Measures of genuine multipartite entanglement for graph states*

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Graph states are special multipartite entangled states that have been proven useful in a variety of quantum information tasks. We address the issue of characterizing and quantifying the genuine multipartite entanglement of graph states up to eight qubits. The entanglement measures used are the geometric measure, the relative entropy of entanglement, and the logarithmic robustness, have been proved to be equal for the genuine entanglement of a graph state. We provide upper and lower bounds as well as an iterative algorithm to determine the genuine multipartite entanglement.

Keywords: genuine entanglement, graph state, iterative algorithm

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1. Introduction

Multipartite entanglement lies at the center of quantum computation, quantum secret sharing, and quantum metrology. It has wide applications in quantum many-body physics and quantum steganography.[1,2] Quantifying multipartite quantum entanglement has drawn a great deal of attention recently.[3,4] Unlike the case of bipartite entanglement, quantifying multipartite entanglement is difficult even for a pure multipartite state. Graph states are among several kinds of multipartite states for which the entanglement can be determined exactly by some entanglement measures. Apart from the usual cases of full separability and full entanglement, sometimes, we have to face the intermediate cases that only some particles are entangled and the other particles are not.[5] A variety of research about k-separable,[5,6] especially bi-separable,[6,7] have been carried out.

For a tripartite quantum state ρ, suppose the three parties are denoted as A, B, and C; if it can be written as $\rho^{\mathrm{fs}} = \sum_k p_k \left|\psi_k^{\mathrm{fs}}\right\rangle \left\langle\psi_k^{\mathrm{fs}}\right|$, where $\left|\psi^{\mathrm{fs}}\right\rangle = |a\rangle|b\rangle|c\rangle$ denotes a fully separable pure state and p_k form a probability distribution, then the tripartite state is called fully separable. A tripartite state is called biseparable if it can be written as $\rho^{\mathrm{bs}} = \sum_k p_k \left|\psi_k^{\mathrm{bs}}\right\rangle \left\langle\psi_k^{\mathrm{bs}}\right|$, where $\left|\psi^{\mathrm{bs}}\right\rangle = |a\rangle\left|\phi^{\mathrm{bc}}\right\rangle$ and $\left|\phi^{\mathrm{bc}}\right\rangle$ is an entangled state on the parties B and C; this partition is A|BC. The other two bipartitions are B|AC and C|AB, and we have $\left|\psi^{\mathrm{bs}}\right\rangle = |b\rangle|\phi^{\mathrm{ac}}\rangle$ and $\left|\psi^{\mathrm{bs}}\right\rangle = \left|\phi^{\mathrm{ab}}\right\rangle|c\rangle$, respectively. A biseparable state can be the mixture of all these three kinds of biseparable pure states. A state that is not biseparable is called genuinely tripartite entangled. For a general multipartite system, we have the definition of biseparability:[8] a biseparable state is a multipartite state that can be expressed as a convex sum of the projectors of product vectors and bipartite entangled vectors. Note that the definition of biseparability is different from 2-separability in the k-separable theory.[5,6] To elucidate the difference, let us take a four-partite system with parties A, B, C, and D for example. The partitions for a biseparable state are AB|CD, AC|BD, and AD|BC. While the partitions for a 2-separable state[6] can be A|BCD, B|ACD, C|ABD, and D|ABC in additional to AB|CD, AC|BD, and AD|BC. For partition A|BCD, there is a chance that the group BCD has tripartite entanglement.[9]

Many entanglement measures have been proposed for multipartite entanglement, among them are the robustness of entanglement, the relative entropy of entanglement, and the geometric measure.[10–14] In this paper, we will prove that these three kinds of entanglement measures are equal for the genuine entanglement of graph states. The genuine entanglement will be calculated for all the graph states of eight or less vertices. Upper and lower bounds of the genuine entanglement will be given. When the upper and the lower bounds are equal, the exact value of entanglement is determined.[15] Otherwise we will resort to an iterative algorithm.

2. Preliminary

2.1. Graph state and partitions

An n-qubit graph state $|G\rangle$[16,17] is a pure state associated with a graph $G = (V, E)$ which consists of a set of n vertices V and a set of edges E characterized by the adjacency matrix $\boldsymbol{\Gamma}$. An arbitrary vertex stands for a qubit. An edge $(i, j) \in E$ represents the interaction between qubits i and j. The $n \times n$ symmetric matrix $\boldsymbol{\Gamma}$ has nullified diagonal elements and if vertices i and j are connected, $\Gamma_{ij} = 1$, otherwise, $\Gamma_{ij} = 0$. An n-qubit graph state with adjacency matrix $\boldsymbol{\Gamma}$ can be written as

$$|G\rangle = \frac{1}{\sqrt{2^n}} \sum_{\mu} (-1)^{\frac{1}{2}\boldsymbol{\mu}\boldsymbol{\Gamma}\boldsymbol{\mu}^{\mathrm{T}}} |\mu\rangle, \tag{1}$$

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where $\mu = (\mu_1, \ldots, \mu_n)$ is a binary vector with $\mu_i = 0, 1$ for $i = 1, \ldots, n$. Clearly, a graph state is only determined by its adjacency matrix. A graph state is stabilized by the stabilizer group S with generators $K_i = X_i \prod_{j \in \{a|\Gamma_{ai}=1\}} Z_j$, where X_i and Z_i are the Pauli X and Z matrices of the i-th qubit, respectively. For a given vertex number n, the number of adjacency matrices or the number of graph states is $2^{\binom{n}{2}}$. For $n = 8$, the number of graph states is 2^{28}. However, many graphs can be converted to each other by local unitary transformations or by permutations of the vertices. The number of local unitary inequivalent graphs is greatly reduced. There are: an inequivalent two-qubit graph, an inequivalent three-qubit graph, two four-qubit graphs (No. 3, No. 4), four five-qubit graphs (No. 5–No. 8), 11 six-qubit graphs (No. 9–No. 19), 26 seven-qubit graphs (No. 20–No. 45), and 101 eight-qubit graphs (No. 46–No. 146).[18,19] The graph basis states are defined as

$$\left|G_{\overline{g}}\right\rangle = \left|G_{g_1 g_2 \cdots g_n}\right\rangle = Z_1^{g_1} Z_2^{g_2} \cdots Z_n^{g_n} |G\rangle,$$

where $g_i = 0, 1$ and $\overline{g} = g_1 g_2 \ldots g_n$. We have $K_i \left|G_{\overline{g}}\right\rangle = (-1)^{g_i} \left|G_{\overline{g}}\right\rangle$. All the graph basis states produced from $|G\rangle$ are locally unitary equivalent to $|G\rangle$. Thus they have the same value of entanglement.

If a density operator ρ of an n-qubit system does not contain genuine one-qubit entanglement,[8] we can write ρ in the following form:[9]

$$\rho = \sum_{\alpha} p_\alpha \rho_\alpha, \tag{2}$$

with $p_\alpha \geq 0$, $\sum_\alpha p_\alpha = 1$, and

$$\rho_\alpha = \rho_{1\alpha} \otimes \rho_{2\alpha} \otimes \rho_{3\alpha} \otimes \cdots \otimes \rho_{k\alpha}, \tag{3}$$

where $\rho_{i\alpha}$ $(i = 1, 2, \ldots, k)$ stands for a component state which contains $m_{i\alpha}$ $(1 \leq m_{i\alpha} < l)$ qubits with $\sum_{i=1}^{k} m_{i\alpha} = n$. Namely, in an arbitrary component α, the n qubits are divided into k parties with $m_{i\alpha}$ qubits for the i-th component, and the component state ρ_α is a tensor product of the partite states. For a given component α, each qubit belongs to a fixed party, nevertheless, for different α components, the partite divisions of the qubits are different.

In general, for a multipartite system V, any sequence $(A_1, \ldots, A_N)$ of disjointed subsets $A_i \in V$ with $\bigcup_{i=1}^{N} A_i = V$ will be called a partition of V. We write

$$(A_1, \ldots, A_N) \leq (B_1, \ldots, B_M), \tag{4}$$

if $(A_1, \ldots, A_N)$ is a finer partition than $(B_1, \ldots, B_M)$. We call $(B_1, \ldots, B_M)$ a coarse grain partition. Among all the partitions, there is a finest partition. The full separability is with respect to the finest partition. Any coarse grain partition will lead to a kind of separability. A separable state without genuine l-partite entanglement can be written as the mixture of coarse grain product states. The size of the coarse grain should be limited to $l - 1$ or less. We will call the separable state $l - 1$ coarse separable. The biseparability defined in Ref. [6] is the 2 coarse separability here. While the k-separable state is a mixture of k-partite product states.

2.2. Entanglement inequality

In order to study the entanglement depth of the graph states, we review the entanglement measures first. For a pure state $|\psi\rangle$, the geometric measure of entanglement[12,20,21] is defined as

$$E_{\mathrm{g}}(|\psi\rangle) = \min_{|\phi\rangle \in \mathrm{Pro}} -\log_2 |\langle\phi|\psi\rangle|^2, \tag{5}$$

where Pro is the set of product states. This is the distance between the product state $|\phi\rangle$ and state $|\psi\rangle$ in terms of fidelity. The entanglement measures other than the geometric measure involved in this paper are defined as follows. For a state ρ, the global robustness of entanglement $R(\rho)$[11] is defined as

$$R(\rho) = \min_{\Omega} t,$$

where Ω is a state such that $\omega = (\rho + t\Omega)/(1 + t)$ is separable. The logarithmic robustness[20,21] is

$$R_{\mathrm{L}}(\rho) = \log_2(1 + R(\rho)). \tag{6}$$

The relative entropy of entanglement[13] is defined as the distance to the closest separable state in terms of the relative entropy

$$E_{\mathrm{r}}(\rho) = \min_{\chi \in \mathrm{Sep}} S(\rho||\chi), \tag{7}$$

where $S(\rho||\chi)$ is the relative entropy, and Sep is the set of separable states.

For pure states, the relationship among geometric measure, logarithmic robustness, and relative entropy of entanglement is[20]

$$R_{\mathrm{L}}(|\psi\rangle) \geq E_{\mathrm{r}}(|\psi\rangle) \geq E_{\mathrm{g}}(|\psi\rangle). \tag{8}$$

The difficulty in calculating these measures usually increases from right to left as the defining optimization problem becomes harder.[21] For a given N-partite system with a qubit in each party, an entangled pure state $|\psi\rangle = \sum_{i_1 \cdots i_n = 0}^{1} a_{i_1 \cdots i_n} |i_1 \cdots i_n\rangle$ has 2^n terms at most. In order to obtain the geometric measure, we should determine the closest state $|\phi\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle) \cdots (\alpha_n|0\rangle + \beta_n|1\rangle)$, the number of its parameters is about $\mathrm{O}(n)$. However, in terms of the relative entropy of entanglement and the global robustness of entanglement, the yet to be determined separable states ω and χ are $2^n \times 2^n$ matrices, the number of the parameters is about $\mathrm{O}(4^n)$. Furthermore, we need the separable conditions of ω and χ. So we prefer to use the geometric measure for the entanglement of graph states.

For a graph state $|G\rangle$, we have[20,21]

$$n-\log_2 N \geq R_{\rm L}(|G\rangle)=E_{\rm r}(|G\rangle)=E_{\rm g}(|G\rangle), \tag{9}$$

where N is the number of graph diagonal states that can be discriminated by local operations and classical communication (LOCC). We thus call $E_{\rm u}=n-\log_2 N$ the LOCC upper bound of entanglement for a graph state. The equalities in expression (9) can be proven by constructing the separable state ω from the closest separable state $|\phi\rangle$ with respect to graph state $|G\rangle$ in geometric measure[21]

$$\omega=\sum_{S_i\in K} S_i|\phi\rangle\langle\phi|S_i=\sum_{\bar{g}}|\langle G_{\bar{g}}|\phi\rangle|^2|G_{\bar{g}}\rangle\langle G_{\bar{g}}|, \tag{10}$$

where $K=\{S_i\}_{i=1}^{2^n}$ is the stabilize group. Assuming $t=1/|\langle G|\phi\rangle|^2-1$, we have $R_{\rm L}(|G\rangle)\leq E_{\rm g}(|G\rangle)$, together with inequality (8), expression (9) is proved.

In convention, the product state and the separable state in this subsection are n-partite fully product state and fully separable state, respectively. The separability is with respect to the partition of the system into n parties, each party has one qubit. However, if the system of n qubits is divided into k parties with $m_{i\alpha}$ qubits for the i-th party, the closest product state in the definition of geometric measure and the separable states in the definitions of the relative entropy and the logarithmic robustness are defined with respect to this partition, now they are the k-partite fully product state and fully separable states, respectively. Thus inequalities (8) and (9) remain true.

3. Entanglement measures of genuine one-qubit entanglement

The genuine three-qubit entanglement measure for a graph state $|G\rangle$ can be the geometric measure. It is based on the inner product of $|G\rangle$ with the closest pure state $|\Phi\rangle$ that is not genuine three-qubit entangled. Here $|\Phi\rangle$ is the product of two-qubit or one-qubit vectors that maximizes $|\langle G|\Phi\rangle|$. The genuine three-qubit geometric measure of the graph state is defined as[20,21]

$$E_{3\rm g}(|G\rangle)=\min_{|\Phi\rangle\in{\rm Bipro}}\{-\log_2|\langle G|\Phi\rangle|^2\}, \tag{11}$$

where Bipro is the set of vectors which are the product of two-qubit or one-qubit vectors. For instance, we consider the genuine three-qubit entanglement of a four-qubit graph state. Suppose the four qubits are marked as A, B, C, and D, they can be divided into AB|CD, AC|BD, and AD|BC. The possible closest pure states for these partitions are $|\Phi_1\rangle=|\Phi_{\rm AB}\rangle|\Phi_{\rm CD}\rangle$, $|\Phi_2\rangle=|\Phi_{\rm AC}\rangle|\Phi_{\rm BD}\rangle$, and $|\Phi_3\rangle=|\Phi_{\rm AD}\rangle|\Phi_{\rm BC}\rangle$, respectively. They are the products of two-qubit vectors. We calculate and compare the inner products $\langle G|\Phi_\alpha\rangle$ ($\alpha=1,2,3$) to obtain the genuine three-qubit entanglement. Denote $E_{3{\rm g}\alpha}(|G\rangle)=\min_{|\Phi_\alpha\rangle}\{-\log_2|\langle G|\Phi_\alpha\rangle|^2\}$, then $E_{3\rm g}(|G\rangle)=\min_\alpha E_{3{\rm g}\alpha}(|G\rangle)$.

For a given partition, we can define the logarithmic robustness and the relative entropy of entanglement for a multipartite state ρ as far as the separable states appeared in the definitions are in the sense of the particular partition. In the case of the genuine three-qubit entanglement of a four-qubit graph state, we have the following kinds of separable states: (i) ω_α ($\alpha=1,2,3$ for partitions AB|CD, AC|BD, AD|BC, respectively); (ii) $\omega=\{\omega_1,\omega_2,\omega_3\}$, the separable set is a union of the three separable state sets of the particular partitions; (iii) $\omega'=\sum_\alpha p_\alpha\omega_\alpha$, the probability mixture of the three separable states of the particular partitions. Hence we have three kinds of logarithmic robustnesses and three kinds of relative entropies of entanglement. They are denoted as $R_{\rm L3\alpha}$, $R_{\rm L3}$, $R'_{\rm L3}$ and $E_{3{\rm r}\alpha}$, $E_{3\rm r}$, $E'_{3\rm r}$, respectively. Then $R_{\rm L3}=\min_\alpha R_{\rm L3\alpha}$, $E_{3\rm r}=\min_\alpha E_{3{\rm r}\alpha}$. For each particular partition, we have

$$n-\log_2 N_\alpha \geq R_{\rm L3\alpha}(|G\rangle)=E_{3{\rm r}\alpha}(|G\rangle)=E_{3{\rm g}\alpha}(|G\rangle), \tag{12}$$

where N_α is the maximal number of graph basis states that can be discriminated by LOCC of the particular partition. Denote $N=\max_\alpha N_\alpha$, we have

$$n-\log_2 N \geq R_{\rm L3}(|G\rangle)=E_{3\rm r}(|G\rangle)=E_{3\rm g}(|G\rangle). \tag{13}$$

From the definitions, we have $R'_{\rm L3}\leq R_{\rm L3}$ and $E'_{3\rm r}\leq E_{3\rm r}$, this is due to the fact that the range of the biseparable state ω' is larger than that of ω, the logarithmic robustness and the relative entropy of entanglement are defined as the minimum over the range. The larger the range, the smaller the entanglement value. On the other hand, we can prove $R'_{\rm L3}\geq R_{\rm L3}$. Suppose $\omega_\alpha=(|G\rangle\langle G|+t_\alpha\Omega_\alpha)/(1+t_\alpha)$ are biseparable states, then $\omega=\sum_\alpha p_\alpha\omega_\alpha$ is biseparable. Let $\omega=(|G\rangle\langle G|+t\Omega)/(1+t)$ for some state Ω and positive number t, then

$$\frac{1}{1+t}=\sum_{\alpha=1}^{3}\frac{p_\alpha}{1+t_\alpha},$$

from which we can easily obtain $1+t\geq 1+\min_\alpha t_\alpha$. Hence $R'_{\rm L3}\geq R_{\rm L3}$. The inequality $E'_{3\rm r}\geq E_{3\rm r}$ follows from the convex property of the relative entropy function. Thus

$$R'_{\rm L3}=R_{\rm L3},\ E'_{3\rm r}=E_{3\rm r}. \tag{14}$$

So it is sufficient to be limited to the biseparable states of case (ii). An extension to the genuine one-qubit entanglement measure is straightforward. We have

$$n-\log_2 N \geq R_{{\rm L}l}(|G\rangle)=E_{l\rm r}(|G\rangle)=E_{l\rm g}(|G\rangle), \tag{15}$$

where $N=\max_\alpha N_\alpha$. Here N_α is the maximal number of the graph basis states that can be discriminated by the LOCC operations and classical communication with respect to the α-th partition. Each party of the partition has $l-1$ qubits or less.

4. Bounds of genuine one-qubit entanglement

4.1. The upper bound

The upper bound of the genuine one-qubit entanglement is given in inequality (15). Given the α-th coarse grain partition, the problem is how to discriminate a maximal number of graph state bases. Within each party of the coarse grain, we can remove or add inner edges. Namely, we can freely change the matrix element Γ_{ij} of the adjacency matrix $\boldsymbol{\Gamma}$ from 0 to 1 or 1 to 0 if qubits i and j belong to the same party of the α-th partition. For simplicity, we remove all the inner edges within each party. Thus the adjacency matrix $\boldsymbol{\Gamma}$ will become a matrix $\boldsymbol{\Gamma}_\alpha$, which corresponds to a graph state $|G_\alpha\rangle$. The graph state $|G_\alpha\rangle$ is local unitary equivalent to the original graph state $|G\rangle$. Here the local operation means the operation acting within a party.

Let A_α be a set of vertices of graph G_α in which the vertices are not connected with each other. To find the maximal number of graph basis states corresponding to graph state $|G_\alpha\rangle$, we search for all the unconnected subsets of graph G_α so that $|A_\alpha|$ is maximized. For a graph state of a binary system, each vertex is assigned a qubit. We then locally measure X on all A_α qubits and measure Z on all others (say B_α qubits). The Z measurement results of the B_α qubits are transmitted to the A_α qubits by classical communication. Hence we can use the stabilizer expression $K_i = X_i \prod_{j\in\{v|\Gamma_{vi}=1\}} Z_j$ and LOCC to determine the eigenvalues $\{k_i|i\in A_\alpha\}$ corresponding to the subset of generators $\{K_i|i\in A_\alpha\}$, where we have used the fact that A_α is an unconnected set, so the neighbor qubit $j\in B_\alpha$. Let the graph basis states of $|G_\alpha\rangle$ be $\left|G_{\alpha k_1 k_2\cdots k_n}\right\rangle$, now the total number of the subscript k_i is n, there are $|A_\alpha|$ subscripts that can be determined by LOCC. Thus we can discriminate $2^{|A_\alpha|}$ graph state bases. Inequality (15) leads to

$$n-|A_\alpha| \geq E_{l\alpha}(|G\rangle),$$

where $E_{l\alpha}(|G\rangle) = R_{\mathrm{L}l\alpha}(|G\rangle) = E_{l\mathrm{r}\alpha}(|G\rangle) = E_{l\mathrm{g}\alpha}(|G\rangle)$. Maximizing A_α with respect to all the partitions, we then have

$$n-|A| \geq E_l(|G\rangle), \tag{16}$$

where $E_l(|G\rangle) = R_{\mathrm{L}l}(|G\rangle) = E_{l\mathrm{r}}(|G\rangle) = E_{l\mathrm{g}}(|G\rangle)$, and $|A| = \max_\alpha |A_\alpha|$. Hence the upper bound of the genuine entanglement is $E_U = n-|A|$.

4.2. The lower bound

The lower bound of the genuine one-qubit entanglement is obtained by the bipartite entanglement of the graph state. Here the bipartition is a coarser grain partition of the $l-1$ coarse partition, each party of the $l-1$ coarse partition contains at most $l-1$ qubits. The bipartition is obtained by merging the $l-1$ coarse partition such that only one of the borders of the partition remains and all the other borders are removed. Let $E_{\mathrm{L}\alpha} = E_{gbi\alpha}$ be the lower bound of the genuine one-qubit entanglement for a given α-th partition, where $E_{gbi\alpha}$ is the maximal of the bipartite entanglement of the graph state measured by the geometric measure and the cut in the bipartition is one of the cuts in the α-th partition, the maximization is over all the permissive bipartitions based on the α-th partition. Maximizing the bipartite entanglement is for finding the tightest lower bound. For a given α-th partition, it is clear that the bipartite entanglement is less than or equal to the genuine one-qubit entanglement since the range of the product state in the geometric measure is enlarged. The lower bound of the genuine one-qubit entanglement is $E_{\mathrm{L}} = \min_\alpha E_{\mathrm{L}\alpha}$. So we have

$$E_{\mathrm{L}} = \min_\alpha \{\max_{bi} E_{gbi\alpha}\}\}, \tag{17}$$

it is a kind of mini-max calculation. The lower bound of the genuine one-qubit entanglement is obtained by maximizing the permissive bipartite entanglement of a given coarse grain partition, then minimizing over all permissive coarse grain partitions.

Notice that the entanglement[22] and the genuine entanglement[23,24] can characterize the critical points of quantum phase transitions in some spin models. Our lower bound of the genuine entanglement can also be a proper means to study the nature of quantum phase transitions in ground and thermal states of spin models.

In Table 1, we give the lower bounds of the genuine three-qubit entanglement for graph states up to six qubits. By an inspection of the table, we can see that when the upper bound coincides with the lower bound, the genuine three-qubit entanglement value of the graph state can be ascertained. Otherwise, the genuine three-qubit entanglement value is not determined and we only know that the range is from the lower bound to the upper bound. At this circumstance, we should utilize an iterative algorithm to obtain the genuine entanglement value.

Table 1. Lower and upper bounds of the genuine three-qubit entanglement for graph states of six qubits or less. Here, No is the cardinal number of graph states, E_{U} is the genuine entanglement upper bound, E_{L} is the lower bound.

No	E_{L}	E_{U}	No	E_{L}	E_{U}	No	E_{L}	E_{U}
2	1	1	8	2	2	14	2	2
3	1	1	9	1	1	15	1	2
4	1	1	10	1	1	16	1	2
5	1	1	11	2	2	17	2	2
6	1	1	12	2	2	18	2	2
7	1	1	13	1	1	19	2	3

From Table 1, we find that the genuine three-qubit entanglement values of Nos. 2–7, 9, 10, 13 graph states are 1, the genuine three-qubit entanglement values of Nos. 8, 11, 12, 14, 17, 18 graph states are 2, and the values of Nos. 15, 16, 19 graph states are not determined yet. We will resort to the iterative algorithm.

5. Iterative algorithm

The iterative algorithm that is used to calculate the genuine three-qubit entanglement of graph states can be expressed as

$$E_{3g}(|G\rangle) = \min_{|\Phi\rangle\in \text{Bipro}} -\log_2 |\langle G|\Phi\rangle|^2 = -\log_2(\max_{|\Phi\rangle\in \text{Bipro}} |f_\Phi|^2), \quad (18)$$

where $|\Phi\rangle = |\phi_{12}\rangle\otimes|\phi_{34}\rangle\otimes\cdots\otimes|\phi_{(n-1)n}\rangle$ when n is even, $|\Phi\rangle = |\phi_{12}\rangle\otimes|\phi_{34}\rangle\otimes\cdots\otimes|\phi_n\rangle$ when n is odd, and the qubits permutate. Where $|\phi_{12}\rangle$ is a two-qubit state vector of the first two qubits, the biseparable state $|\Phi\rangle$ is the product of two-qubit or one-qubit vectors. Take $n=4$ for example, the biseparable pure state $|\Phi\rangle$ can be denoted as

$$\begin{aligned}|\Phi\rangle &= |\phi_{12}\rangle\otimes|\phi_{34}\rangle \\ &= (\alpha_{00}|00\rangle+\alpha_{01}|01\rangle+\alpha_{10}|10\rangle+\alpha_{11}|11\rangle) \\ &\quad\times(\beta_{00}|00\rangle+\beta_{01}|01\rangle+\beta_{10}|10\rangle+\beta_{11}|11\rangle), \quad (19)\end{aligned}$$

with the normalization conditions $\sum_{i,j=0}^{1}|\alpha_{ij}|^2=1$ and $\sum_{i,j=0}^{1}|\beta_{ij}|^2=1$. We can write f as

$$f_\phi = \frac{1}{\sqrt{2^n}}\sum_{\mu=0}^{1}(-1)^{\frac{1}{2}\boldsymbol{\mu}\boldsymbol{\Gamma}\boldsymbol{\mu}^{\mathrm{T}}}\alpha_{00}^{(1-\mu_1)(1-\mu_2)}\alpha_{01}^{(1-\mu_1)\mu_2}\alpha_{10}^{\mu_1(1-\mu_2)} \times\alpha_{11}^{\mu_1\mu_2}\beta_{00}^{(1-\mu_3)(1-\mu_4)}\beta_{01}^{(1-\mu_3)\mu_4}\beta_{10}^{\mu_3(1-\mu_4)}\beta_{11}^{\mu_3\mu_4}. \quad (20)$$

Utilize the Lagrange multiplier method with

$$L = |f|^2-\lambda_1\Big(\sum_{i,j=0}^{1}|\alpha_{ij}|^2-1\Big) -\lambda_2\Big(\sum_{i,j=0}^{1}|\beta_{ij}|^2-1\Big). \quad (21)$$

Let $\partial L/\partial\alpha_{ij}=0$ and $\partial L/\partial\beta_{ij}=0$ to obtain the iterative equations. For a given coarse grain partition, we can obtain the candidate genuine three-qubit entanglement of a graph state. Minimizing over all the permissive partitions, we then obtain the genuine three-qubit entanglement of the graph state.

A convenient way to perform the maximization over various permissive biseparable states is to maintain the positions of cuts and permutate the qubits. The permutation of qubits then can be converted to the transformation of the adjacency matrix. For the genuine three-qubit entanglement of graph states, the results of the iterative algorithm are shown in Table 2.

Table 2. Genuine three-qubit entanglement of 3–8 vertex graph states. Here No represents the number of graph states, and E_{3g} is the iterative value.

E_{3g}	No
1	2–7, 9, 10, 13, 15, 16, 20, 21, 25, 29, 46, 47, 52, 63, 69, 124
2	8, 11, 12, 14, 17–19, 22–24, 26–28, 30–39, 41–44, 48–51, 53–62, 64–65
2	67–68, 70–82, 86–94, 96, 101, 103, 107-110, 120–123, 126, 141, 143
3	40, 45, 66, 83–85, 95, 97–100, 102, 104–106, 111–119, 125, 127–140, 142, 144–146

From Table 2, we can confirm that the genuine three-qubit entanglement of graph states with six or less qubits obtained by the coincidence of the upper and the lower bounds in the last section. The iterative algorithm shows that E_{3g} of No. 15 and No. 16 is one, and E_{3g} of No. 19 is two. We find that the genuine three-qubit entanglement of all the graph states with six or less qubits is not greater than two. For seven-qubit graph states, $E_{3g}=3$ appears for No. 40 and No. 45 graph states.

We also calculate the genuine four-qubit entanglement of geometric measure E_{4g} of 5–7 vertex graph states using the iterative algorithm. The formula for E_{4g} is

$$E_{4g}(|G\rangle) = \min_{|\Phi\rangle\in \text{Tripro}} -\log_2 |\langle G|\Phi\rangle|^2 = -\log_2\big(\max_{|\Phi\rangle\in \text{Tripro}} |f_\Phi|^2\big), \quad (22)$$

where Tripro is the set of products of three-qubit, two-qubit, or one-qubit vectors. For example, $|\Phi\rangle = |\phi_{123}\rangle\otimes|\phi_{456}\rangle\otimes\cdots\otimes|\phi_{(n-2)(n-1)n}\rangle$, where $|\phi_{123}\rangle$ is a three-qubit state vector of the first three qubits. By analyzing and comparing all of the data, the genuine four-qubit entanglement of 5–7 vertices is determined and shown in Table 3.

Table 3. Genuine four-qubit entanglement of 5–7 vertices graph states.

E_{4g}	No
1	5–7, 9–16, 20–32, 34, 36
2	8, 17–19, 33, 35, 37–45

By analyzing Table 3, we can conclude that the genuine four-qubit entanglement E_{4g} of five-qubit graph states is one except No. 8, E_{4g} of the later is two, E_{4g} of six-qubit and seven-qubit graph states is less than or equal to two.

6. Conclusion

We have clarified the definitions of partial separability and the corresponding genuine one-qubit entanglement for the multipartite quantum system. For the genuine one-qubit entanglement of graph states, we use the LOCC method to obtain the upper bound. The lower bound of the genuine one-qubit entanglement is obtained by maximizing the permis-

sive bipartite entanglement of a given coarse grain partition and then minimizing over all permissive coarse grain partitions, the number of permissive coarse grain partitions expands quickly with the number of qubits in the graph states. When the upper bound matches the lower bound, the genuine one-qubit entanglement value of the graph state can be ascertained. Otherwise, we have utilized the iterative algorithm to obtain the genuine one-qubit entanglement value. We calculate all the genuine three-qubit entanglements for graph states up to eight qubits, and the genuine four-qubit entanglements of graph states up to seven qubits.

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