Exact solutions and linear stability analysis for two-dimensional Ablowitz–Ladik equation*

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The Ablowitz–Ladik equation is a very important model in nonlinear mathematical physics. In this paper, the hyperbolic function solitary wave solutions, the trigonometric function periodic wave solutions, and the rational wave solutions with more arbitrary parameters of two-dimensional Ablowitz–Ladik equation are derived by using the (G'/G)-expansion method, and the effects of the parameters (including the coupling constant and other parameters) on the linear stability of the exact solutions are analysed and numerically simulated.

Keywords: two–dimensional Ablowitz–Ladik equation, linear stability, exact solution, numerical simulation

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1. Introduction

Many phenomena including nonlinear optics (waveguide arrays), matter waves (Bose–Einstein condensates trapped in optical lattices), plasma physics (Langmuir wave), and molecular biology (modeling the DNA double strand) are often described by nonlinear partial differential equations (PDEs),[1–3] especially nonlinear Schrödinger-type equations. In the recent years, a number of powerful methods have been proposed, namely the inverse scattering method,[3,4] Hirota's bilinear method,[6,7] the Bäcklund transformation method,[3–5,8] the Darboux transformation method,[3,4] the Hyperbolic function method,[9] and the homogeneous balance method.[10–12]

The discrete nonlinear Schrödinger-type equation is a relevant model for a wide range of applications, and has received attention by many researchers.[13–19] Dai *et al.*,[13] Dai and Zhang[20] used the extended tanh-function approach to derive the exact solutions of discrete complex cubic Ginzburg–Landau equations, discrete complex cubic–quintic Ginzburg–Landau equations with a non-local quintic term respectively. Chow *et al.*[15] presented the exact solutions of the Ablowitz–Ladik model in terms of elliptic functions. Huang and Liu[21] obtained Jacobi elliptic function solutions of the Ablowitz–Ladik system via the Jacobi elliptic function expansion approach. Maruno *et al.*[22] studied a new quintic discrete nonlinear Schrödinger equation by singularity confinement criteria and growth properties. In Ref. [23], the exact localized and periodic solutions of the discrete complex Ginzburg–Landau equation were given, and in Ref. [24], Maruno *et al.* derived a set of exact solutions which include, as particular cases, bright and dark soliton solutions, constant magnitude solutions with phase shifts, periodic solutions in terms of elliptic Jacobi functions in general forms, and various particular periodic solutions to discrete complex cubic–quintic Ginzburg–Landau equation with a non-local quintic term. Aslan[25,26] used the G'/G-expansion method to derive the discrete solitons of the discrete nonlinear Schrödinger-type equations.

In the practical application, only stable soliton is valuable. Thus the investigation of the soliton stability is very important. Kevrekidis *et al.*[27] used the V-K criterion to analyze the stabilities of the stationary solutions of the higher-dimensional Ablowitz–Ladik model. Khare *et al.*[28,29] considered the saturable descrete nonlinear Schrödinger equation, two-dimensional cubic–quintic discrete nonlinear Schrödinger equations and analyzed the stabilities of the stationary solutions. Zhang and Liu,[30] and Zhang *et al.*[31] used the (G'/G)-expansion method[32–34] to derive the exact solutions of the discrete complex cubic Ginzburg–Landau equation, the quintic discrete nonlinear Schrödinger equation respectively, and studied their linear stabilities.

Due to important applications in nonlinear optics, Bose–Einstein condensates, etc., the two-dimensional Ablowitz–Ladik equation is studied in this paper, the exact solutions and their linear stabilities are investigated in Sections 2 and 3 respectively.

2. The exact solutions of the (2+1)-dimensional Ablowitz–Ladik equation

In this paper, we consider the (2+1)-dimensional Ablowitz–Ladik equation as

$$\mathrm{i}\frac{\partial u_{n,m}}{\partial t}+\varepsilon(u_{n+1,m}+u_{n-1,m}+u_{n,m+1}+u_{n,m-1}-4u_{n,m})$$

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$$-\frac{1}{4}|u_{n,m}|^2(u_{n+1,m}+u_{n-1,m}+u_{n,m+1}+u_{n,m-1})=0. \quad (1)$$

According to the (G'/G)-expansion method, the exact solutions of Eq. (1) are derived.

Case 1 When

$$\lambda^2-4\mu>0,$$

the trigonometric function periodic wave solutions are obtained as follows:

$$u_{n,m}(t) = \pm 2\sqrt{\frac{\varepsilon\left[\cos(d_1)\sinh^2(\Delta_1 h_2)+\cos(h_1)\sinh^2(\Delta_1 d_2)\right]}{\left[\cos(d_1)\coth^2(\Delta_1 d_2)\sinh^2(\Delta_1 h_2)+\cos(h_1)\coth^2(\Delta_1 h_2)\sinh^2(\Delta_1 d_2)\right]}}\left(\frac{C_1+C_2\tanh(\Delta_1\xi_{n,m})}{C_1\tanh(\Delta_1\xi_{n,m})+C_2}\right)$$
$$\times\exp\left(\mathrm{i}\left(d_1 n+h_1 m+2\varepsilon\left(\frac{(\cos(d_1)\operatorname{csch}(\Delta_1 d_2)\sinh(\Delta_1 h_2)+\cos(h_1)\operatorname{csch}(\Delta_1 h_2)\sinh(\Delta_1 d_2))^2}{\cos(d_1)\coth^2(\Delta_1 d_2)\sinh^2(\Delta_1 h_2)+\cos(h_1)\coth^2(\Delta_1 h_2)\sinh^2(\Delta_1 d_2)}-2\right)t+\zeta_1\right)\right),$$

where

$$\xi_{n,m} = d_2 n+h_2 m+\zeta_2-\frac{4\varepsilon\left[\cos(d_1)\sinh^2(\Delta_1 h_2)+\cos(h_1)\sinh^2(\Delta_1 d_2)\right]}{\sqrt{\lambda^2-4\mu}\tanh(\Delta_1 d_2)\tanh(\Delta_1 h_2)}$$
$$\times\frac{\sin(d_1)\tanh(\Delta_1 h_2)+\sin(h_1)\tanh(\Delta_1 d_2)}{\cos(d_1)\coth^2(\Delta_1 d_2)\sinh^2(\Delta_1 h_2)+\cos(h_1)\coth^2(\Delta_1 h_2)\sinh^2(\Delta_1 d_2)}t,$$

$\Delta_1=\sqrt{\lambda^2-4\mu}/2$, C_1, C_2, ζ_1, ζ_2, d_2, and h_2 are constants, and d_1 and h_1 satisfy $\cos(d_1)\cos(h_1)>0$.

The above exact solutions are simplified into

$$u^{(1)}_{n,m}(t) = 2\sqrt{\frac{\varepsilon\left[\sinh^2(h)+\sinh^2(d)\right]}{\coth^2(d)\sinh^2(h)+\coth^2(h)\sinh^2(d)}}\frac{C_1+C_2\tanh(dn+hm)}{C_1\tanh(dn+hm)+C_2}$$
$$\times\exp\left(2\mathrm{i}\left(\frac{(\operatorname{csch}(d)\sinh(h)+\operatorname{csch}(h)\sinh(d))^2}{\coth^2(d)\sinh^2(h)+\coth^2(h)\sinh^2(d)}-2\right)\varepsilon t\right), \quad (2)$$

where C_1, C_2, d, and h are constants, and $n,m\in\mathbb{N}$.

From Eq. (2), we can derive two special-type exact solutions of Eq. (1).

Case 1.1 Dark soliton:

$$u^{(11)}_{n,m}(t) = 2\sqrt{\frac{\varepsilon\left[\sinh^2(h)+\sinh^2(d)\right]}{\coth^2(d)\sinh^2(h)+\coth^2(h)\sinh^2(d)}}\tanh(dn+hm)$$
$$\times\exp\left(2\mathrm{i}\left(\frac{(\operatorname{csch}(d)\sinh(h)+\operatorname{csch}(h)\sinh(d))^2}{\coth^2(d)\sinh^2(h)+\coth^2(h)\sinh^2(d)}-2\right)\varepsilon t\right), \quad (3)$$

where d and h are constants (as shown in Fig. 1).

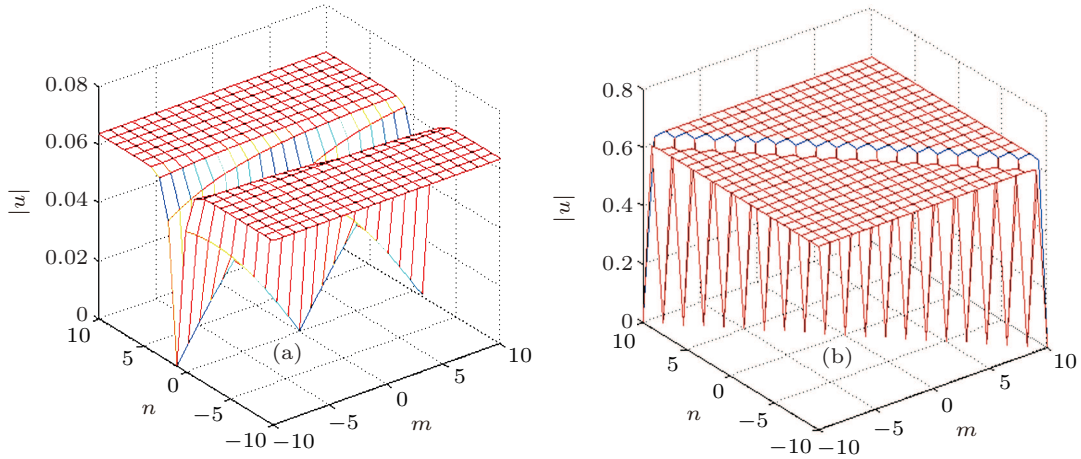


Fig. 1. (color online) Dark solitons from Eq. (3): (a) when $d=1$, $h=0.1$, $\varepsilon=0.1$, $t=0$ and (b) when $d=1$, $h=1$, $\varepsilon=0.1$, $t=0$.

Case 1.2 Bright soliton (Fig. 2):

$$u_{n,m}^{(12)}(t) = 2\sqrt{\frac{\varepsilon\left[\sinh^2(h)+\sinh^2(d)\right]}{\coth^2(d)\sinh^2(h)+\coth^2(h)\sinh^2(d)}}\frac{2+\tanh(dn+hm)}{2\tanh(dn+hm)+1} \times \exp\left(2\mathrm{i}\left(\frac{(\mathrm{csch}(d)\sinh(h)+\mathrm{csch}(h)\sinh(d))^2}{\coth^2(d)\sinh^2(h)+\coth^2(h)\sinh^2(d)}-2\right)\varepsilon t\right). \tag{4}$$

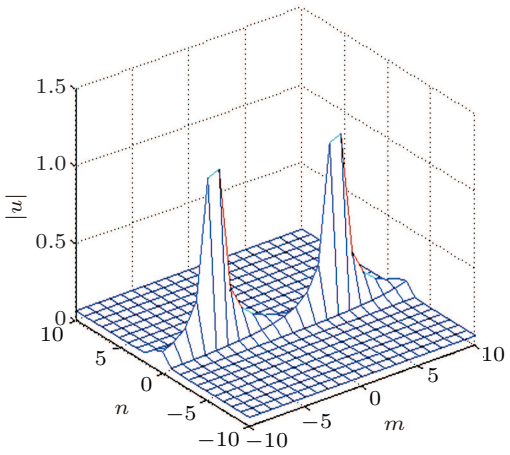


Fig. 2. (color online) Bright solitons from Eq. (4) when $d = 1$, $h = 0.1$, $\varepsilon = 0.1$, $t = 0$.

Case 2 When $\lambda^2 - 4\mu < 0$, the trigonometric function periodic wave solutions are obtained as follows:

$$u_{n,m}(t) = \pm 2\sqrt{\frac{\varepsilon\left[\cos(d_1)\sin^2(\Delta_2 h_2)+\cos(h_1)\sin^2(\Delta_2 d_2)\right]}{\left[\cos(d_1)\cot^2(\Delta_2 d_2)\sin^2(\Delta_2 h_2)+\cos(h_1)\cot^2(\Delta_2 h_2)\sin^2(\Delta_2 d_2)\right]}}\left(\frac{-C_1\tan(\Delta_2\xi_{n,m})+C_2}{C_1+C_2\tan(\Delta_2\xi_{n,m})}\right) \times \exp\left(\mathrm{i}\left(d_1 n+h_1 m+2\varepsilon\left(\frac{(\cos(d_1)\csc(\Delta_2 d_2)\sin(\Delta_2 h_2)+\cos(h_1)\csc(\Delta_2 h_2)\sin(\Delta_2 d_2))^2}{\cos(d_1)\cot^2(\Delta_2 d_2)\sin^2(\Delta_2 h_2)+\cos(h_1)\cot^2(\Delta_2 h_2)\sin^2(\Delta_2 d_2)}-2\right)t+\zeta_1\right)\right),$$

where

$$\xi_{n,m} = d_2 n+h_2 m+\zeta_2-\frac{4\varepsilon\left[\cos(d_1)\sin^2(\Delta_2 h_2)+\cos(h_1)\sin^2(\Delta_2 d_2)\right]}{\sqrt{4\mu-\lambda^2}\tan(\Delta_2 d_2)\tan(\Delta_2 h_2)} \times \frac{\sin(d_1)\tan(\Delta_2 h_2)+\sin(h_1)\tan(\Delta_2 d_2)}{\cos(d_1)\cot^2(\Delta_2 d_2)\sin^2(\Delta_2 h_2)+\cos(h_1)\cot^2(\Delta_2 h_2)\sin^2(\Delta_2 d_2)}t,$$

$\Delta_2 = \sqrt{4\mu-\lambda^2}/2$, C_1, C_2, ζ_1, ζ_2, d_2, and h_2 are constants, and d_1 and h_1 satisfy $\cos(d_1)\cos(h_1) > 0$.

The exact solutions above are simplified into

$$u_{n,m}^{(2)}(t) = -2\sqrt{\frac{\varepsilon\left[\sin^2(h)+\sin^2(d)\right]}{\left[\cot^2(d)\sin^2(h)+\cot^2(h)\sin^2(d)\right]}}\left(\frac{-C_1\tan(dn+hm)+C_2}{C_1+C_2\tan(dn+hm)}\right) \times \exp\left(\mathrm{i}\left(2\varepsilon\left(\frac{(\csc(d)\sin(h)+\csc(h)\sin(d))^2}{\cot^2(d)\sin^2(h)+\cot^2(h)\sin^2(d)}-2\right)t\right)\right), \tag{5}$$

where C_1, C_2, d, and h are constants, and $n, m \in \mathbb{N}$.

From Eq. (3), we can derive two special-type exact solutions of Eq. (1).

Case 2.1 Oscillatory 'tan' solution (Fig. 3):

$$u_{n,m}^{(21)}(t) = 2\sqrt{\frac{\varepsilon\left[\sin^2(h)+\sin^2(d)\right]}{\left[\cot^2(d)\sin^2(h)+\cot^2(h)\sin^2(d)\right]}}\tan(dn+hm) \times \exp\left(2\mathrm{i}\left(\frac{(\csc(d)\sin(h)+\csc(h)\sin(d))^2}{\cot^2(d)\sin^2(h)+\cot^2(h)\sin^2(d)}-2\right)\varepsilon t\right), \tag{6}$$

where d and h are constants.

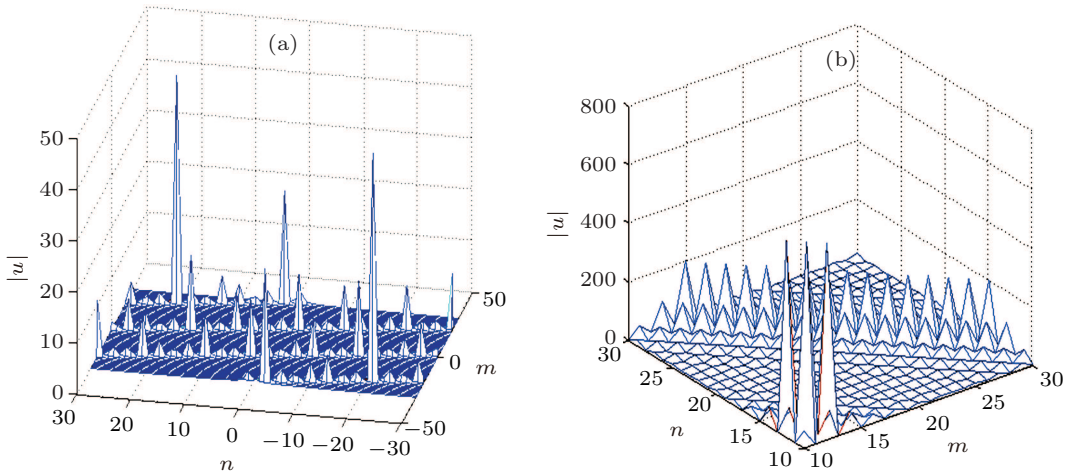


Fig. 3. (color online) Oscillatory 'tan' solutions from Eq. (6): (a) when $d = 3.2$, $h = 3.3$, $\varepsilon = 0.1$, $t = 0$; and (b) when $d = 1.5$, $h = 1.5$, $\varepsilon = 0.1$, $t = 0$.

Case 2.2 The trigonometric function periodic wave solutions (Fig. 4):

$$u_{n,m}^{(22)}(t) = -2\sqrt{\frac{\varepsilon\left[\sin^2(h)+\sin^2(d)\right]}{\left[\cot^2(d)\sin^2(h)+\cot^2(h)\sin^2(d)\right]}}\frac{-\tan(dn+hm)+2}{1+2\tan(dn+hm)} \times \exp\left(2\mathrm{i}\left(\frac{(\csc(d)\sin(h)+\csc(h)\sin(d))^2}{\cot^2(d)\sin^2(h)+\cot^2(h)\sin^2(d)}-2\right)\varepsilon t\right). \tag{7}$$

Case 3 when $\lambda^2 - 4\mu = 0$,

$$u_{n,m}(t) = \pm 2d_2h_2\sqrt{\frac{\varepsilon\left(d_2^2\cos(h_1)+h_2^2\cos(d_1)\right)}{\left(d_2^4\cos(h_1)+h_2^4\cos(d_1)\right)}}\left(\frac{C_1}{C_1\xi_{n,m}+C_2}\right) \times \exp\left(\mathrm{i}\left(d_1n+h_1m+2\varepsilon\left(\frac{\left(d_2^2\cos(h_1)+h_2^2\cos(d_1)\right)^2}{d_2^4\cos(h_1)+h_2^4\cos(d_1)}-2\right)t+\zeta_1\right)\right),$$

where

$$\xi_{n,m} = d_2n+h_2m-\frac{2\varepsilon d_2h_2\left(d_2^2\cos(h_1)+h_2^2\cos(d_1)\right)\left(d_2\sin(h_1)+h_2\sin(d_1)\right)}{d_2^4\cos(h_1)+h_2^4\cos(d_1)}t+\zeta_2,$$

C_1, C_2, ζ_1, ζ_2, d_2, and h_2 are constants, and d_1 and h_1 satisfy $\cos(d_1)\cos(h_1) > 0$.

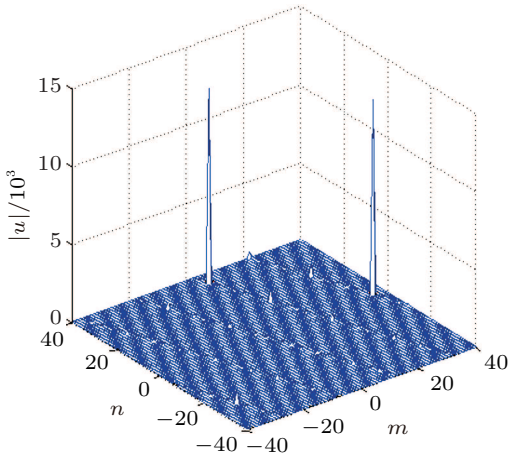


Fig. 4. (color online) Trigonometric function periodic wave solutions (7) when $d = 3.2$, $h = 3.3$, $\varepsilon = 0.1$, $t = 0$.

The exact solution above is simplified into (Fig. 5)

$$u_{n,m}^{(3)}(t) = 2dh\sqrt{\frac{\varepsilon(d^2+h^2)}{(d^4+h^4)}}\left(\frac{1}{dn+hm+0.5}\right) \times \exp\left(\mathrm{i}\left(2\varepsilon\left(\frac{\left(d^2+h^2\right)^2}{d^4+h^4}-2\right)t\right)\right). \tag{8}$$

Note that the other types of exact solutions of Eq. (1) can be obtained if the parameters are set to be appropriate values.

3. Linear stability analyses of exact solutions for the (2+1)-dimensional Ablowitz–Ladik equation

In order to analyze the linear stabilities of exact solutions $u_{n,m}^{(j)}$ $(j = 1,2,3)$, we assume that

$$u_{n,m} = u_{n,m}^{(j)}\mathrm{e}^{-\mathrm{i}\omega t} + \delta u_{n,m}(t)\mathrm{e}^{-\mathrm{i}\omega t}, \tag{9}$$

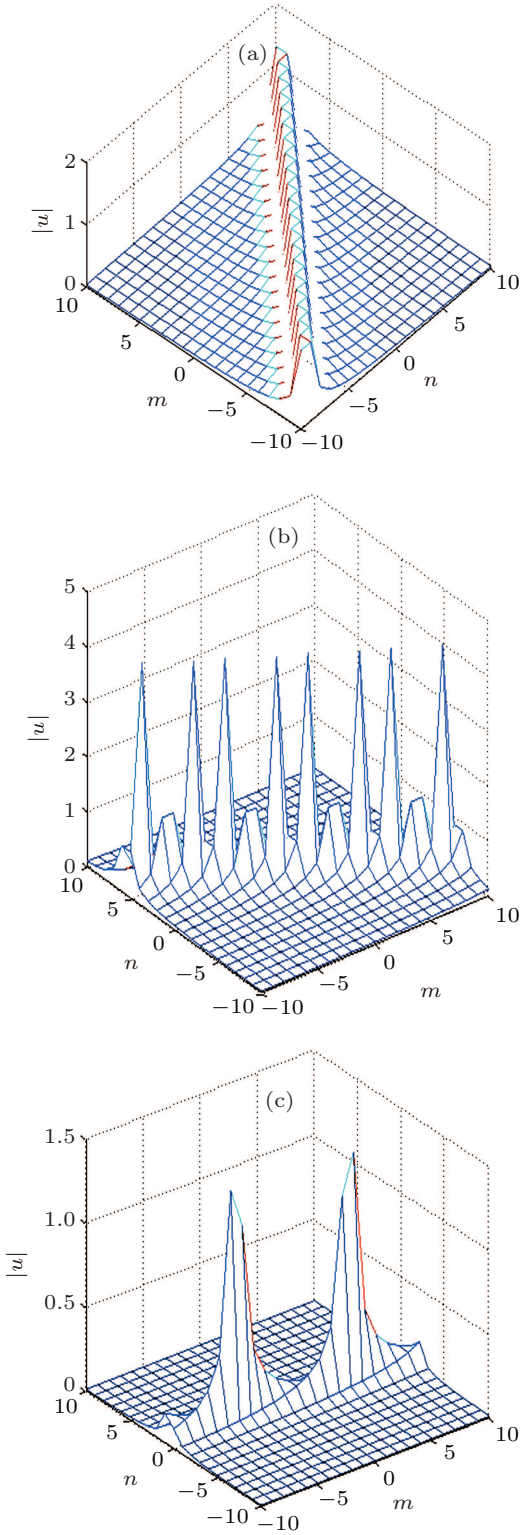


Fig. 5. (color online) Rational wave solution (8): (a) when $d=1$, $h=1$, $\varepsilon=0.1$, $t=0$; (b) when $d=1$, $h=0.6$, $\varepsilon=0.1$, $t=0$; and (c) when $d=1$, $h=0.11$, $\varepsilon=0.1$, $t=0$.

where $\delta u_{n,m}(t)$ is a small perturbation. Substituting Eq. (9) into Eq. (1) and retaining only linear terms in the perturbation, we obtain

$$\begin{aligned}&\mathrm{i}\delta\dot{u}_{n,m}+\left(\varepsilon-\frac{1}{4}\left|u_{n,m}^{(j)}\right|^2\right)\\&\times\left(\delta u_{n+1,m}+\delta u_{n-1,m}+\delta u_{n,m+1}+\delta u_{n,m-1}\right)\\&+(\omega-4\varepsilon)\,\delta u_{n,m}-\frac{1}{4}u_{n,m}^{(j)}\left(\delta u_{n,m}^{*}+\delta u_{n,m}\right)\\&\times\left(u_{n+1,m}^{(j)}+u_{n-1,m}^{(j)}+u_{n,m+1}^{(j)}+u_{n,m-1}^{(j)}\right)=0.\end{aligned}\tag{10}$$

Setting $\delta u_{n,m}=\delta u_{n,m}^{r}+\mathrm{i}\delta u_{n,m}^{i}$, equation (10) can be changed into

$$\begin{aligned}&\delta\dot{u}^{r}+(\omega-4\varepsilon)\delta u_{n,m}^{i}+\left(\varepsilon-\frac{1}{4}\left|u_{n,m}^{(j)}\right|^2\right)\\&\times\left(\delta u_{n+1,m}^{i}+\delta u_{n-1,m}^{i}+\delta u_{n,m+1}^{i}+\delta u_{n,m-1}^{i}\right)=0,\end{aligned}\tag{11}$$

$$\begin{aligned}&-\delta\dot{u}_{n,m}^{i}+\left[\omega-4\varepsilon-\frac{1}{2}u_{n,m}^{(j)}\left(u_{n+1,m}^{(j)}+u_{n-1,m}^{(j)}\right.\right.\\&\left.\left.+u_{n,m+1}^{(j)}+u_{n,m-1}^{(j)}\right)\right]\delta u_{n,m}^{r}+\left(\varepsilon-\frac{1}{4}\left|u_{n,m}^{(j)}\right|^2\right)\\&\times\left(\delta u_{n+1,m}^{r}+\delta u_{n-1,m}^{r}+\delta u_{n,m+1}^{r}+\delta u_{n,m-1}^{r}\right)=0.\end{aligned}\tag{12}$$

Assume $J=n+(m-1)N$, and introduce the two real vectors: $\delta U^{r}=\{\delta u_{J}^{r}\}$, $\delta U^{i}=\{\delta u_{J}^{i}\}$, where N and M are periodical parameters of solitons, and the two real matrices

$$\boldsymbol{A}=\left\{A_{J,J'}\right\},\quad \boldsymbol{B}=\left\{B_{J,J'}\right\},$$

in which

$$\begin{aligned}A_{J,J'}&=(\omega-4\varepsilon)\,\delta_{J,J'}+\left(\varepsilon-\frac{1}{4}\left|u_{n,m}^{(j)}\right|^2\right)\\&\quad\times\left(\delta_{J,J'-1}+\delta_{J,J'+1}+\delta_{J,J'-N}+\delta_{J,J'+N}\right)\\&=(\omega-4\varepsilon)\,\delta_{J,J'}+\left(\varepsilon-\frac{1}{4}\left|u_{J}^{(j)}\right|^2\right)\\&\quad\times\left(\delta_{J,J'-1}+\delta_{J,J'+1}+\delta_{J,J'-N}+\delta_{J,J'+N}\right),\end{aligned}\tag{13}$$

$$\begin{aligned}B_{J,J'}&=\left[\omega-4\varepsilon-\frac{1}{2}u_{n,m}^{(j)}\left(u_{n+1,m}^{(j)}+u_{n-1,m}^{(j)}\right.\right.\\&\quad\left.\left.+u_{n,m+1}^{(j)}+u_{n,m-1}^{(j)}\right)\right]\delta_{J,J'}+\left(\varepsilon-\frac{1}{4}\left|u_{n,m}^{(j)}\right|^2\right)\\&\quad\times\left(\delta_{J,J'-1}+\delta_{J,J'+1}+\delta_{J,J'-N}+\delta_{J,J'+N}\right)\\&=\left[\omega-4\varepsilon-\frac{1}{2}u_{J}^{(j)}\left(u_{J+1}^{(j)}+u_{J-1}^{(j)}\right.\right.\\&\quad\left.\left.+u_{J+N}^{(j)}+u_{J-N}^{(j)}\right)\right]\delta_{J,J'}+\left(\varepsilon-\frac{1}{4}\left|u_{J}^{(j)}\right|^2\right)\\&\quad\times\left(\delta_{J,J'-1}+\delta_{J,J'+1}+\delta_{J,J'-N}+\delta_{J,J'+N}\right),\end{aligned}\tag{14}$$

where $J'\pm1$ in the Kronecker δ means: $J'\pm1$ mod N, and $J'\pm N$ in the Kronecker δ means: $J'\pm N$ mod NM, then equations (11) and (12) can be written compactly as

$$\delta\dot{U}^{r}+\boldsymbol{A}\delta U^{i}=0,\quad -\delta\dot{U}^{i}+\boldsymbol{B}\delta U^{r}=0.\tag{15}$$

From Eq. (15), we have

$$\delta\ddot{U}^{r}+\boldsymbol{AB}\delta U^{r}=0,\quad \delta\ddot{U}^{i}+\boldsymbol{BA}\delta U^{i}=0.\tag{16}$$

The eigenvalue spectra of the matrices $\boldsymbol{AB}$ and $\boldsymbol{BA}$ determine the stabilities of the exact solutions. If all the eigenvalues are positive, the solutions are stable, and if eigenvalues contain a negative (or complex) eigenvalue, the solution is unstable.

3.1. Linear stability analyses of hyperbolic function solitary wave solutions

3.1.1. Effects of parameters d and h on the linear stability of $u_{n,m}^{(11)}$

From Fig. 6, it is easy to see that the stability region of $u_{n,m}^{(11)}$ is along the coordinate axes of d and h, and is symmetrical with respect to line $d = h$ when $M = N$. When $N = 3$, $M = 3$, $u_{n,m}^{(11)}$ is linearly stable in the region of $d > 1$, $h < 0.5$ and $h > 1$, $d < 0.5$. With the increments of N and M, the stability region becomes small.

With the increments of N and M, the stability region $(d < 1, h < 1)$ is discretized. This is similar to the cases of the cubic–quintic discrete nonlinear Schrödinger equation.[22]

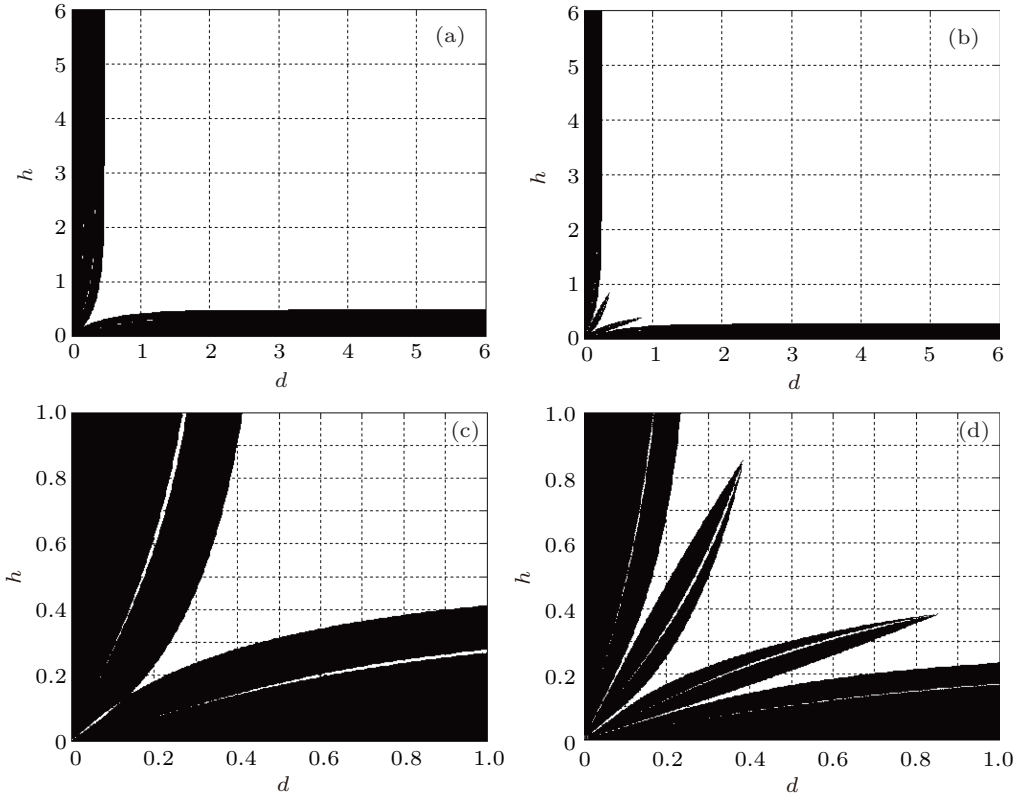


Fig. 6. Effects of parameters d and h on the linear stability of $u_{n,m}^{(11)}$. Panel (a) $N = 3$, $M = 3$, $\varepsilon = 0.1$; Panel (b) $N = 5$, $M = 5$, $\varepsilon = 0.1$ (The black regions are the stable regions); Panel (c) partial enlargement of panel (a); Panel (d) partial enlargement of panel (b) (The black regions are the stable regions).

Similar to the analysis of $u_{n,m}^{(11)}$, the effects of the parameters d and h on the linear stability region of $u_{n,m}^{(12)}$ are shown in Fig. 7 when $\varepsilon = 0.1$, $N = 3$, $M = 3$ (the black regions are the stable regions).

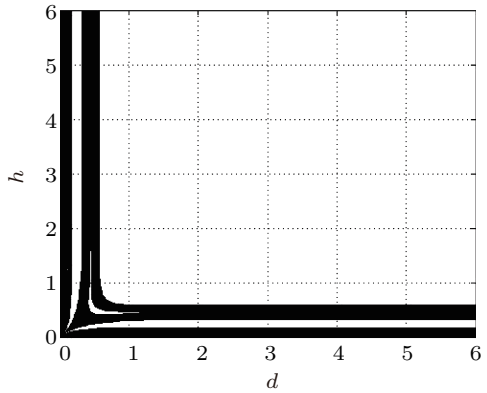


Fig. 7. Effects of parameters d and h on the linear stability of $u_{n,m}^{(12)}$ when $\varepsilon = 0.1$, $N = 3$, $M = 3$.

3.1.2. Effects of parameters C_1 and C_2 on the linear stability of $u_{n,m}^{(1)}$

For simplicity, we suppose that parameters C_1 and C_2 are nonnegative and set to be $(C_1, C_2) = (\sin\theta, \cos\theta)$ and $\theta \in [0, \pi/2]$. The distributions of the minimal eigenvalues of $\boldsymbol{AB}$ of the exact solutions varying with parameters C_1 and C_2 are shown in Fig. 8.

3.1.3. Effect of coupling constant ε on the linear stability of $u_{n,m}^{(11)}$

The coupling constant ε is very important for the Ablowitz–Ladik system, and when $\varepsilon \to 0$, the Ablowitz–Ladik system is changed into the discrete limiting form and the coupling constant ε influences the stabilities of the solutions.[20]

The distribution of the minimal eigenvalues of $\boldsymbol{AB}$ of the exact solutions varying with coupling constant ε is shown in Fig. 9. From Fig. 9, it is easy to see that the coupling constant ε has no influence on the stability of the solution $u_{n,m}^{(11)}$ essen-

tially except for the strength of the stability of the solution $u_{n,m}^{(11)}$.

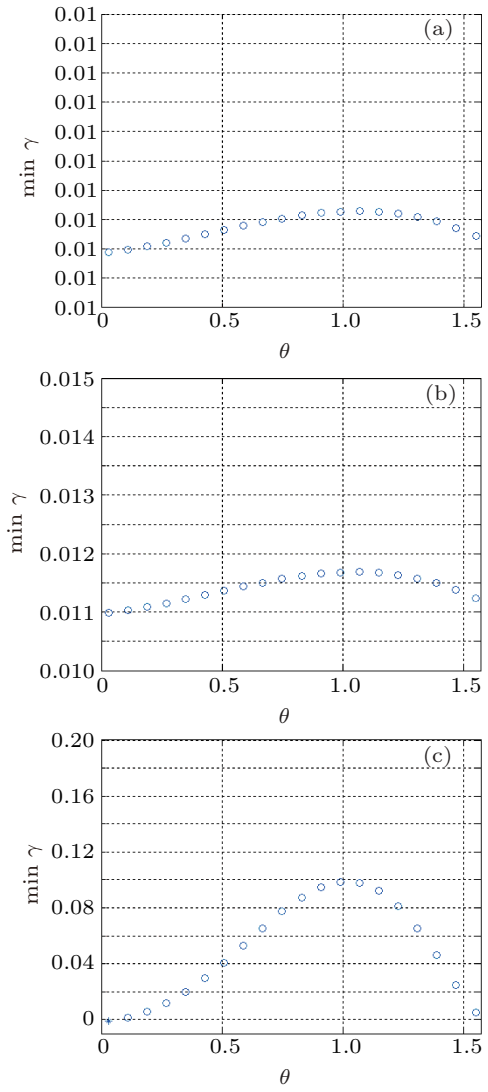


Fig. 8. (color online) Distributions of the minimal eigenvalues of AB when $N = 3$, $M = 3$, $\varepsilon = 0.1$. Panel (a) $d = 1$, $h = 0.01$; Panel (b) $d = 1$, $h = 0.1$; Panel (c) $d = 1$, $h = 1$ (o – positive eigenvalue; $*$ – negative eigenvalue; ∇ – complex eigenvalue).

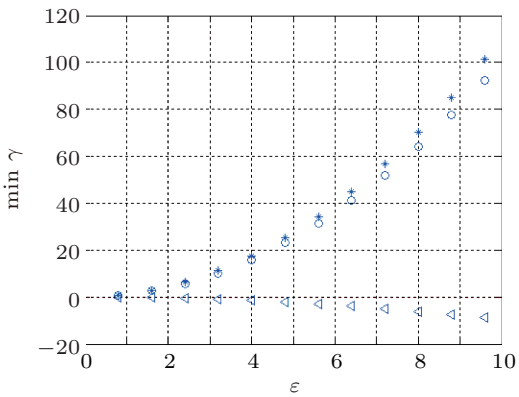


Fig. 9. (color online) Distributions of the minimal eigenvalues of AB when $N = 3$, $M = 3$. Case (a) $d = 1$, $h = 0.01$ (o); Case (b) $d = 1$, $h = 0.1$ ($*$); Case (c) $d = 1$, $h = 1$ ($\triangleleft$).

3.2. Linear stability analyses of the trigonometric function periodic wave solutions

3.2.1. Effects of parameters d and h on the linear stability of $u_{n,m}^{(21)}$

From Fig. 10, it is easy to see that the stable regions of $u_{n,m}^{(21)}$ are massive and discretized, and with the increments of N and M, the stable region becomes small, and the instable points appear in the massive stable region. The large massive stable regions appear in the neighbor of the points $(d,h) = (3n,3n)$. By using the numerical simulation, we have that $u_{n,m}^{(21)}$ is instable at the point $(d,h) = (3.2,3.3)$ if $N > 10$, $M > 10$.

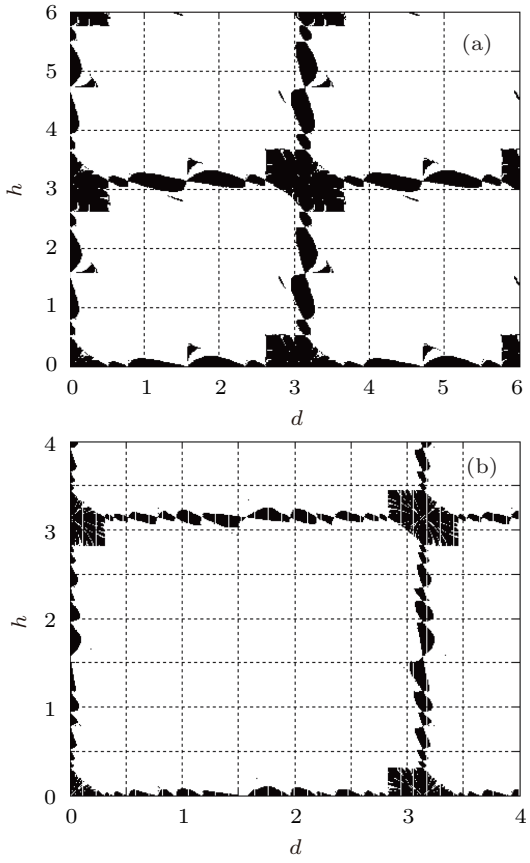


Fig. 10. Effects of parameters d and h on the linear stability of $u_{n,m}^{(21)}$. Panel (a) $N = 3$, $M = 3$; Panel (b) $N = 5$, $M = 5$ (The black regions are the stable region).

The stable region almost exclusively appears in the neighbor of the points $(d,h) = (3,3)$ if $N \neq M$ (Fig. 11) and the stable region remarkably decreases. This is similar to the case of the two-dimensional cubic–quintic discrete nonlinear Schrödinger equation.[22]

The effects of parameters d and h on the stable region of $u_{n,m}^{(22)}$ when $N = 3$, $M = 3$ are shown in Fig. 12.

3.2.2. Effect of the parameters C_1 and C_2 on the linear stability of $u_{n,m}^{(2)}$

In a way similar to Subsection 3.1.2, the distribution of the minimal eigenvalues of AB varying with parameters C_1 and C_2 are shown in Fig. 13.

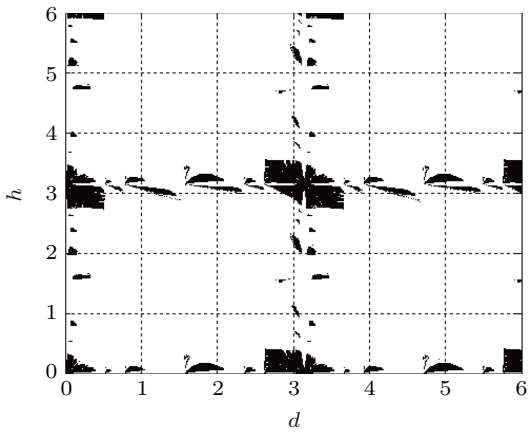


Fig. 11. Effects of parameters d and h on the linear stability of $u_{n,m}^{(21)}$ when $N = 3$, $M = 4$.

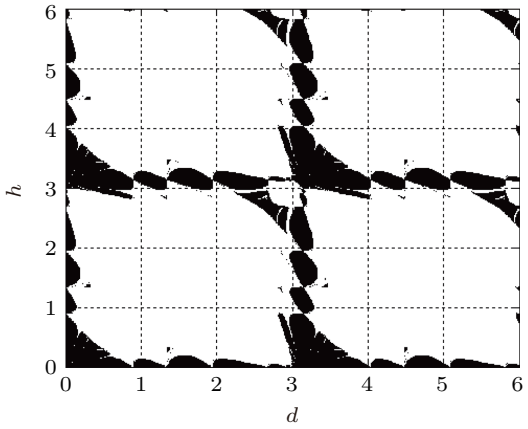


Fig. 12. Effects of parameters d and h on the linear stability of $u_{n,m}^{(22)}$ when $N = 3$, $M = 3$.

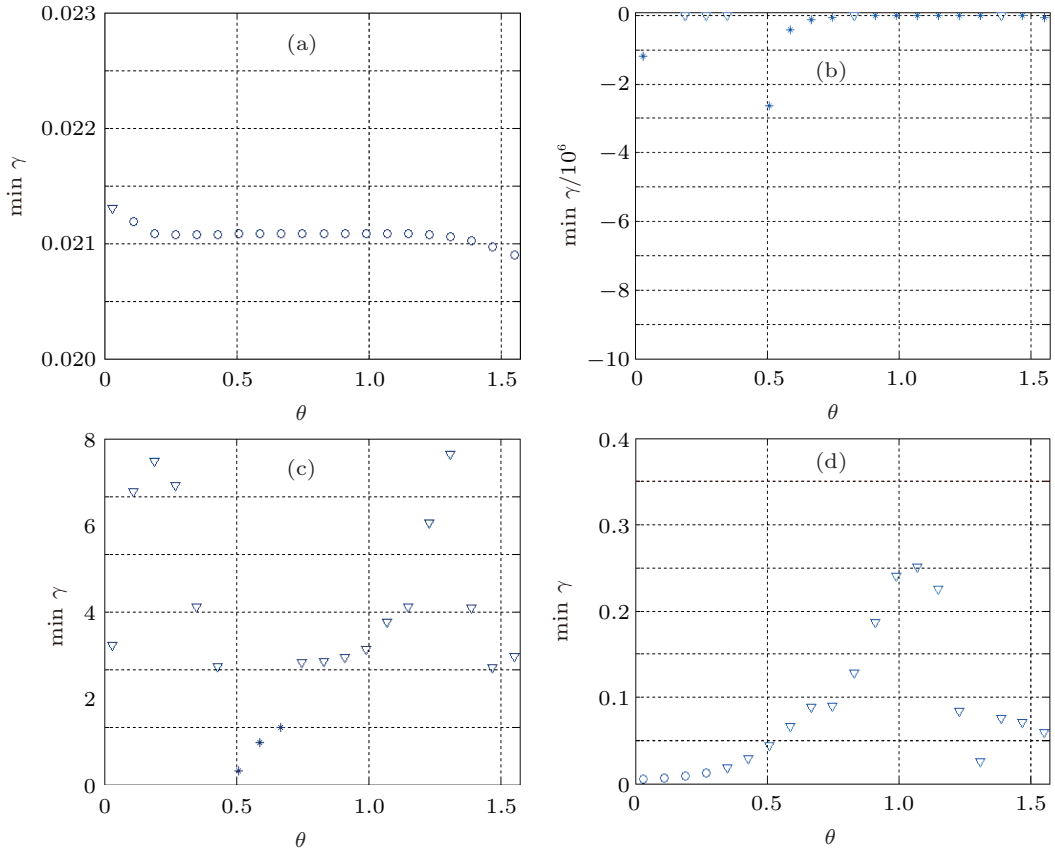


Fig. 13. (color online) Distributions of the minimal eigenvalues of AB when $N = 3$, $M = 3$, $\varepsilon = 0.1$. Panel (a) $d = 3.2$, $h = 3.3$; Panel (b) $d = 1.5$, $h = 1.5$; Panel (c) $d = 1$, $h = 1$; Panel (d) $d = 2.8$, $h = 2.8$. (o – positive eigenvalue; $*$ – negative eigenvalue; ∇ – complex eigenvalue).

3.2.3. Effect of the coupling constant ε on the linear stability of $u_{n,m}^{(21)}$

The distributions of the minimal eigenvalues of AB of the exact solutions varying with the coupling constant ε are shown in Fig. 14. They are similar to the case in Subsection 3.1.3.

3.3. Linear stability analysis of the rational wave solution $u_{n,m}^{(3)}$

We use the method above to analyze the linear stability of $u_{n,m}^{(3)}$. From Fig. 15, the stable region of $u_{n,m}^{(3)}$ is larger than those of the other solutions, and the stable regions are discretized with the increments of N and M; when $M \to \infty$, $N \to \infty$, $u_{n,m}^{(3)}$ is linearly stable in the neighbor of the line $d = h$.

The distributions of the minimal eigenvalues of AB of the exact solutions varying with the coupling constant ε are shown in Fig. 16. They are similar to the case in Subsection 3.1.3.

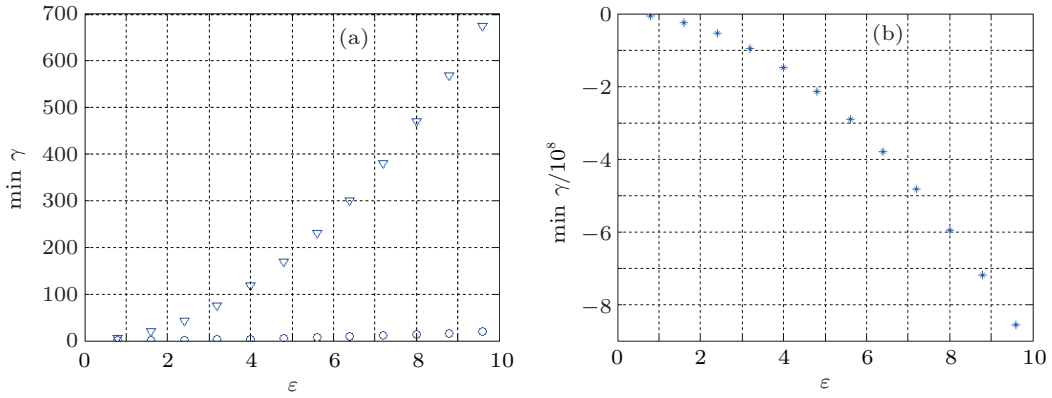


Fig. 14. (color online) Distributions of the minimal eigenvalues of AB when $N = 3$, $M = 3$. Case a: $d = 3.2$, $h = 3.3$ (o); Case b: $d = 1$, $h = 0.1$ (∇); Case c: $d = 1.5$, $h = 1.5$ (*).

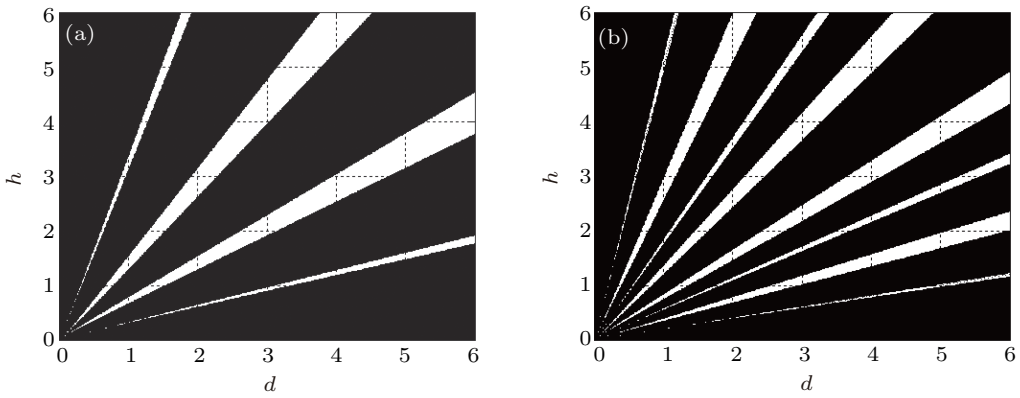


Fig. 15. Effects of parameters d and h on the linear stability of $u^{(3)}_{n,m}$. Panel (a) $N = 3$, $M = 3$; Panel (b) $N = 5$, $M = 5$.

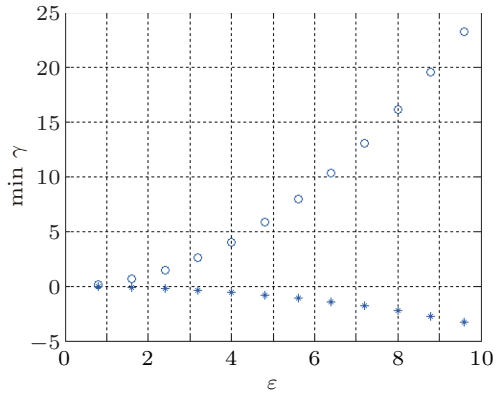


Fig. 16. (color online) Distributions of the minimal eigenvalues of AB when $N = 3$, $M = 3$. Case a: $d = 3$, $h = 3$ (o); Case b: $d = 2$, $h = 3$ (*).

4. Conclusions and discussion

In this paper, the hyperbolic function solitary wave solutions, the trigonometric function periodic wave solutions, and the rational wave solutions with more arbitrary parameters of the two-dimensional Ablowitz–Ladik equation are derived using the (G'/G)-expansion method, and the effects of the parameters on the linear stabilities of the exact solutions are discussed. From the analysis and numerical simulation, we obtain the following points.

(i) The coupling constant ε has no influence on the stabilities of the solutions includinig the bright solitons, dark solitons, and rational wave solutions except for the strength of the stability.

(ii) With the increments of N and M, the stability region of the solutions including the bright solitons, dark solitons, and rational wave solutions is discretized if the parameters d and h are considered.

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