Linear and nonlinear generalized consensuses of multi-agent systems*

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Consensus in directed networks of multiple agents, as an important topic, has become an active research subject. Over the past several years, some types of consensus problems have been studied. In this paper, we propose a novel type of consensus, the generalized consensus (GC), which includes the traditional consensus, the anti-consensus, and the cluster consensus as its special cases. Based on the Lyapunov's direct method and the graph theory, a simple control algorithm is designed to achieve the generalized consensus in a network of agents. Numerical simulations of linear and nonlinear GC are used to verify the effectiveness of the theoretical analysis.

Keywords: generalized consensus, multi-agent system, directed network, connection topology

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1. Introduction

During the past decades, consensus problems for networks of dynamic agents have been extensively studied by researchers from distinct points of view, partly due to a broad application of multi-agent systems, including wireless sensor networks, satellite formation,[1] the directions of fish and birds,[2,3] and distributed sensor filtering values.[4] One critical issue arising in multi-agent systems is to develop control strategies based only on relatively local information that can guarantee the whole system to evolve to have a coordinated behavior.

Classically, the agent dynamics in the multi-agent system is described as

$$\dot{\boldsymbol{x}}_i = \boldsymbol{u}_i, \quad \boldsymbol{x}_i, \boldsymbol{u}_i \in \mathbf{R}^n, \quad i = 1,2,\ldots,N, \tag{1}$$

where $\boldsymbol{x}_i$ is the state of agent i $(i = 1,2,\ldots,N)$, and $\boldsymbol{u}_i$ is the interconnected consensus protocol, which is only based on the information of the neighbors of agent i. This multi-agent system solves a consensus problem if for any given initial states and any j, $i = 1,2,\ldots,N$, $\lim_{t\to\infty} \|\boldsymbol{x}_j(t) - \boldsymbol{x}_i(t)\| = 0$. A typical consensus control algorithm is

$$\boldsymbol{u}_i(t) = \alpha \sum_{j=1}^{N} a_{ij}[\boldsymbol{x}_j(t) - \boldsymbol{x}_i(t)],$$

where a_{ij} is the (i,j)-th entry of the corresponding adjacency matrix, and α is the control parameter. Some types of consensuses, such as the classical consensus,[5–7] the anti-consensus, the phase consensus,[8,9] and the cluster consensus, have been studied. Vicsek *et al.*[10,11] proposed a simple model for the phase transition of a group of self-driven particles and numerically depicted the evolution of the model. Some literatures further investigated the cluster consensus[12–15] of multi-agent systems with different subgroups.

On the other hand, in nonlinear science, the generalized synchronization[16–22] phenomena of chaotic systems have been intensively studied recently. In complete synchronization, the dynamics of two coupled systems totally coincide with each other; in generalized synchronization, a certain known or unknown functional relation exists between the dynamics of two coupled systems which are usually nonidentical. The generalized synchronization has been deeply studied for complex networks.[23,24] In fact, the generalized synchronization is more natural to describe some phenomena of coherence in biological and physical systems consisting of multiple interacting components.

Inspired by the above works, we propose the definition of a generalized consensus of multi-agent systems, which includes the traditional consensus, the anti-consensus, and the cluster consensus as its cases. The corresponding control algorithms are designed to achieve linear and nonlinear generalized consensuses of the multi-agent systems. Under the updated distributed protocols, it is meaningful and practical that each agent moves to the average of the functional relation of states of its neighbors. Furthermore, if there is a leader in a network, each agent moves to the functional relation of the state of the virtual leader.

The paper is organized as follows. Section 2 introduces some background on the algebraic directed graph theory. The definition of a new type of generalized consensus and a designed control algorithm are presented in Section 3. Generalized consensuses in networks with different connected topologies are discussed in Section 4. Some simulation examples are shown to illustrate the linear and the nonlinear generalized consensuses in Section 5. Section 6 concludes the paper.

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2. Preliminaries: algebraic directed graph theory

The detailed preliminaries are similar to those in Ref. [25]. A directed graph $G(V,\varepsilon,A)$ consists of a set of vertices $V=\{v_1,v_2,\ldots,v_N\}$ and a set of directed edges $\varepsilon=\{(i,j):i,j\in V\}$. A weighted adjacency matrix $\boldsymbol{A}=[a_{ij}]_{N\times N}$ has non-negative entries. An edge $e_{ij}\in\varepsilon$ if and only if $a_{ij}>0$. A directed graph is strongly connected if and only if there is a directed path between any pair of distinct vertices. Furthermore, a directed graph contains a directed spanning tree if there exists a vertex. The Laplacian matrix $\boldsymbol{L}=[l_{ij}]_{N\times N}$ of $G(\boldsymbol{A})$ is defined as $l_{ii}=\sum_{j\neq i}a_{ij}$, $l_{ij}=-a_{ij}$ for $i\neq j$. For a directed graph, the Laplacian matrix $\boldsymbol{L}$ has the following properties.

Lemma 1[26] Suppose that a directed graph $G(\boldsymbol{A})$ is strongly connected, then, its Laplacian matrix $\boldsymbol{L}$ is irreducible and satisfies $\boldsymbol{L}\boldsymbol{1}_N=0$. Furthermore, there exists a positive vector $\boldsymbol{\xi}=(\xi_1,\xi_2,\ldots,\xi_N)^{\mathrm{T}}$ such that $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{L}=0$ and $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{1}_N=1$.

Lemma 2 (Frobenius normal form)[27] There exists a permutation matrix $\boldsymbol{W}$ with order N and integer $m\geq 1$, such that

$$\boldsymbol{W}^{\mathrm{T}}\boldsymbol{L}\boldsymbol{W}=\begin{pmatrix}\bar{\boldsymbol{L}}_{11} & 0 & \cdots & 0\\ \bar{\boldsymbol{L}}_{21} & \bar{\boldsymbol{L}}_{22} & \cdots & 0\\ \vdots & \vdots & \ddots & 0\\ \bar{\boldsymbol{L}}_{m1} & \bar{\boldsymbol{L}}_{m1} & \cdots & \bar{\boldsymbol{L}}_{mm}\end{pmatrix}, \tag{2}$$

where $\bar{\boldsymbol{L}}_{11}\in\mathbf{R}^{q_1\times q_1}$, $\bar{\boldsymbol{L}}_{22}\in\mathbf{R}^{q_2\times q_2}$, …, $\bar{\boldsymbol{L}}_{mm}\in\mathbf{R}^{q_m\times q_m}$ are irreducible square matrices, and q_k are uniquely determined within a simultaneous permutation of their lines with $\sum_{k=1}^{m}q_k=N$, but their ordering is not necessarily unique.

The process of permuting the Laplacian matrix $\boldsymbol{L}$ of $G(\boldsymbol{A})$ into its Frobenius normal form is a standard problem in graph algorithms, which can be solved by the depth-first search algorithm in linear time.[28]

Suppose that $G(\boldsymbol{A})$ is a directed network and its Laplacian matrix $\boldsymbol{L}$ is in the Frobenius normal form and $G_1,G_2,\ldots,G_m$ are the strongly connected components of $G(\boldsymbol{A})$ with the adjacency matrices $\bar{\boldsymbol{A}}_1=\mathrm{diag}(\bar{\boldsymbol{L}}_{11})-\bar{\boldsymbol{L}}_{11}$, $\bar{\boldsymbol{A}}_2=\mathrm{diag}(\bar{\boldsymbol{L}}_{22})-\bar{\boldsymbol{L}}_{22}$, …, $\bar{\boldsymbol{A}}_m=\mathrm{diag}(\bar{\boldsymbol{L}}_{mm})-\bar{\boldsymbol{L}}_{mm}$, then, $G^*(\boldsymbol{A}^*)=0$ is called a condensation network of $G(\boldsymbol{A})$. If there is a connection from a node in $V(G_j)$ to a node in $V(G_i)$, $i\neq j$, then weight $A^*_{ij}>0$; otherwise, $A^*_{ij}=0$ for $i,j=1,2,\ldots,m$; $A^*_{ii}=0$ for $i=1,2,\ldots,m$. Furthermore, for each $i=2,3,\ldots,m$, there is an integer $j<i$ such that $A^*_{ij}>0$ if and only if the directed network $G(\boldsymbol{A})$ contains a directed spanning tree.

3. Generalized consensus model

Consider the multi-agent system (1), we will investigate the following generalized consensus phenomenon.

Definition 1 System (1) or protocol $\boldsymbol{u}_i$ is said to achieve the generalized consensus or agreement if for any initial conditions,

$$\lim_{t\to\infty}\|\boldsymbol{h}_j(\boldsymbol{x}_j(t))-\boldsymbol{h}_i(\boldsymbol{x}_i(t))\|=0, \tag{3}$$

where $\boldsymbol{h}_i(\boldsymbol{x}_i)$: $\boldsymbol{x}_i=(x_{i_1},x_{i_2},\ldots,x_{i_n})\in\mathbf{R}^n\to(\boldsymbol{h}_i(\boldsymbol{x}_{i_1}),\boldsymbol{h}_i(\boldsymbol{x}_{i_2}),\ldots,\boldsymbol{h}_i(\boldsymbol{x}_{i_n}))\in\mathbf{R}^n$ $(i=1,2,\ldots,N)$ are smooth and invertible functions. If $\boldsymbol{h}_i(\boldsymbol{x}_i)=\boldsymbol{x}_i/k_i$ ($k_i\neq 0$ are constants) for all $i=1,2,\ldots,N$, then system (1) achieves the linear generalized consensus; if at least one of $\boldsymbol{h}_i(\boldsymbol{x}_i)$ $(i=1,2,\ldots,N)$ is nonlinear, then system (1) achieves the nonlinear generalized consensus.

Remark 1 In fact, the generalized consensus is a more general concept in comparison with the traditional consensus. When $k_i=1$ $(i=1,2,\ldots,N)$, the linear generalized consensus reduces to the normal consensus; if $a_{ij}\neq 0$, for $k_i=-k_j$, the linear generalized consensus reduces the anti-consensus; if $k_i=c_l$, $i\in N_l$, $l=1,\ldots,p$, $\cup_{l=1}^{p}N_l=\{1,2,\ldots,N\}$, $p<N$, c_l are the cluster parameters, it reduces to the cluster consensus.[12–15]

Correspondingly, we put forward an updating protocol of system (1)

$$\boldsymbol{u}_i(t)=\alpha(\boldsymbol{h}_i'(\boldsymbol{x}_i))^{-1}\sum_{j=1}^{N}a_{ij}[\boldsymbol{h}_j(\boldsymbol{x}_j(t))-\boldsymbol{h}_i(\boldsymbol{x}_i(t))], \tag{4}$$

where α denotes the control strength factor, $(\boldsymbol{h}_i'(\boldsymbol{x}_i))^{-1}=\mathrm{diag}((\boldsymbol{h}_i'(x_{i_1}))^{-1},(\boldsymbol{h}_i'(x_{i_2}))^{-1},\ldots,(\boldsymbol{h}_i'(x_{i_n}))^{-1})_n$. Substituting Eq. (4) into system (1) yields

$$\dot{\boldsymbol{x}}_i=\alpha(\boldsymbol{h}_i'(\boldsymbol{x}_i))^{-1}\sum_{j=1}^{N}a_{ij}[\boldsymbol{h}_j(\boldsymbol{x}_j(t))-\boldsymbol{h}_i(\boldsymbol{x}_i(t))],\quad i=1,2,\ldots,N. \tag{5}$$

Clearly, if the generalized consensus can be achieved, it is natural to require a generalized consensus state $\boldsymbol{x}_0(t)\in\mathbf{R}^n$ of system (5) satisfying $\boldsymbol{h}_i(\boldsymbol{x}_i(t))\to\boldsymbol{x}_0(t)$. Especially, if $\boldsymbol{h}_i(\boldsymbol{x}_i(t))=\boldsymbol{x}_i(t)/k_i$ for all $i=1,2,\ldots,N$, the protocol u_i is linear as follows:

$$\dot{\boldsymbol{x}}_i=\alpha k_i\sum_{j=1}^{N}a_{ij}[\boldsymbol{x}_j(t)/k_j-\boldsymbol{x}_i(t)/k_i],\quad i=1,2,\ldots,N. \tag{6}$$

4. Main results

4.1. Generalized consensus in strongly connected networks

Let $\boldsymbol{e}_i(t)=\boldsymbol{h}_i(\boldsymbol{x}_i(t))-\boldsymbol{x}_0(t)$ represent the position vector of the i-th agent relative to the weighted average position of all the agents in system (5), where $\boldsymbol{x}_0(t)=\sum_{j=1}^{N}\xi_j\boldsymbol{h}_j(\boldsymbol{x}_j(t))$, and $\boldsymbol{\xi}=(\xi_1,\xi_2,\ldots,\xi_N)$ is the positive left eigenvector of Laplacian matrix $\boldsymbol{L}$ associated with its zero eigenvalue, satisfying $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{1}_N=1$. Then, one has the following error dynamical system:

$$\begin{aligned}\dot{\boldsymbol{e}}_i(t)&=\boldsymbol{h}_i'(\boldsymbol{x}_i(t))\dot{\boldsymbol{x}}_i(t)-\sum_{j=1}^{N}\xi_j\boldsymbol{h}_j'(\boldsymbol{x}_j(t))\dot{\boldsymbol{x}}_j(t)\\&=-\alpha\sum_{j=1}^{N}l_{ij}\boldsymbol{h}_j(\boldsymbol{x}_j(t))-\sum_{j=1}^{N}\xi_j\Big[-\alpha\sum_{k=1}^{N}l_{jk}\boldsymbol{h}_k(\boldsymbol{x}_k(t))\Big]\\&=-\alpha\sum_{j=1}^{N}l_{ij}(\boldsymbol{h}_j(\boldsymbol{x}_j(t))-\boldsymbol{x}_0(t)).\end{aligned} \tag{7}$$

Let $\boldsymbol{e}(t) = [\boldsymbol{e}_1^{\mathrm{T}}(t), \boldsymbol{e}_2^{\mathrm{T}}(t), \ldots, \boldsymbol{e}_N^{\mathrm{T}}(t)]$, then system (7) can be written as

$$\dot{\boldsymbol{e}}(t) = -\alpha(\boldsymbol{L} \otimes \boldsymbol{I}_n)\boldsymbol{e}(t). \tag{8}$$

Theorem 1 Suppose that the network topology $G(\boldsymbol{A})$ is strongly connected, then the generalized consensus in system (5) is globally asymptotically achieved.

Proof Construct the following Lyapunov–Krasovskii functional:

$$V(t) = \boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{I}_N \otimes \boldsymbol{I}_n)\boldsymbol{e}(t). \tag{9}$$

Taking the time derivative of $V(t)$ along the trajectories of system (5) gives

$$\dot{V}(t) = -\alpha \boldsymbol{e}^{\mathrm{T}}(t)[(\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}}) \otimes \boldsymbol{I}_n]\boldsymbol{e}(t). \tag{10}$$

It is easy to verify that all the eigenvalues of $\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}}$ are positive, except for the single zero eigenvalue. Since $S = \{\boldsymbol{e}(t) = [\boldsymbol{e}_1^{\mathrm{T}}(t), \boldsymbol{e}_2^{\mathrm{T}}(t), \ldots, \boldsymbol{e}_N^{\mathrm{T}}(t)] \in \mathbf{R}^{Nn} : V \le V_0\}$ is a compact invariant set, according to the LaSalle invariance principle, all $\boldsymbol{e}_i(t) \to 0$ as $t \to \infty$, i.e., all $\boldsymbol{x}_i(t)$ asymptotically converge to $\boldsymbol{h}_i^{-1}(\boldsymbol{x}_0(t))$ as $t \to +\infty$. This implies that the generalized consensus is globally asymptotically reached.

4.2. Consensus in networks containing a directed spanning tree

Consider multi-agent system (5) whose topology contains a directed spanning tree. According to Lemma 2, without loss of generality, we assume that the Laplacian matrix $\boldsymbol{L}$ is in its Frobenius normal form. Furthermore, let $\bar{\boldsymbol{L}}_{ii} = \bar{\boldsymbol{L}}_i + \boldsymbol{A}_i$ ($i = 1, \ldots, m$), where $\bar{\boldsymbol{L}}_i$ is a zero-row-sum matrix, and $\boldsymbol{A}_i \ge 0$ is a diagonal matrix. By Lemma 1, there exists a positive vector $\bar{\boldsymbol{\xi}}_1 = (\bar{\xi}_{1_1}, \ldots, \bar{\xi}_{N_{q_1}})^{\mathrm{T}}$ such that $\bar{\boldsymbol{\xi}}_1^{\mathrm{T}} \bar{\boldsymbol{L}}_1 = 0$ and $\bar{\boldsymbol{\xi}}_1^{\mathrm{T}} \boldsymbol{1}_{q_1} = 1$.

Theorem 2 Suppose that the communication topology $G(\boldsymbol{A})$ contains a directed spanning tree, then the generalized consensus in system (5) is globally asymptotically achieved.

Proof It is obvious that the condensation network of $G(\boldsymbol{A})$, indicated by $G^*(\boldsymbol{A}^*)$, itself is a directed spanning tree. The dynamics of the agents belonging to the vertex set of the root of $G^*(\boldsymbol{A}^*)$ will not be affected by the others, and the local topology among them is strongly connected. By Theorem 1, the states of these agents will reach the generalized consensus asymptotically, that is, there exists $\epsilon_1 > 0$ such that $\boldsymbol{h}_{1i}(\boldsymbol{x}_{1i}(t)) = \bar{\boldsymbol{x}}(t) + \mathrm{O}(t^{-\epsilon_1})$, $i = 1, 2, \ldots, q_1$, where $\bar{\boldsymbol{x}}(t)$ represents the asymptotic generalized consensus state.

Based on the above analysis, consider the dynamics of the agents denoted by $v_{i1}, v_{i2}, \ldots, v_{iq_i}$, $2 \le i \le m$ belonging to the i-th vertex in $G^*(\boldsymbol{A}^*)$. It is only affected by the vertices which have directed paths to v_{i_s}, $s = 1, 2, \ldots, q_i$. Suppose that such agents excluding v_{is}, $s = 1, 2, \ldots, q_i$ are $v_{j1}, v_{j2}, \ldots, v_{jk}$. Furthermore, assume that the states of agents $v_{j1}, v_{j2}, \ldots, v_{jk}$ have already achieved the generalized consensus, and the generalized consensus position state is $\bar{\boldsymbol{x}}(t)$. Let $\hat{\boldsymbol{e}}_{ir}(t) = \boldsymbol{h}_{i_r}(\boldsymbol{x}_{i_r}(t)) - \bar{\boldsymbol{x}}(t)$. Simple calculations give the following error dynamical system:

$$\begin{aligned}\dot{\hat{\boldsymbol{e}}}_{ir}(t) \;=\; & -\alpha \sum_{j=1}^{q_i} l_{i_r i_j}(\boldsymbol{h}_{i_j}(\boldsymbol{x}_{i_j}(t)) - \boldsymbol{h}_{i_r}(\boldsymbol{x}_{i_r}(t))) \\ & -\alpha \sum_{p=1}^{k_i} l_{i_r j_p}(\bar{\boldsymbol{x}}(t) - \boldsymbol{h}_{i_r}(\boldsymbol{x}_{i_r}(t))) + \mathrm{O}(\mathrm{e}^{-\epsilon t}), \\ & r = 1, 2, \ldots, q_i. \end{aligned} \tag{11}$$

Let $\hat{\boldsymbol{e}}(t) = [\hat{\boldsymbol{e}}_{i_1}(t), \ldots, \hat{\boldsymbol{e}}_{i_{q_i}}(t)]^{\mathrm{T}}$, then, system (11) can be written as

$$\dot{\hat{\boldsymbol{e}}}(t) = -\alpha(\bar{\boldsymbol{L}}_{ii} \otimes \boldsymbol{I}_n)\hat{\boldsymbol{e}}(t) + \mathrm{O}(\mathrm{e}^{-\epsilon t}). \tag{12}$$

According to Eq. (12) and following the proof of Theorem 1, one can show that the states of agents $v_{i1}, v_{i2}, \ldots, v_{iq_i}$, $2 \le i \le m$ will reach the generalized consensus asymptotically, converging to $\boldsymbol{h}_{ir}^{-1}(\bar{\boldsymbol{x}}(t))$.

4.3. Generalized consensus in networks with a leader

Consider how to guarantee the position states of multi-agent system (5) to achieve the generalized consensus, i.e., $\lim_{t\to\infty} \|\boldsymbol{h}_i(\boldsymbol{x}_i(t)) - \boldsymbol{x}^*\| \to 0$, for all $i = 1, 2, \ldots, N$, where the communication topology does not contain any directed spanning tree. One way to achieve this goal is to assume that there is a virtual leader and a fraction of agents inform the state of the leader so that a navigational feedback term could be added to each of them. We have

$$\begin{aligned}\dot{\boldsymbol{x}}_i = (\boldsymbol{h}_i'(\boldsymbol{x}_i))^{-1}\Big[-\alpha \sum_{j=1}^{N} l_{ij}\boldsymbol{h}_j(\boldsymbol{x}_j(t)) - c_i\beta(\boldsymbol{h}_i(\boldsymbol{x}_i(t)) - \boldsymbol{x}^*(t))\Big], \\ i = 1, 2, \ldots, N, \end{aligned} \tag{13}$$

where $c_i = 1$ if agent i is informed and 0 otherwise. When $\boldsymbol{h}_i(\boldsymbol{x}(t)) = \boldsymbol{x}_i(t)/k_i$, the linear form is

$$\begin{aligned}\dot{\boldsymbol{x}}_i \;=\; & -k_i\alpha \sum_{j=1}^{N} l_{ij}(\boldsymbol{x}_j(t)/k_j) - k_i c_i \beta(\boldsymbol{x}_i(t)/k_i - \boldsymbol{x}^*(t)), \\ & i = 1, 2, \ldots, N. \end{aligned} \tag{14}$$

Let $\tilde{\boldsymbol{e}}_i = \boldsymbol{h}_i(\boldsymbol{x}_i(t)) - \boldsymbol{x}^*$, then by Eq. (13), one has

$$\dot{\tilde{\boldsymbol{e}}}(t) = -(\tilde{\boldsymbol{L}} \otimes \boldsymbol{I}_n)\tilde{\boldsymbol{e}}(t), \tag{15}$$

where $\tilde{\boldsymbol{e}}(t) = (\tilde{\boldsymbol{x}}_1^{\mathrm{T}}(t), \ldots, \tilde{\boldsymbol{x}}_N^{\mathrm{T}}(t))$, $\tilde{\boldsymbol{L}} = \alpha\boldsymbol{L} + \beta\mathrm{diag}\{c_1, \ldots, c_N\}$.

Theorem 3 Suppose that there exists a directed path from the virtual leader to each agent, then the generalized consensus in system (13) is globally asymptotically achieved.

Obviously, $(\tilde{\boldsymbol{L}}^{\mathrm{T}} + \tilde{\boldsymbol{L}}) \ge 0$, the proof is similar to that for Theorem 2 and omitted here for brevity.

5. Simulations

Example 1 (linear case) Consider linear multi-agent system (6) with topology $G(\boldsymbol{A}_1)$ as shown in Fig. 1. Topology $G(\boldsymbol{A}_1)$ contains a directed spanning tree. Let the coupling strength be $\alpha = 0.50$, the linear projective parameters are $\boldsymbol{k} = (k_1, k_2, \ldots, k_{10}) = (1, -3, 0.5, 5, 6, 7, -1, 3, 0.2, 2)$. The projective position states of all agents are obtained as shown in Fig. 2, where the given initial condition is $\boldsymbol{x}(0) = (1.25, 1, -1, 0.5, 0, 2, 10, 3, 4, 0.65)$. The stable state is

$\boldsymbol{x}(t) \to (2.0127, -6.0382, 1.0064, 10.0637, 12.0764, 14.0892, -2.0127, 6.0382, 0.4025, 4.0255)$ as $t \to \infty$, i.e., $\boldsymbol{x}(t) \to x_0(t) \cdot \boldsymbol{k}$, here the weighted average position $x_0(t) \to 2.0125$ as $t \to +\infty$. Figure 3 indicates that the global error

$$\sqrt{\sum_{i=1}^{N-1} \left(\frac{\boldsymbol{x}_i(t)}{k_i} - \frac{\boldsymbol{x}_{i+1}(t)}{k_{i+1}} \right)^2} \to 0,$$

and the linear generalized consensus is achieved.

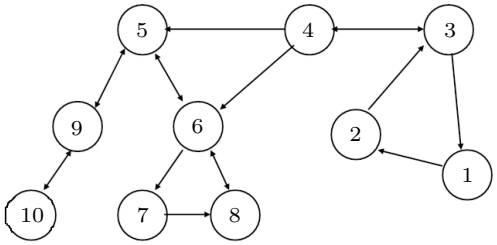


Fig. 1. Communication topology $G(\boldsymbol{A}_1)$.

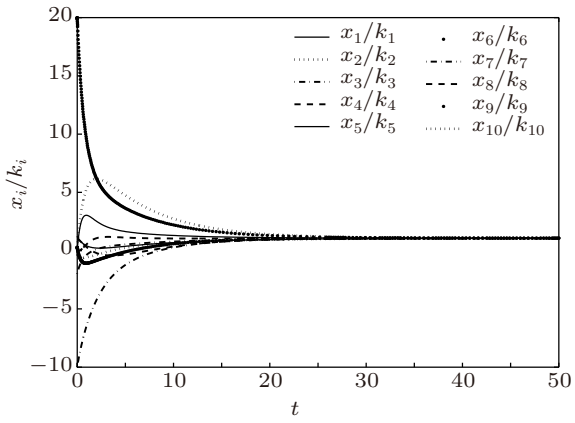


Fig. 2. Linear generalized consensus is achieved for example 1.

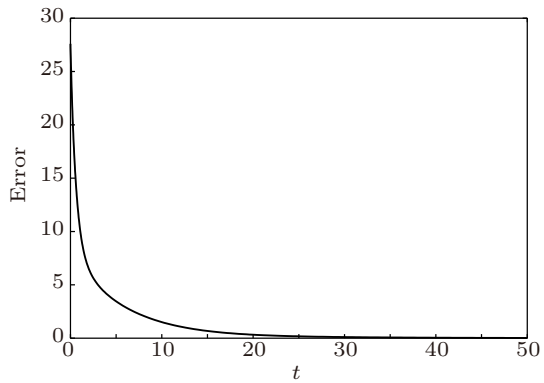


Fig. 3. Evolution of global error $\sqrt{\sum_{i=1}^{N-1}[(x_i(t)/k_i) - (x_{i+1}(t)/(k_{i+1}))]^2}$.

Example 2 (nonlinear case) Consider multi-agent system (13) with topology $G(\boldsymbol{A}_2)$ as shown in Fig. 4, topology $G(\boldsymbol{A}_2)$ does not contain any spanning tree. Suppose that there is a virtue leader labeled V in system (13), where agents 1 and 4 belonging to the first and the second strongly connected components are informed agents, as indicated in Fig. 4. Let $x^*(t) \equiv 4$, $h_1(x) = 4x$, $h_2(x) = x^2$, $h_3(x) = 4x/3$, $h_4(x) = 2x - 4$, $h_5(x) = \sqrt{x+11}$, with the given initial condition $\boldsymbol{x}(0) = (1.25, 1, -0.5, 0.5, 0)$. Figure 5 shows that the designed stable state is $\boldsymbol{x}(t) \to (h_1^{-1}(x^*), h_2^{-1}(x^*), h_3^{-1}(x^*), h_4^{-1}(x^*), h_5^{-1}(x^*)) = (1,2,3,4,5)$ as $t \to \infty$, i.e., all $\boldsymbol{x}_i(t) \to h_i^{-1}(x^*)$, $i = 1,2,3,4,5$. Figure 6 indicates that the global error $\sqrt{\sum_{i=1}^{4}(h_i(x_i(t)) - h_{i+1}(x_{i+1}(t)))^2} \to 0$, and the nonlinear generalized consensus is achieved.

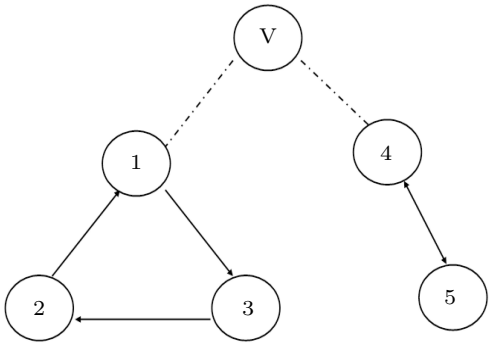


Fig. 4. Communication topology $G(\boldsymbol{A}_2)$.

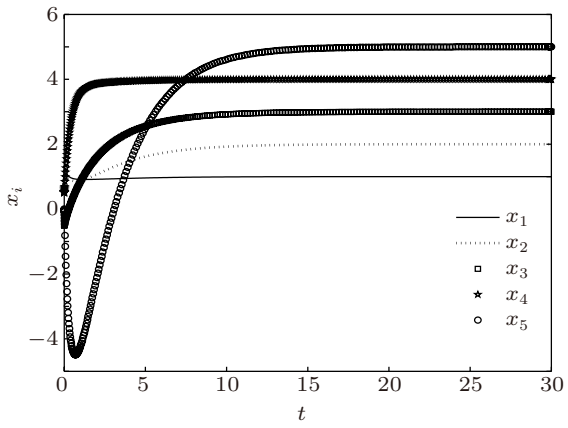


Fig. 5. Nonlinear generalized consensus is achieved for example 2.

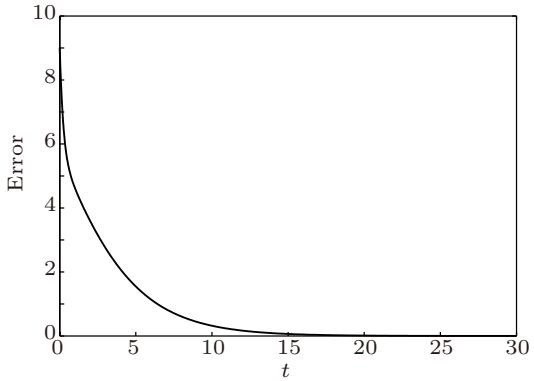


Fig. 6. Evolution of global error $\sqrt{\sum_{i=1}^{4}(h_i(x_i(t)) - h_{i+1}(x_{i+1}(t)))^2}$.

6. Conclusion

In summary, we have given a new concept of consensus: the generalized consensus in continuous-time multi-agent systems. A general simple distributed control scheme has been

proposed for the GC of multi-agent systems in directed networks. Based on the Lyapunov–Krasovskii functions, we have theoretically proved the generalized consensus in strongly connected networks, the topology contains and does not contain a directed spanning tree, respectively. Simulation examples of linear and nonlinear cases have been reported to verify the theoretical results. Our future works will focus on the generalized consensus behaviors of more practical models, such as multi-agent systems with nonlinear dynamics and second-order multi-agent systems. Continued research would be desirable.

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