Collective dynamics in a non-dissipative two-coupled pendulum system*

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We study the collective dynamics of a non-dissipative two-coupled pendulum system, including phase synchronization (PS) and measure synchronization (MS). We find that as the coupling intensity between the two pendulums increases, the PS happens prior to the MS. We also present a three-dimensional phase space representation of MS, from which a more detailed information about evolution can be obtained. Furthermore, the order parameters are introduced to describe the phase transition between PS and MS. Finally, through the analysis of the Poincaré sections, we show that the system exhibits separatrix crossing behavior right at the MS transition point, and as the total initial energy increases, the Hamiltonian chaos will arise with separatrix chaos at the chaotic MS transition point.

Keywords: two-coupled pendulum system, phase synchronization, measure synchronization, Poincaré section analysis

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1. Introduction

The dynamics of coupled nonlinear systems can exhibit a generic form of cooperative behavior called synchronization, a universal phenomenon which was first reported in 1665 by Huygens's experimental observation of two pendulum clocks closely hanged on the wall.[1–4] Since then, many significant characteristics of synchronization have been revealed. However, most previous studies mainly focused on the dissipative systems, and the synchronization between two trajectories is normally associated with the contraction of phase space volume. Unlike dissipative systems, coupled Hamiltonian systems conserve the phase space volume and do not allow synchronization in the original sense, since Liouville's theorem prevents the trajectories from fully collapsing. Therefore, the collective dynamics of coupled Hamiltonian systems would be multifarious and show more interesting behaviors.

In 1999, a novel collective behavior called measure synchronization (MS) was proposed by Hampton and Zanette.[5] They found that the two-coupled Hamiltonian system experiences a dynamic phase transition from a non-synchronized state to a synchronized state, and the transition is characterized by a commonly shared phase space that has an invariant measure as the coupling intensity increases. Until now, such a phenomenon has been investigated in coupled φ^4 Hamiltonian systems,[6–8] coupled Duffing Hamiltonian systems,[9] Frenkel–Kontorova models,[10,11] and a two-species Bosonic Josephson junction.[12,13]

Recently, based on Huygens's famous observation in the seventeenth century, experiments and numerical simulations show that two-coupled pendulums with dissipation can synchronize both in-phase and anti-phase.[1–3] The conservative version of a two-coupled pendulum system involves the studies of the bifurcation behavior and Hamiltonian chaos.[4] The goal of the present paper is to explore the phase transition including synchronization (PS) and measure synchronization (MS) in a non-dissipative coupled pendulum system.

The remainder of this paper is organized as follows. In Section 2 we introduce the non-dissipative two-coupled pendulum system and phase transition. In Section 3 we present a detailed dynamical analysis of PS and MS. In Section 4 we analyze the Poincaré section to reveal the essence of MS. In Section 5, we draw some conclusions.

2. Model and numerical results

For simplicity, we consider that the two pendulums are comprised of the same rigid bar of a fixed length and the same bob, and assume that each rigid bar has no mass and each bob is a point-like object. The two pendulums are connected by a torsion bar with torsion constant k. Then, the total Hamiltonian of this two-coupled pendulum system can be written as follows:[4]

$$H = \frac{1}{2}J_1\omega_1^2 - a_1\cos\theta_1 + \frac{1}{2}J_2\omega_2^2 - a_2\cos\theta_2 + \frac{1}{2}k(\theta_1-\theta_2)^2, \quad (1)$$

which can be divided into two parts. $H_\sigma = \frac{1}{2}J_\sigma\omega_\sigma^2 - a_\sigma\cos\theta_\sigma$ ($\sigma = 1,2$) describes the energy of each pendulum, where parameters J_σ and a_σ represent rotary inertia and the length of

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each pendulum respectively, and the last term on the right-hand side is the interaction energy, with k being an adjustable parameter which determines the intensity of the coupling between pendulums. The dynamic behavior is governed by the corresponding canonical equations:

$$\begin{aligned}
\dot{\theta}_1 &= J_1\omega_1,\\
\dot{\omega}_1 &= -a_1\sin\theta_1 - k(\theta_1-\theta_2),\\
\dot{\theta}_2 &= J_2\omega_2,\\
\dot{\omega}_2 &= -a_2\sin\theta_2 + k(\theta_1-\theta_2).
\end{aligned} \tag{2}$$

Let $J_1 = J_2 = J = 1$ and $a_1 = a_2 = a = 1$ in the simulation. The dynamics of the system depends on parameter k and the initial condition. In this paper, we fix $\theta_1(0) = \theta_2(0) = 0$ because the interaction energy $H_{\rm I} = \frac{1}{2}(\theta_1-\theta_2)^2$ keeps at zero at the beginning. For any initial choice of $\omega_1(0)$ and $\omega_2(0)$, the variation of k does not change the total energy. We set $\omega_1(0) = 0.6$ and $\omega_2(0) = 0.3$, so that the initial total energy is $E = -1.775$, and the standard fourth-order Runge–Kutta method is used to obtain the numerical solution.

Figure 1 shows the coupled dynamics of two pendulums. We take $k = 0$, 0.08, 0.023, and 0.02539 in Figs. 1(a)–1(d). When $k = 0$, the libration amplitude of pendulum 1 (θ_1 shown in Fig. 1) is larger than that of pendulum 2 (θ_2 shown in Fig. 1), but as time increases both libration amplitudes remain unchanged (see Fig. 1(a)). When $k = 0.08$ (see Fig. 1(b)), one can find that the libration amplitudes of two pendulums begin to change, and the libration amplitude of pendulum 1 is larger than that of pendulum 2, with an irregular phase difference between them. By increasing coupling intensity further (see Fig. 1(c)), one can observe that the libration amplitude of pendulum 1 is larger than that of pendulum 2, and both pendulums librate in-phase, which means that the coupled system is in a phase synchronized state. In Fig. 1(d), apart from the libration in phase, the variations of two libration amplitudes are the same but with the π-phase difference between the maximum of adjacent amplitudes. As time increases the average libration amplitudes of two pendulums are the same, which corresponds to a novel synchronization on the basis of PS in Fig. 1(c) and exhibits the collective dynamical mechanism of the coupled system.

In order to explain how the coupling intensity affects the system, we also explore the trajectories of the two pendulums in a $(\theta_i - \omega_i\,(i = 1,2))$ phase plane for different coupling intensities corresponding to the initial conditions and coupling intensity in Fig. 1.[5] Both trajectories of the pendulums are smooth for zero coupling (see Fig. 2(a)). As the coupling intensity k increases, these two trajectories will first evolve in such a way that the internal border of the inner trajectory shrinks into the center of the phase plane, while the external

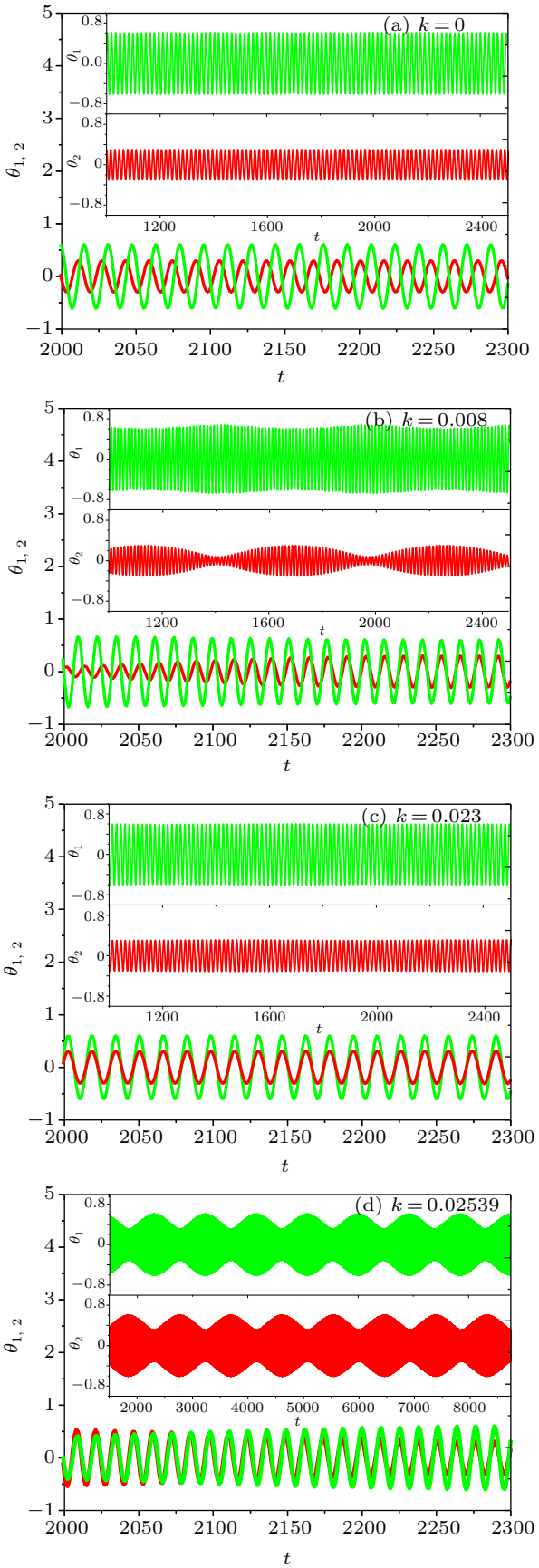


Fig. 1. (color online) Dynamic behaviors of two coupled pendulums with different coupling intensities. (a) and (b) Irregular phase difference between two pendulums and the libration amplitude changes as coupling intensity increases; (c) and (d) both pendulums librates in-phase, which means that phase synchronization is reached.

border of the outer trajectory extends outward, until the inner trajectory reaches the center of the (θ, ω) plane while the outer trajectory ceases extending outward (see Figs. 2(b) and 2(c)). As the coupling intensity k continues to increase, especially when $k = 0.023$, the trajectories are similar to those with $k = 0$ (see Fig. 2(d)). From Fig. 1(c) one can identify that the coupled system is in a phase synchronized state. Then, as k grows, the evolution process shows that the external border of the inner trajectory approaches to the internal border of the outer trajectory (see Fig. 2(f)). A sudden transition occurs at the point $k_c \approx k = 0.02539$, and the two trajectories share the same phase space with the invariant measure. This dynamic phase transition marks the transition to MS, and it corresponds to the coupled dynamics of two pendulums in Fig. 1(d). The aforementioned evolution process is similar to that in a two-species Bosonic–Josephson junction with specific initial conditions and interaction mechanisms.[12,13]

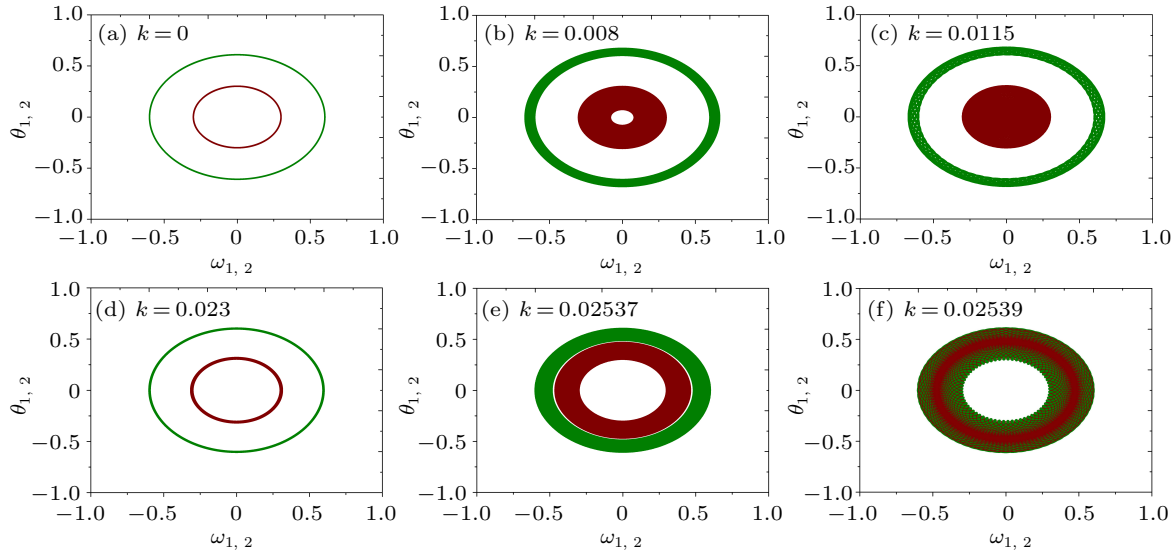


Fig. 2. (color online) Dynamic behaviors of a coupled pendulum shown in projected phase spaces ((θ_1, ω_1) and (θ_2, ω_2)). (a) $k = 0$, no MS; (b) $k = 0.008$, the internal border of the inner trajectory shrinks into the center of the phase plane; (c) $k = 0.0115$, the inner trajectory reaches the center of the (θ, ω) plane; (d) $k = 0.023$, the trajectories are periodic; (e) $k = 0.02537$, the two trajectories approach to each other; (f) $k = 0.02539$, the two trajectories share the same phase space and reach MS.

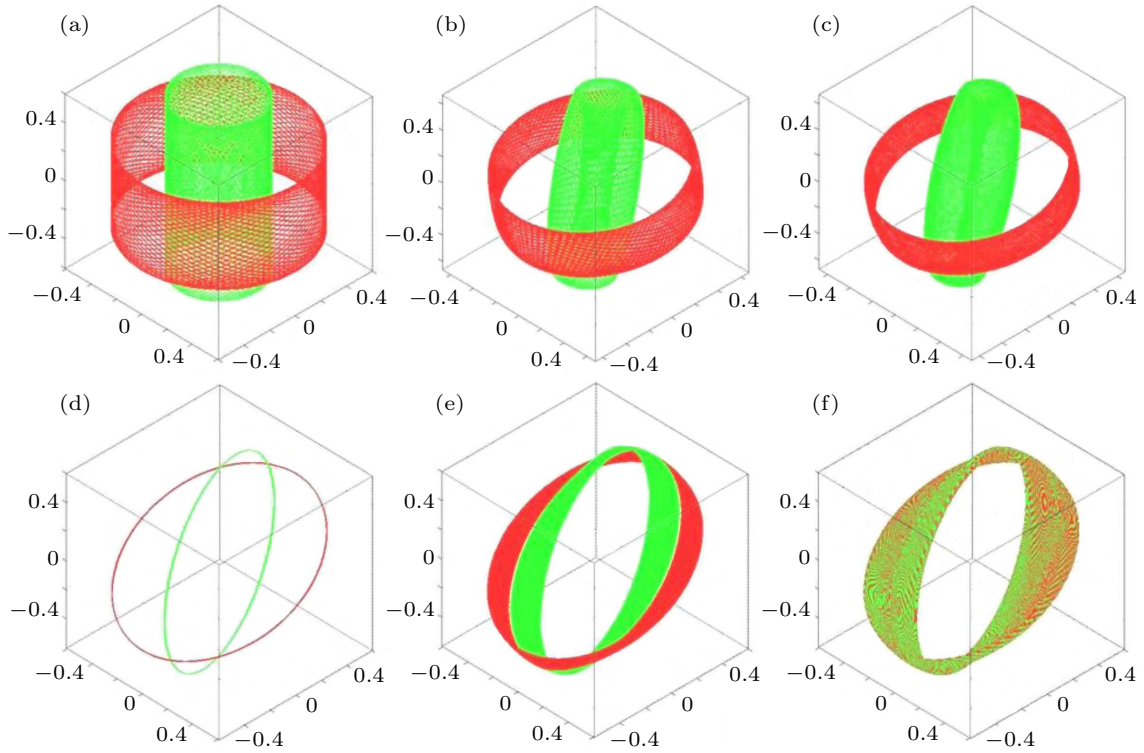


Fig. 3. (color online) Energy manifolds of the coupled pendulum in three-dimensional phase space, the initial condition and the coupling intensity are the same as those in Fig. 2. The outer manifold corresponds to $(\theta_1, \omega_1, \omega_2)$ and the inner one corresponds to $(\theta_2, \omega_2, \omega_1)$. The MS will appear when the two energy manifolds are overlapped (see panel (f)). Especially, the manifolds in panel (d) are periodic and the others are quasiperiodic, which cannot be seen in Figs. 2(a) and 2(d).

Until now, the previous studies of MS have focused only on a two-dimensional projected phase-space domain in a coupled Hamiltonian system. The dynamic behavior of the coupled system needs to be given a complete and accurate illustration because the system actually takes part in a four-dimensional phase space $(\theta_1, \omega_1, \theta_2, \omega_2)$. In the absence of dissipation, the energy manifolds of a four-dimensional phase space are restricted to a three-dimensional phase space. Here we choose two different three-dimensional planes $(\theta_1, \omega_1, \omega_2)$ and $(\theta_2, \omega_2, \omega_1)$ to observe such a phase transition. As shown in Fig. 3, with the same initial condition and the coupling intensity as those in Fig. 2, the more detailed information can be observed in three-dimensional phase space. In Figs. 3(a)–3(e), there are two separated manifolds with one manifold around the other. Initially, the inner phase-space volume shrinks in size (Figs. 3(a)–3(c)). Then, as the coupling intensity increases, this inner volume expands continuously until MS is achieved, and the two manifolds approach each other(Figs. 3(d) and 3(e)). Figure 3(f) shows that the two manifolds completely overlap, which means that a measure synchronized state is reached. In particular, in Fig. 3(a), the trajectories are quasiperiodic while in Fig. 3(d) they are periodic, which cannot be found in the (θ, ω) plane because the two trajectories are the same in a two-dimensional phase plane (see Figs. 2(a) and 2(d)).

3. Dynamic analyses

In Section 2, we revealed how the coupling intensity affects the two-coupled pendulum system. Now, we analyze the dynamic mechanism of the phase transition.

The critical behavior of MS has been proposed and studied. Hampton and Zanette[5] have already introduced the concept of an order parameter to describe the transition from a non-synchronized state to a measure synchronized state. First, we compute the order parameter from the perspective of its energy characteristics. The energies of the pendulums are

$$E_{1,2} = \frac{1}{2} J_{1,2} \omega_{1,2}^2 - a_{1,2} \cos \theta_{1,2}. \tag{3}$$

Figures 4(a)–4(d) show the time evolutions of the energy function for each pendulum with different coupling intensities. Before MS transition (see Figs. 4(a) and 4(b)), E_1 and E_2 do not overlap at all and the two pendulums have different energy variations. As coupling intensity increases, the lower boundary of the initially higher energy pendulum approaches the upper boundary of the initially lower energy pendulum. At the MS transition point (see Fig. 4(c)), $E_{1,2}$ have the same range of energy variations. The de-mixing-to-mixing feature of MS transition can be clearly observed in Fig. 3.

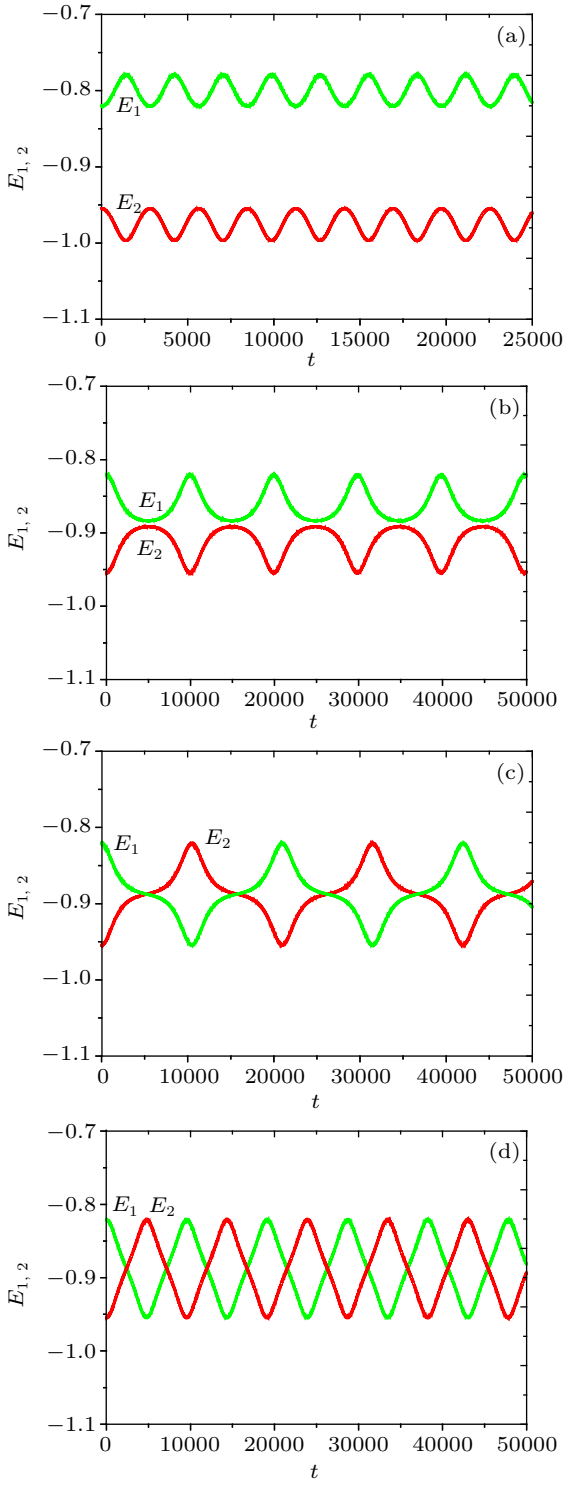


Fig. 4. (color online) Time evolutions of energy functions for the two-coupled pendulum system. Panels (a) and (b) show that before the transition point of MS, the two pendulums have different energy variations. Panels (c) and (d) display that after the transition point of MS, the energy variations of the two pendulums are the same.

Then, we introduce the average energy of each pendulum as follows:

$$\langle E_{1,2} \rangle = \frac{1}{T} \int_0^T E_{1,2} \, \mathrm{d}t. \tag{4}$$

The difference in averaged energy between the two pendu-

lums, $\Delta\langle E\rangle$, is

$$\Delta\langle E\rangle = |\langle E_1\rangle - \langle E_2\rangle|. \tag{5}$$

Figure 5(a) shows the averaged energies $\langle E_1\rangle$ and $\langle E_2\rangle$ versus coupling intensity. There exists a sharp transition at $k = k_c = 0.02539$. Before k_c, there is a finite difference between $\langle E_1\rangle$ and $\langle E_2\rangle$, after k_c, the two pendulums have identical averaged energies. Figure 5(b) shows $\Delta\langle E\rangle$ versus coupling intensity, and the discontinuous jump of $\Delta\langle E\rangle$ is identified to occur at the critical coupled intensity $k = k_c = 0.02539$. Obviously, $\Delta\langle E\rangle$ can act as the order parameter to describe the MS transition.

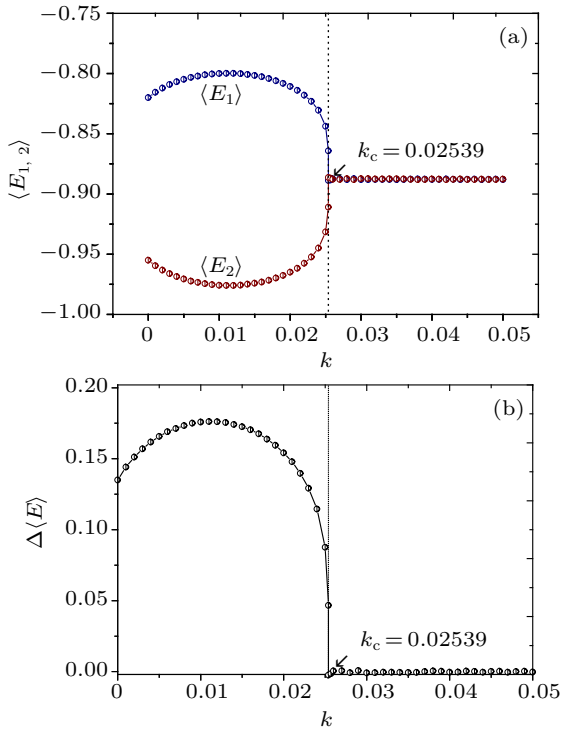


Fig. 5. (color online) (a) Averaged energies $\langle E_1\rangle$ and $\langle E_2\rangle$ versus coupling intensity. $k > k_c = 0.02539$, the two pendulums have identical average energies; (b) $\Delta\langle E\rangle$ versus coupling intensity, where the MS transition occurs at $k = k_c = 0.02539$.

Since $\Delta\langle E\rangle$ can be employed only to illustrate the MS transition, we need to seek a method to describe both PS transition and MS transition. Hampton and Zanette[5] have used the order parameters η and ω to illustrate the MS transition. First, they introduced the phase $\varphi_i(t)$ of each element, which can be defined as the angle included between the segment from the origin to the position of the element at time t and the specific axis. Numerically, the phase difference is $\Delta\varphi(t) = \varphi_1(t) - \varphi_2(t)$. Before the transition point, the phase difference increases (or decreases) linearly with time $\varphi(t) \approx \eta t + \varphi_0$ as the coupling intensity increases. This suggests that the trajectories have different main frequencies, and η is defined as

$$\eta = \left|\lim_{t\to\infty} t^{-1}\varphi(t)\right|. \tag{6}$$

After the two trajectories are synchronized, the phase difference will oscillate with a constant frequency $\varphi(t) = \varphi(t+\omega)$ and ω is the Fourier transform of $\varphi(t)$.

We compute the order parameters η and ω. In Fig. 6, we can find that there are two discontinuous jumps at $k = 0.0116$ and $k = 0.02539$. From Fig. 2, we know that the MS transition occurs at $k = 0.02539$. When $k = 0.0116$, the numerical result demonstrates that the phase difference oscillates with a fixed frequency. From Figs. 1 and 2, we find that $k = 0.0116$ is the PS transition point and the order parameter ω illustrates that the two pendulums posess the same characteristic frequency and are in a phase synchronized state. One can also find MS transition at $k = 0.02539$, and the two pendulums still share the same characteristic frequency.

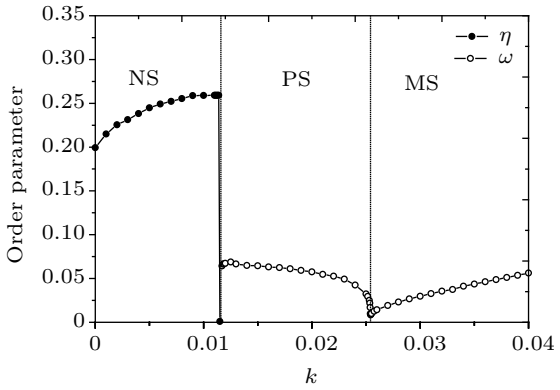


Fig. 6. Order parameters η and ω versus coupling intensity k. The transition point from non-synchronized state to PS is at $k = 0.0115$, and the MS transition point is at $k = k_c \approx 0.02539$.

4. Poincaré section analysis

MS is a special dynamical phase transition phenomenon in coupled Hamiltonian systems. In this section, we will use the Poincaré section method to analyze the MS transition in the non-dissipative two-coupled pendulum system to reveal the corresponding mechanism.

We set the initial total energy at $E = -1.775$ and the coupling intensity at $k = 0.02539$. With fifteen different initial conditions chosen randomly, the Poincaré map of phase space diagram is plotted in Fig. 7(a). The Poincaré map shows that the ω_1 axis divides the region into two parts: upper and lower, and each region consists of 'closed trajectories' circling around a center. Except for the quasiperiodic motion, in Fig. 8(a), one can also find the specific initial conditions gathered in the upper region which correspond to the point $k = 0.02539$ where the MS occurs. From Fig. 8(a) one can see that the specific initial condition is ($\theta_1(0) = \theta_2(0) = 0$, $\omega_1(0) = 0.6$, $\omega_2(0) = 0.3$), which is the same as that in Fig. 2. Figure 7(b) shows the Poincaré section analysis for MS with different coupling intensities. In the vicinity of separatrix, each trajectory corresponds to a fixed coupling intensity, and the separatrix corresponds to the critical coupling intensity

$k_c = 0.02539$. When $k < k_c$, the ring-shaped trajectories in both upper and lower regions will expand in size as the coupling intensity increases; when $k > k_c$ the two trajectories will cross the separatrix and keep expanding outwards (see the arrows in Fig. 7(b)), and both the trajectories with the same coupling intensity will completely merge together. From the analysis of Fig. 7(b), one can see that the MS transition is the process of separatrix crossing, i.e., the appearance of separatrix marks the MS transition point.[14] In addition, the overlap of the both trajectories can be an indication of measure synchronized state.

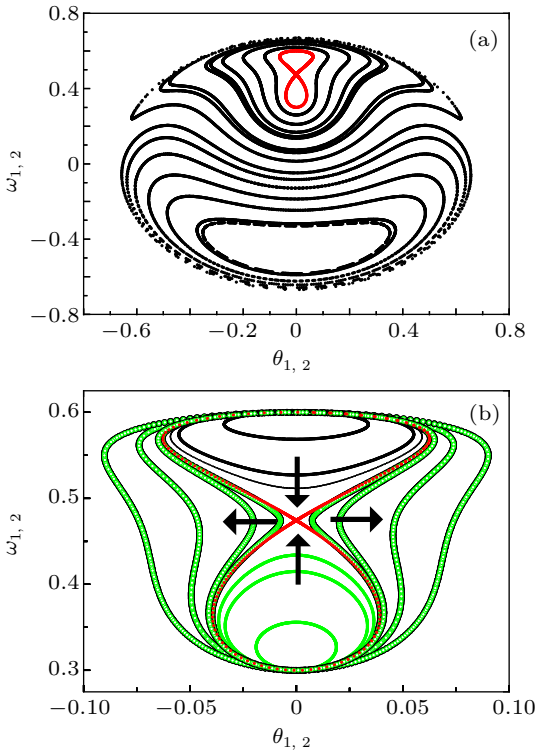


Fig. 7. (color online) (a) The Poincaré map with $k = 0.02539$ and 15 different random initial conditions that share the same total energy $E = -1.775$. There exists a separatrix in the upper region and one can find that MS occurs in specific initial conditions (the same as those in Fig. 2) when $k = 0.02539$. (b) The Poincaré section analysis for MS, with the same initial conditions as those in Fig. 2, and 8 different coupling intensities. Arrows point to the direction along which intensity k increases. The separatrix corresponds to the critical coupling intensity $k_c \approx k = 0.02539$ where the MS occurs.

Similarly, we perform Poincaré section analysis for chaotic MS transition in a two-coupled pendulum system. Figure 8(a) shows the Poincaré map with fixed total energy $E = 1.005$ and the coupling intensity $k = 0.090$. The figure shows that chaos arises when the total energy increases.[2] Figure 8(b) shows the result of Poincaré section analysis for chaotic MS. The initial conditions for chaotic MS are $\theta_1(0) = \theta_2(0) = 0$, $\omega_1(0) = 0.5$, and $\omega_2(0) = 2.4$. We note that with the increase of coupling intensity, there occurs chaotic separatrix crossing behavior (random dots), which corresponds to the emergence of chaotic MS. With k increasing ($k = 0.090$), the regular orbits are destroyed in the neighborhood of the separatrix, which is called separatrix chaos. This is a sign of a chaotic measure synchronized state.[14] In summary, we have found that if the separatrix chaos occurs in the separatrix crossing process, chaotic MS will be achieved, and this is closely related to the total energy of the coupled pendulum system, i.e., the initial conditions.

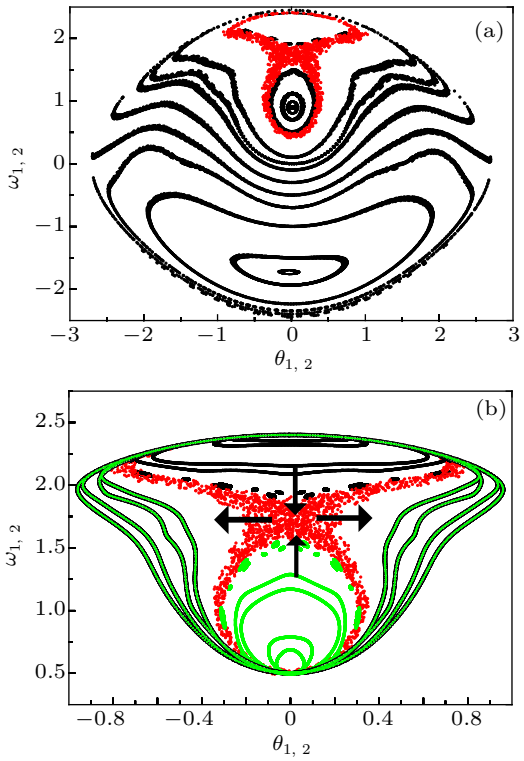


Fig. 8. (color online) (a) Poincare map with initial total energy $E = 1.005$ in (θ, ω) phase plane, and (b) the Poincaré section analysis for chaotic MS. Arrows point to the direction along which coupling intensity k increases. Separatrix chaos occurs at $k = 0.090$, which corresponds to the critical coupling intensity $k_c \approx k = 0.090$ where chaotic MS occurs.

Figure 9 shows the largest Lyapunov exponent λ_m versus the changing coupling intensity k, with the same initial conditions as those in Fig. 9(b). λ_m changes from zero to a positive constant at the critical point $k_c = 0.09$, which marks the transition from quasiperiodic state to chaotic state.

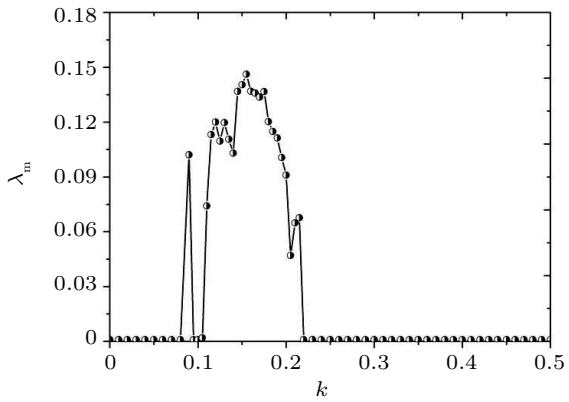


Fig. 9. The largest Lyapunov exponent λ_m versus coupling intensity k.

5. Conclusions

The collective dynamics of a non-dissipative two-coupled pendulum system are investigated in this paper. The numerical results and analyses reveal that for the case of such a system discussed in this paper, as the coupling intensity increases, the system must experience the phase synchronized state before reaching the measure synchronized state. A three-dimensional view of MS has also been presented, which provides detailed and useful information that can not be observed in the two-dimensional phase plane. In addition, the difference in averaged energy between the two pendulums, ΔE, and η can act as the order parameter of MS and PS respectively, and ω can be used to identify the MS transition point. Moreover, the essence of MS is investigated by using Poincaré section analysis. The results show that the two-coupled pendulum system exhibits separatrix crossing behavior at the MS transition. If the total energy of the coupled-Hamiltonian system increases, chaotic MS would arise due to separatrix chaos. Thus, the Poincaré section analysis becomes a general method to explain MS.

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References

[1] Dlião R 2009 *Chaos* **19** 023118
[2] Czołczyński K, Perlikowski P, Stefański A, and Kapitaniak T 2011 *Int. J. Bifurcat. Chaos* **21** 2047
[3] Czołczyński K, Perlikowski P, Stefański A, and Kapitaniak T 2011 *Chaos* **21** 023129
[4] Huynh H N and Chew L Y 2010 *Int. J. Bifurcat. Chaos* **20** 2427
[5] Hampton A and Zanette D H 1999 *Phys. Rev. Lett.* **83** 2179
[6] Chen S Y, Xu H B, Wang G R and Chen S G 2004 *Acta Phys. Sin.* **53** 4098 (in Chinese)
[7] Wang X G, Zhan M, Lai C H and Hu G 2003 *Phys. Rev. E* **67** 066215
[8] Wang X G, Zhang Y and Hu G 2002 *Phys. Lett. A* **298** 383
[9] Vincent U E 2005 *New J. Phys* **7** 209
[10] Yang Y, Wang C L, Jiang H, Chen J M and Duan W S 2012 *Phys. Scr.* **86** 015003
[11] Zhang J R, Jiang H, Yang Y, Duan W S and Chen J M 2012 *Phys. Scr.* **86** 065602
[12] Qiu H B, Tian J and Fu L B 2010 *Phys. Rev. A* **81** 043613
[13] Tian J, Qiu H B, Wang G F, Chen Y and Fu L B 2013 *Phys. Rev. E* **88** 032906
[14] Tian J, Qiu H B, Chen Z C and Chen Y 2013 *Mod. Phys. Lett. B* **27** 1350036