-\tau_1(t)) Q_{4k} \bar{e}_k(t-\tau_1(t)) \\
&\quad + \bar{e}_k^{\mathrm{T}}(t) Q_{5k} \bar{e}_k(t) - (1-\mu_2) \bar{e}_k^{\mathrm{T}} \\
&\quad \times (t-\tau_2(t)) Q_{5k} \bar{e}_k(t-\tau_2(t))], \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
LV_2(t) &= \sum_{k=1}^{n} [\dot{\bar{e}}_k^{\mathrm{T}}(t) Z_k \dot{\bar{e}}_k(t) - p \int_{t+\theta}^{t} \dot{\bar{e}}_k^{\mathrm{T}}(s) Z_{1k} \dot{\bar{e}}_k(s) \mathrm{d}s \\
&\quad - h_1 \int_{t-h_1}^{t} \dot{\bar{e}}_k^{\mathrm{T}}(s) Z_{2k} \dot{\bar{e}}_k(s) \mathrm{d}s - (h_2 - h_1) \\
&\quad \times \int_{t-h_2}^{t-h_1} \dot{\bar{e}}_k^{\mathrm{T}}(s) Z_{3k} \dot{\bar{e}}_k(s) \mathrm{d}s]. \qquad (18)
\end{aligned}
$$

Using Lemma 1, we have that

$$
\begin{aligned}
&- p \int_{t-p}^{t} \dot{\bar{e}}_k^{\mathrm{T}}(s) Z_{1k} \dot{\bar{e}}_k(s) \mathrm{d}s \\
&\le -[\bar{e}_k(t) - \bar{e}_k(t-d(t))]^{\mathrm{T}} Z_{1k} [\bar{e}_k(t) - \bar{e}_k(t-d(t))] \\
&\quad - [\bar{e}_k(t-d(t)) - \bar{e}_k(t-p)]^{\mathrm{T}} \\
&\quad \times Z_{1k} [\bar{e}_k(t-d(t)) - \bar{e}_k(t-p)], \qquad (19) \\
&- h_1 \int_{t-h_1}^{t} \dot{\bar{e}}_k^{\mathrm{T}}(s) Z_{2k} \dot{\bar{e}}_k(s) \mathrm{d}s \\
&\le -[\bar{e}_k(t) - \bar{e}_k(t-\tau_1(t))]^{\mathrm{T}} Z_{2k} [\bar{e}_k(t) - \bar{e}_k(t-\tau_1(t))] \\
&\quad - [\bar{e}_k(t-\tau_1(t)) - \bar{e}_k(t-h_1)]^{\mathrm{T}} \\
&\quad \times Z_{2k} (\bar{e}_k(t-\tau_1(t)) - \bar{e}_k(t-h_1)), \qquad (20) \\
&- (h_2 - h_1) \int_{t-h_2}^{t-h_1} \dot{\bar{e}}_k^{\mathrm{T}}(s) Z_{3k} \dot{\bar{e}}_k(s) \mathrm{d}s \\
&\le -[\bar{e}_k(t-h_1) - \bar{e}_k(t-\tau_2(t))]^{\mathrm{T}} \\
&\quad \times Z_{3k} [\bar{e}_k(t-h_1) - \bar{e}_k(t-\tau_2(t))] \\
&\quad - [\bar{e}_k(t-\tau_2(t)) - \bar{e}_k(t-h_2)]^{\mathrm{T}} \\
&\quad \times Z_{3k} [\bar{e}_k(t-\tau_2(t)) - \bar{e}_k(t-h_2)]. \qquad (21)
\end{aligned}
$$

According to Eq. (12), we obtain

$$
\begin{aligned}
&E\{\dot{\bar{e}}_k^{\mathrm{T}}(t) Z_k \dot{\bar{e}}_k(t)\} \\
&= \phi_k^{\mathrm{T}}(t) Z_k \phi_k(t) + \beta_0 (1-\beta_0) \omega_k^{\mathrm{T}}(t) Z_k \omega_k(t). \qquad (22)
\end{aligned}
$$

On the other hand, based on assumption 1, for any $\varepsilon_k > 0$, we have

$$
\begin{aligned}
y_k(t) &= -\varepsilon_k [\bar{g}_k(\bar{e}_{ik}(t)) - F_k^- \bar{e}_{ik}(t)] \\
&\quad \times [\bar{g}_k(\bar{e}_{ik}(t)) - F_k^+ \bar{e}_{ik}(t)] \ge 0. \qquad (23)
\end{aligned}
$$

Combining Eqs. (17)–(23) and taking mathematical exceptions on both sides of Eq. (16) gives that

$$
E\{LV(t)\} \le \sum_{k=1}^{n} \delta_k^{\mathrm{T}}(t) \Theta_k \delta_k(t), \tag{24}
$$

where

$$
\begin{aligned}
\Theta_k &= \Sigma_k + \Omega_{1k}^{\mathrm{T}} Z_k \Omega_{1k} + \beta_0 (1-\beta_0) \Omega_{2k}^{\mathrm{T}} Z_k \Omega_{2k}, \\
\delta_k(t) &= [\bar{e}_k^{\mathrm{T}}(t) \ \ \bar{e}_k^{\mathrm{T}}(t-d(t)) \ \ \bar{e}_k^{\mathrm{T}}(t-p) \bar{e}_k^{\mathrm{T}}(t-\tau_1(t)) \\
&\quad \bar{e}_k^{\mathrm{T}}(t-\tau_2(t)) \ \ \bar{e}_k^{\mathrm{T}}(t-h_1) \ \ \bar{e}_k^{\mathrm{T}}(t-h_2) \ \ \bar{g}_k(\bar{e}_k(t))].
\end{aligned}
$$

From Lemma 2, it is easy to derive that condition (14) can guarantee $\Theta_k < 0$. Let $\lambda = \min\{\lambda_{\min}(-\Theta_k)\}$, $\bar{e}(t) = [\bar{e}_1(t) \ \ \bar{e}_2(t) \ \ \cdots \ \ e_k(t)]^{\mathrm{T}}$. It follows from Eq. (24) that

$$
E\{LV(t)\} \le -\lambda E\{\|\bar{e}(t)\|^2\}. \tag{25}
$$

Then, by the generalized Itô's formula, we have

$$\begin{aligned} & E\{V(t)\}-E\{V(0)\} \\ &= E\left\{\int_0^t LV(s)\mathrm{d}s\right\} \le -\lambda\int_0^t E\{\|\bar{e}(s)\|^2\}\mathrm{d}s. \end{aligned} \tag{26}$$

Therefore, by virtue of Eq. (26) and the discussion in Ref. [33], we have $\lim_{t\to\infty} E\{\|\bar{e}_i(t)\|^2\}=0$, $i=1,2,\ldots,n$. Thus, error system (12) is asymptotically stable in mean square sense. Based on remark 5, the arrangement of error state does not change the error system's stability, i.e., $\lim_{t\to\infty} E\{\|e_i(t)\|^2\}=0$, $i=1,2,\ldots,N$. According to definition 1, the controlled complex network (6) can achieve synchronization in mean square sense under the given pinning sampled-data controllers. This completes the proof.

Remark 6 Under the given pinning sampled-data control scheme, Theorem 1 gives a sufficient condition to guarantee the synchronization of complex network (6) with probabilistic coupling time-delay. The derived synchronization criterion is formulated in terms of LMIs and can be easily verified by using LMI toolbox of MATLAB. Moreover, the proposed condition in Theorem 1 is delay-dependent because it depends on the biggest sampling period p. Thus, we can find the sampling period as big as possible to ensure synchronizing the delayed complex network (6) in mean square sense. However, any pinning schemes (low-degree, random or high-degree) have not been employed in Theorem 1. How to select appropriate pinning scheme to achieve synchronization remains a challenging problem in pinning sampled-data control. This will be one of our future topics.

Remark 7 It is noted that our synchronization criterion is obtained without using the Kronecker product. The whole decision variables of positive matrix P_k (Z_{ik}, Q_{ik}) in Theorem 1, $k=1,\ldots,n$, is $0.5Nn(N+1)$. Compared with the traditional Kronecker product method discussed in remark 4, the proposed condition (14) in Theorem 1 has smaller decision variables by rearranging the error states. In addition, the LMIs in Theorem 1 with smaller dimension can be built via the error systems (12) without using Kronecker product. One can see that the rearrangement of error states in the network is crucial and important in reducing the computational burden. This is why we perform the rearrangement of network error states in our paper.

When $\beta(t)=1$ or $\beta(t)=0$, $\tau(t)$ takes value in the interval $[0,h]$ and satisfies $\dot{\tau}(t)\le\mu$. Then, we can immediately have the following error dynamics:

$$\begin{aligned} \dot{\bar{e}}_k(t) &= \bar{g}_k(\bar{e}_k(t))+d_kG\bar{e}_k(t)+a_kG\bar{e}_k(t-\tau(t)) \\ &\quad +K\bar{e}_k(t-d(t)), \quad i=1,2,\ldots,N. \end{aligned} \tag{27}$$

Based on Theorem 1, by removing Q_{3k}-dependent, Q_{5k}-dependent, and Z_{3k}-dependent terms in Lyapunov functional (16), we can easily derive the following delay-dependent pinning sampled-data synchronization criterion.

Corollary 1 Under assumptions 2–3, the complex network (6) with sampled-data controllers can achieve pinning synchronization, if there exist positive definition matrices P_k, Q_{ik} $(i=1,2,4)$, Z_{ik} $(i=1,2)$, and some scalars $\varepsilon_k>0$, $k=1,2,\ldots,n$, such that

$$\begin{bmatrix} \hat{\Sigma}_k & \Upsilon_k^{\mathrm{T}}\hat{Z}_k \\ * & -\hat{Z}_k \end{bmatrix} < 0, \tag{28}$$

where

$$\hat{\Sigma}_k = \begin{bmatrix} \hat{\Sigma}_{11k} & P_kK_k+Z_{1k} & 0 & \hat{\Sigma}_{14k} & 0 & \Sigma_{18k} \\ * & -2Z_{1k} & Z_{1k} & 0 & 0 & 0 \\ * & * & -Q_{1k}+Z_{1k} & 0 & 0 & 0 \\ * & * & * & \hat{\Sigma}_{44k} & Z_{2k} & 0 \\ * & * & * & * & -Q_{2k}-Z_{2k} & 0 \\ * & * & * & * & * & -\varepsilon_kI \end{bmatrix},$$

$$\begin{aligned} \hat{\Sigma}_{11} &= d_kP_kG+d_kG^{\mathrm{T}}P_k+Q_{1k}+Q_{2k} \\ &\quad +Q_{3k}-Z_{1k}-Z_{2k}-\varepsilon_kF_k^-F_k^+I, \\ \hat{\Sigma}_{14} &= a_kP_kG+Z_{2k}, \quad \hat{\Sigma}_{44k}=-(1-\mu)Q_{4k}-2Z_{2k}, \\ \Upsilon_k &= [D_k \ 0 \ K \ A_k \ 0 \ I], \quad \hat{Z}_k=p^2Z_{1k}+h^2Z_{2k}. \end{aligned}$$

Remark 8 The number of decision variables and maximal order of LMI of different methods are listed in Table 1. It shows that our proposed method involves smaller decision variables and lower computational complexity than the most existing results.

Remark 9 It is worth pointing out that the main result obtained here can be extended to some more general complex dynamical networks with uncertainties or external disturbances by using robust control theory. Owing to the space limit, it is omitted here. In the next section, the effectiveness of the developed method is shown by numerical simulation examples.

Table 1. Numerical complexity of different synchronization criteria.

Method	Ref. [28]	Ref. [29]	Corollary 1
No. of decision variables	$3.5Nn(Nn+1)+1$	$3.5Nn(Nn+1)+1$	$3Nn(N+1)+n$
Maximal order of LMI	$7Nn$	$7Nn$	$7N$

4. Numerical examples

Consider complex network (1) with four nodes. The nonlinear function f is taken as

$$f(x_i(t)) = \begin{bmatrix} -0.5x_{i1} + \tanh(0.2x_{i1}) \\ 0.65x_{i2} - \tanh(0.45x_{i2}) \end{bmatrix}.$$

It can be found that f satisfies Eq. (2) with $F_1^- = -0.5$, $F_1^+ = -0.3$, $F_2^- = 0.2$, and $F_2^+ = 0.65$. The outer-coupling matrix is assumed to be

$$G = \begin{bmatrix} -2 & 0 & 1 & 1 \\ 0 & -3 & 1 & 2 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{bmatrix}.$$

The inner-coupling matrices are given as $D = A = \mathrm{diag}\{1,1\}$. Let $\beta_0 = 0.9$, $\tau_1(t) = 0.2 + 0.15\sin(t)$, $\tau_2(t) = 0.85 + 0.5\sin(t)$. It followed that $h_1 = 0.35$, $h_2 = 1.35$, $\mu_1 = 0.15$, and $\mu_2 = 0.5$. Figure 1 depicts the random coupling delay.

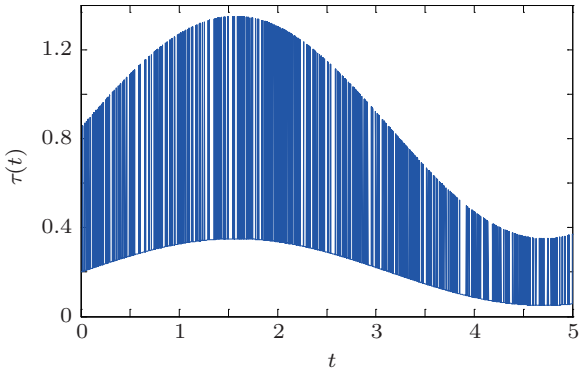


Fig. 1. (color online) Random delay in the simulation.

(I) Let the sampling period $p = 0.1$ and $K = \mathrm{diag}\{-10,0,0,0\}$. It is clear that the control action is only added on the first node. By using the MATLAB LMI toolbox, the LMIs (14) can be solved. Hence, system (6) is globally asymptotically synchronized in mean square sense. In the numerical simulation, the initial values are given as $x_1(0) = [3\ \ -1]^{\mathrm{T}}$, $x_2(0) = [-3\ \ 2]^{\mathrm{T}}$, $x_3(0) = [2\ \ 4]^{\mathrm{T}}$, $x_4(0) = [3\ \ -2]^{\mathrm{T}}$, and $s(0) = [3\ \ 2]^{\mathrm{T}}$. The state trajectories of the synchronization error are shown in Figs. 2(a) and 2(b). The pinning control input is shown in Fig. 2(c). Clearly, the controlled network (6) can achieve synchronization in mean square under the given pinning sampled-data scheme.

(II) Let $K = \mathrm{diag}\{-5,-5,0,0\}$. It means that the first two nodes are controlled. Based on Theorem 1, we can find that the maximum value of sampling period is $p = 0.17$. In the numerical simulation, the initial values are given as $x_1(0) = [4\ \ 3]^{\mathrm{T}}$, $x_2(0) = [-2\ \ 5]^{\mathrm{T}}$, $x_3(0) = [-3\ \ 2]^{\mathrm{T}}$, $x_4(0) = [1\ \ -1]^{\mathrm{T}}$, and $s(0) = [-3\ \ 2]^{\mathrm{T}}$. The state trajectories of the synchronization error are shown in Figs. 3(a) and 3(b). The pinning control input is shown in Fig. 3(c). As seen in Figs. 3(a) and 3(b), the trajectories of error signals are indeed well stabilized. It means that complex network (6) is asymptotically synchronized in mean square under our presented pinning sampled-data method.

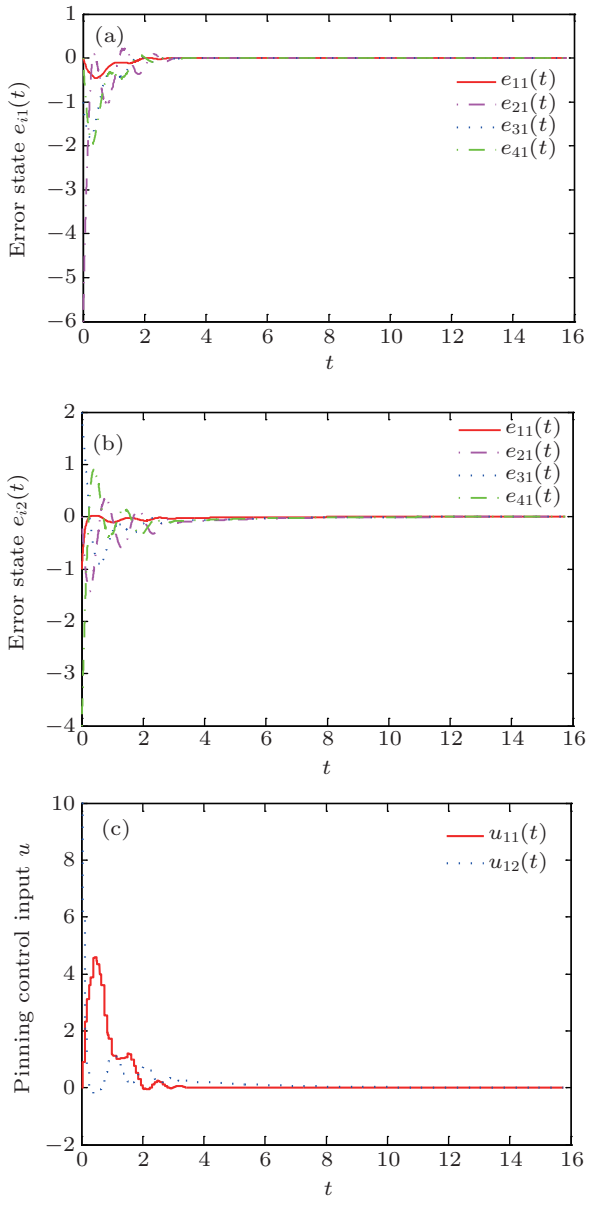


Fig. 2. (color online) (a) Synchronization error states e_{i1}, $i = 1,2,3,4$. (b) Synchronization error states e_{i2}, $i = 1,2,3,4$. (c) Pinning control inputs.

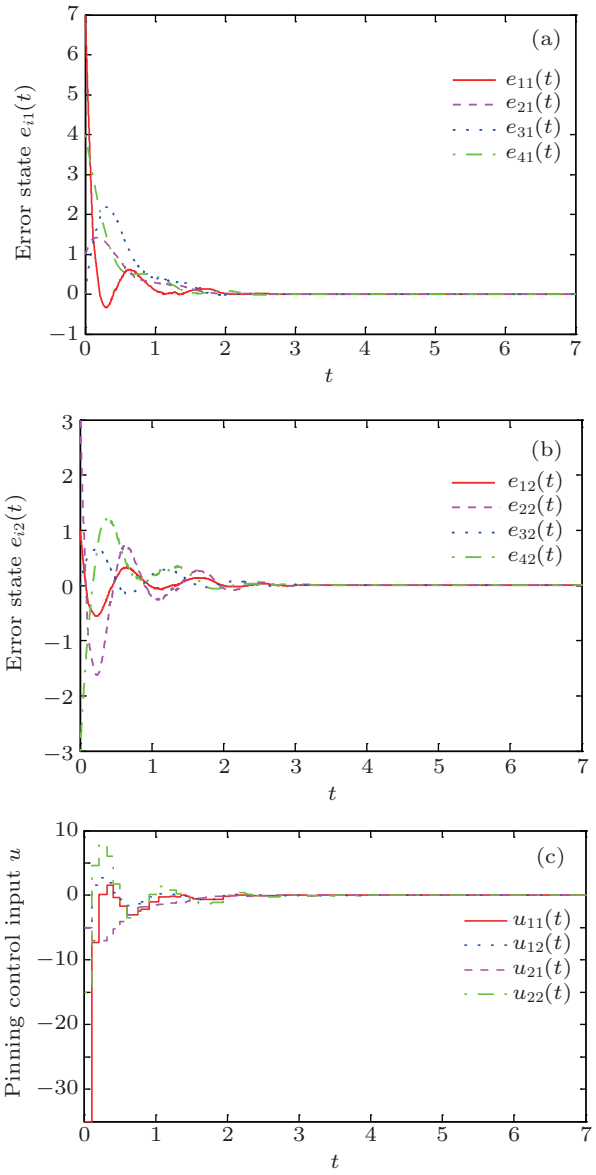


Fig. 3. (color online) (a) Synchronization error states e_{i1}, $i = 1,2,3,4$. (b) Synchronization error states e_{i2}, $i = 1,2,3,4$. (c) Pinning control inputs.

5. Conclusions

In this paper, the problem of pinning sampled-data synchronization for a complex dynamical network with probabilistic time-varying coupling delay is investigated. Without using the Kronecker product, a rearrangement method is used to construct an error system. In the framework of the input delay approach, a delay-dependent pinning sampled-data synchronization condition is derived based on Jensen's inequality. The obtained stability condition is formulated in terms of linear matrix inequalities that can be easily solved by using the MATLABLMI toolbox package. The effectiveness of the proposed synchronization criterion is demonstrated through some numerical examples.

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