Function projective lag synchronization of fractional-order chaotic systems*

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Function projective lag synchronization of different structural fractional-order chaotic systems is investigated. It is shown that the slave system can be synchronized with the past states of the driver up to a scaling function matrix. According to the stability theorem of linear fractional-order systems, a nonlinear fractional-order controller is designed for the synchronization of systems with the same and different dimensions. Especially, for two different dimensional systems, the synchronization is achieved in both reduced and increased dimensions. Three kinds of numerical examples are presented to illustrate the effectiveness of the scheme.

Keywords: fractional order, chaos, function projective lag synchronization

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1. Introduction

Since the inspiring works of Pecora and Carroll,[1,2] chaos synchronization has become one of the hottest topics in nonlinear science because of its great potential applications in secure communication, biological science, chemical reactor, laser systems, and many other fields.[3] A variety of synchronization phenomena are considered, such as complete synchronization,[4] projective synchronization,[5] generalized synchronization,[6] impulsive synchronization,[7] lag synchronization,[8] and anticipated synchronization.[9] Lots of schemes are provided for chaotic synchronization based on the adaptive control,[10] feedback control,[11] backstepping design,[12] sliding mode control,[13] etc.

Fractional calculus focuses on the generalization of integration and differentiation of arbitrary orders.[14] Although it has been a mathematical topic for more than 300 years, its applications to physics and engineering are only a recent subject of interest.[15–17] Compared with the classical integer order models, fractional-order differential models provide a more suitable instrument for the description of memory and hereditary properties of various materials and processes.[18,19] Now, chaos phenomena and chaotic synchronization of fractional-order systems have attracted lots of attention due to its great potential applications especially in secure communication and control processing.[20–22] For example, Mohammad *et al.* presented an active sliding mode controller for the synchronization of fractional-order chaotic systems.[23] Zhou and Zhu studied the function-projective synchronization for the fractional-order chaotic systems with different orders of fractional derivatives.[24] Yuan and Yang discussed the synchronization of the uncertain fractional-order chaotic systems based on the active control method and the particle swarm optimization algorithm.[25] Si *et al.* investigated the projective synchronization of different fractional-order chaotic systems with non-identical orders.[26] Aghababa provided slide mode controllers for the synchronization of fractional-order chaotic systems based on the fractional Lyapunov stability theorem.[27]

From the viewpoint of engineering applications and channel characteristics, complete synchronization is always practically impossible owing to signal propagation delays in the environment. It is therefore reasonable that the slave and master systems synchronize with a time delay τ. Then lag synchronization appears as a coincidence of shift-in-time states of interactive chaotic systems in many different areas including the motor systems,[28] neuron systems,[29] complex networks,[30] and secure communication.[31] In 2009, Li discussed the projective lag synchronization (PLS) of two identical integer-order chaotic systems, where the response system synchronized the past states of the drive system up to a real scaling factor.[32] In 2011, some researchers deliberated on the function-projective lag synchronization (FPLS) of the integer-order hyperchaotic Rössler systems, which meant that the current states of the responser synchronized the past states of the driver up to a scaling function.[33] The complete synchronization, anti-phase synchronization, function-projective synchronization and PLS belong to the FPLS. However, it is unfortunate that most of the existing theoretical results concerning lag synchronization mainly focus on the classical integer-order systems. Lag synchronization of the fractional-order

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chaotic systems is still a challenging and interesting problem at present. In 2011, Zhang and Yang proposed a simple scheme for the lag synchronization and the parameter identification of the fractional-order chaotic systems.[34] Chen, Chai and Wu advised a nonlinear controller for the PLS of the fractional-order chaotic (hyperchaotic) systems based on the pole placement technique.[35] Nonetheless, few works consider the FPLS of the fractional-order chaotic systems.

On the other hand, the synchronization of different dimensional chaotic systems is usually carried out in biological science and social science.[36] Also, most of the existing results concentrate upon the synchronization of different dimensional integer-order chaotic systems[37,38] without considering the synchronization of the fractional-order chaotic systems with different dimensions.[39] Motivated by the above discussion, the FPLS of the fractional-order chaotic systems with the same and different dimensions is investigated in this paper. According to the stability theorem of the linear fractional-order system, a nonlinear fractional-order controller is given for the FPLS of the fractional-order chaotic systems. Besides, the proposed controller can be applied to the synchronization of systems with different orders of fractional derivatives. Compared with the PLS for the fractional-order chaotic systems, the discussed FPLS could get more secure communication in its applications to secure communication, because the unpredictability of the scaling function matrix, the system dimensions, and the fractional orders can additionally enhance the security of communication.

The rest of the paper is organized as follows. In Section 2, a general scheme is provided for the FPLS of fractional-order chaotic systems based on the stability theorem of linear fractional-order systems. In Section 3, numerical examples are presented to manifest the effectiveness of the proposed method. Finally, conclusions are drawn in Section 4.

2. A general method for FPLS

There are many definitions of the fractional derivatives such as Grunwald–Letnikov, Riemann–Liouville, and Caputo derivatives.[14] The Caputo fractional derivative is more popular in practice because the unhomogenous initial conditions are allowed if such conditions are necessary. In this work, we focus on the Caputo derivative

$$D^{\delta}h(t) = J^{k-\delta}h^{(k)}(t), \qquad \delta > 0,$$

where δ is the order of the fractional derivative, $k = \lceil \delta \rceil$, i.e., k is the first integer which is not less than δ, and J^{θ} is the θ-order Riemann–Liouville fractional integral operator which is

$$J^{\theta}s(t) = \frac{1}{\Gamma(\theta)}\int_0^t (t-\tau)^{\theta-1}s(\tau)\,\mathrm{d}\tau, \quad \theta > 0,$$

where $\Gamma(\cdot)$ is the gamma function.

Consider a fractional-order drive system as

$$\begin{aligned} D^{\alpha}x(t) &= F(x(t)), \\ x(t) &= x(0), \quad t \in [-\tau, 0], \end{aligned} \tag{1}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_m(t))^{\mathrm{T}} \in R^m$ is the state vector, m is the dimension of system (1), $\alpha \in (0, 1)$ is the fractional order of system (1), $F : R^m \to R^m$ is a continuous vector function, and $\tau > 0$ denotes the time delay. Choose a fractional-order response system as

$$D^{\beta}y(t) = G(y(t)) + U(x(t-\tau), y(t)), \tag{2}$$

where $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))^{\mathrm{T}} \in R^n$ is the state vector, n is the dimension of system (2), $\beta \in (0, 1)$ is the fractional order of system (2), $G : R^n \to R^n$ is a continuous vector function, and $U = (u_1, u_2, \ldots, u_n)^{\mathrm{T}}$ is a controller to be determined later.

The error state vector between systems (1) and (2) is defined as

$$e(t) = y(t) - K(x(t-\tau))x(t-\tau), \tag{3}$$

where $e(t) = (e_1(t), e_2(t), \ldots, e_n(t))^{\mathrm{T}} \in R^n$. The polynomial matrix $K(x(t-\tau)) \in R^{n\times m}$ is provided for the state transformation of system (1). For simplicity, the notation $x_\tau = x(t-\tau)$ is applied. In particular, if the system dimensions $n \le m$, the polynomial matrix is regarded as

$$K(x_\tau) = \begin{pmatrix} M(x_\tau) & \mathbf{0} \\ \mathbf{0} & N(x_\tau) \end{pmatrix},$$

where the polynomial matrices

$$M(x_\tau) = \mathrm{diag}\{k_1(x_\tau), k_2(x_\tau), \ldots, k_{\sigma-1}(x_\tau)\},$$

$\sigma = \min\{m, n\}$ and $N(x_\tau) = (k_\sigma(x_\tau), k_{\sigma+1}(x_\tau), \ldots, k_\gamma(x_\tau))$, $\gamma = \max\{m, n\}$. If the system dimensions $n > m$, the polynomial matrix is selected as

$$K(x_\tau) = \begin{pmatrix} M(x_\tau) & \mathbf{0} \\ \mathbf{0} & N^{\mathrm{T}}(x_\tau) \end{pmatrix}.$$

Definition 1 For the fractional-order drive system (1) and response system (2), it is said that the drive and response systems (1) and (2) are "function-projective lag synchronization" (FPLS) with respect to a scaling function matrix $K(x_\tau) \in R^{n\times m}$ if there exists a controller $U(x_\tau, y(t))$ such that

$$\lim_{t\to+\infty}\|e(t)\| = \lim_{t\to+\infty}\|y(t) - K(x_\tau)x_\tau\| = 0. \tag{4}$$

Remark 1 If the time delay $\tau = 0$ and the system dimensions $n = m$, the FPLS mentioned above is reduced to the function-projective synchronization for the fractional order chaotic systems (1) and (2) which was discussed in Ref. [24].

Remark 2 If $\tau = 0$, $n = m$, the scaling matrix $K(x_\tau) = \mathrm{diag}\{k, k, \ldots, k\} \in R^{n\times m}$ and $k \neq 0$, the FPLS is changed into

the projective synchronization of fractional-order chaotic systems with non-identical orders.[26]

Remark 3 If the fractional orders $\alpha = \beta$, $n = m$, and $K(x_\tau) = \mathrm{diag}\{k_1, k_2, \ldots, k_n\} \in R^{n\times m}$, the FPLS is considered as the PLS for the fractional-order chaotic (hyperchaotic) systems.[34]

Remark 4 If $\tau = 0$, $n \neq m$, $\alpha = \beta$, and the scaling matrix $K(x_\tau) = C$, where $C \in R^{n\times m}$ is a real matrix, the FPLS is simplified as the hybrid projective synchronization of the different dimensional fractional-order chaotic systems.[39]

With the above parameters, a nonlinear fractional-order controller is assumed as

$$U = D^\beta(K(x_\tau)x_\tau) - G(K(x_\tau)x_\tau) + B(x_\tau, y(t))e(t), \tag{5}$$

where $B(x_\tau, y(t)) \in R^{n\times n}$ is a polynomial matrix to be designed later. Combining Eqs. (2) and (3) with Eq. (5), the fractional-order error system can be obtained as follows:

$$D^\beta e(t) = [A(x_\tau, y(t)) + B(x_\tau, y(t))]e(t), \tag{6}$$

where $A(x_\tau, y(t))e(t) = G(y(t)) - G(K(x_\tau)x_\tau)$, $A \in R^{n\times n}$ is also a polynomial matrix. Then, the FPLS between systems (1) and (2) is transformed into the discussion of the asymptotical stability of the zero solution of system (6).

Proposition 1 For a given scaling matrix $K(x_\tau) \in R^{n\times m}$ and a time delay τ, the FPLS between the drive system (1) and response system (2) can be achieved if there exists a polynomial matrix $B(x_\tau, y(t)) \in R^{n\times n}$ such that

$$A(x_\tau, y(t)) + B(x_\tau, y(t)) = -C(x_\tau, y(t)), \tag{7}$$

where $C(x_\tau, y(t)) = [c_{ij}(x_\tau, y(t))]_{n\times n}$ is a polynomial matrix, $c_{ij} = -c_{ji}$, $i \neq j$ is a polynomial and $c_{ij} > 0$, $i = j$ is a real number.

Proof Assume λ is an arbitrary eigenvalue of the matrix $A(x_\tau, y(t)) + B(x_\tau, y(t))$ and the corresponding nonzero eigenvector is η, i.e.,

$$[A(x_\tau, y(t)) + B(x_\tau, y(t))]\eta = \lambda\eta. \tag{8}$$

Multiplying the above equation left by η^H, we obtain

$$\eta^H[A(x_\tau, y(t)) + B(x_\tau, y(t))]\eta = \lambda\eta^H\eta, \tag{9}$$

where H stands for conjugate transpose of a matrix. $\bar{\lambda}$ is also an eigenvalue of the matrix $A(x_\tau, y(t)) + B(x_\tau, y(t))$, i.e.,

$$\eta^H[A(x_\tau, y(t)) + B(x_\tau, y(t))]^H = \bar{\lambda}\eta^H. \tag{10}$$

Similarly, Eq. (10) becomes

$$\eta^H[A(x_\tau, y(t)) + B(x_\tau, y(t))]^H\eta = \bar{\lambda}\eta^H\eta. \tag{11}$$

Combining Eqs. (9) and (11), the considered eigenvalue λ satisfies

$$\begin{aligned}\lambda + \bar{\lambda} &= \eta^H[(A(x_\tau, y(t)) + B(x_\tau, y(t))) \\ &\quad + (A(x_\tau, y(t)) + B(x_\tau, y(t)))^H]\eta/\eta^H\eta \\ &= -\eta^H[C(x_\tau, y(t)) + C^H(x_\tau, y(t))]\eta/\eta^H\eta \\ &= -\eta^H N\eta/\eta^H\eta,\end{aligned} \tag{12}$$

where $\eta^H\eta > 0$, $N = C(x_\tau, y(t)) + C^H(x_\tau, y(t))$, $N \in R^{n\times n}$. Since the polynomial matrix $C(x_\tau, y(t))$ is subject to $c_{ij} = -c_{ji}$, $i \neq j$ and $c_{ij} \in R^+$, $i = j$, we can derive that N is a real positive-definite diagonal matrix. Thus, $\eta^H N\eta > 0$. Then, $\lambda + \bar{\lambda} = 2\mathrm{Re}(\lambda) < 0$.

Hence, for the fractional-order error system (6), one can get

$$|\arg(\lambda)| > \pi/2 > \beta\pi/2, \tag{13}$$

where $\beta \in (0, 1)$, λ is an arbitrary eigenvalue of the polynomial matrix $A(x_\tau, y(t)) + B(x_\tau, y(t))$. According to the stability theorem of the linear fractional-order autonomous systems,[21] the error system (6) is asymptotically stable at zero point. That is $\lim_{t\to+\infty}\|e(t)\| = 0$, which implies the two different structural fractional-order chaotic systems (1) and (2) realize the FPLS. This completes the proof. □

3. Applications

In this section, the effectiveness and feasibility of the proposed method for FPLS is performed. As mentioned above, m is the dimension of system (1), and n is the dimension of system (2). Then, three groups of examples are respectively applied to consider the FPLS of the same and different dimensional fractional-order chaotic systems. Moreover, a predictor-corrector scheme[40,41] is used for the approximate numerical solutions of the fractional-order differential equations.

3.1. Same dimensional FPLS with $n = m$

It is assumed that the fractional-order Rössler system drives the fractional-order Lü system. The drive system is described by

$$\begin{aligned}D^\alpha x_1(t) &= -x_2(t) - x_3(t), \\ D^\alpha x_2(t) &= x_1(t) + ax_2(t), \\ D^\alpha x_3(t) &= b + x_1(t)x_3(t) - cx_3(t), \\ x(t) &= x(0), \quad t \in [-\tau, 0],\end{aligned} \tag{14}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ is the state vector, a, b, c are the real positive parameters. In Ref. [42], Li and Chen explained that the three-dimensional (3D) fractional-order Rössler system (14) exhibited chaotic behavior for $\alpha \geq 0.8$ and $a > 0.3$. When $\alpha = 0.95$ and $(a, b, c) = (0.4, 0.2, 10)$, the chaotic attractor of system (14) is shown in Fig. 1.

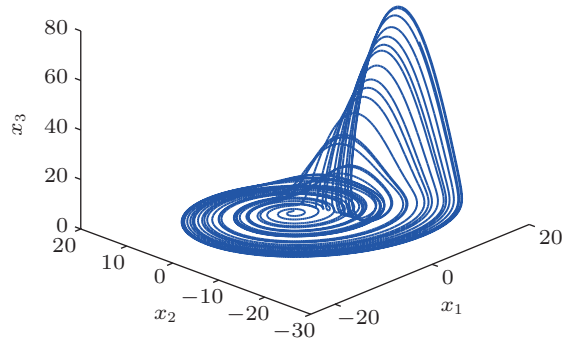


Fig. 1. (color online) The chaotic attractor of the fractional-order Rössler system (14).

The response system is written as

$$\begin{aligned} D^{\beta}y_1(t) &= \delta(y_2(t)-y_1(t))+u_1(x_\tau,y(t)),\\ D^{\beta}y_2(t) &= -y_1(t)y_3(t)+\theta y_2(t)+u_2(x_\tau,y(t)),\\ D^{\beta}y_3(t) &= y_1(t)y_2(t)-\gamma y_3(t)+u_3(x_\tau,y(t)), \end{aligned} \tag{15}$$

where $y(t)=(y_1(t),y_2(t),y_3(t))^{\mathrm{T}}$ is the state vector, δ,θ,γ are the real positive parameters, $U=(u_1,u_2,u_3)^{\mathrm{T}}$ is the controller to be designed later. Mohammad *et al.* introduced that the 3D fractional-order Lü system (15) exhibited chaotic behavior for $\beta\geq 0.81$.[43] When $\beta=0.9$ and $(\delta,\theta,\gamma)=(35,28,3)$, the chaotic attractor of system (15) is displayed in Fig. 2.

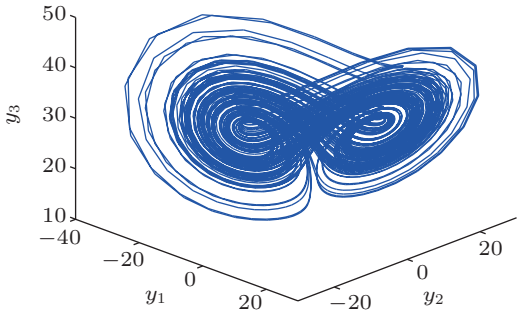


Fig. 2. (color online) The chaotic attractor of the fractional-order Lü system (15).

The error state vector is defined as

$$e(t)=y(t)-K(x_\tau)x_\tau,$$

where $K(x_\tau)=\mathrm{diag}\{k_1(x_\tau),k_2(x_\tau),k_3(x_\tau)\}$. The error states are described by $e_i(t)=y_i(t)-k_i(x_\tau)x_{i\tau}$, $x_{i\tau}=x_i(t-\tau)$, $i=1,2,3$. With the suitable controller (5), the fractional-order error system can be obtained as

$$D^{\beta}e(t)=[A(x_\tau,y(t))+B(x_\tau,y(t))]e(t),$$

where $A(x_\tau,y(t))e(t)=G(y(t))-G(K(x_\tau)x_\tau)$ and the polynomial matrix

$$A(x_\tau,y(t))=\begin{pmatrix} -\delta & \delta & 0\\ -k_3(x_\tau)x_{3\tau} & \theta & -y_1(t)\\ y_2(t) & k_1(x_\tau)x_{1\tau} & -\gamma \end{pmatrix}.$$

Selecting the polynomial matrix

$$B(x_\tau,y(t))=\begin{pmatrix} 0 & k_3(x_\tau)x_{3\tau}-\delta & -y_2(t)\\ 0 & -\theta-d & y_1(t)\\ 0 & -k_1(x_\tau)x_{1\tau} & 0 \end{pmatrix},$$

where $d>0$, we can obtain

$$A(x_\tau,y(t))+B(x_\tau,y(t))=-C(x_\tau,y(t)),$$

in which $C(x_\tau,y(t))=[c_{ij}(x_\tau,y(t))]$, $i,j=1,2,3$, $c_{ij}=-c_{ji}$, $i\neq j$, and $c_{ij}\in R^+$, $i=j$. Then, the fractional-order chaotic Rössler system (14) and chaotic Lü system (15) achieve the same dimensional FPLS based on Proposition 1.

For example, when $\alpha=0.95$, $\beta=0.9$, $(a,b,c)=(0.4,0.2,10)$, $(\delta,\theta,\gamma)=(35,28,3)$, $x(0)=(0.5,0.5,0.5)^{\mathrm{T}}$, and $y(0)=(4.2,3.5,11)^{\mathrm{T}}$, the drive and response systems (14) and (15) are chaotic. Setting $\tau=0.5$, $d=5$, the same dimensional FPLS between systems (14) and (15) can be realized with a scaling function matrix

$$K(x_\tau)=\mathrm{diag}\{10x_{2\tau}+35,\ x_{3\tau}-12.5,\ 4x_{3\tau}-20x_{1\tau}\}.$$

The corresponding error state curves are shown in Fig. 3, which indicate that the same dimensional FPLS for systems (14) and (15) is successfully achieved.

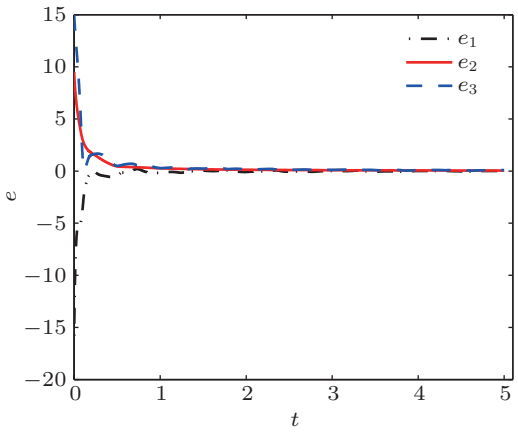


Fig. 3. (color online) The error state curves of the same dimensional FPLS between systems (14) and (15).

3.2. Reduced dimensional FPLS with $n<m$

Consider the four-dimensional (4D) fractional-order hyperchaotic Lorenz system

$$\begin{aligned} D^{\alpha}x_1(t) &= a(x_2(t)-x_1(t))+x_4(t),\\ D^{\alpha}x_2(t) &= bx_1(t)-x_2(t)-x_1(t)x_3(t),\\ D^{\alpha}x_3(t) &= x_1(t)x_2(t)-cx_3(t),\\ D^{\alpha}x_4(t) &= -x_2(t)x_3(t)-rx_4(t),\\ x(t) &= x(0),\quad t\in[-\tau,0], \end{aligned} \tag{16}$$

as the drive system, where $x(t)=(x_1(t),x_2(t),x_3(t),x_4(t))^{\mathrm{T}}$ is the state vector, a,b,c,r are the real positive parameters.

In Ref. [44], Wang and Song pointed out that system (16) exhibited hyperchaotic behavior when $\alpha = 0.98$, $(a, b, c, r) = (10, 28, 8/3, 1)$. The hyperchaotic attractor of the fractional-order Lorenz system (16) is displayed in Fig. 4.

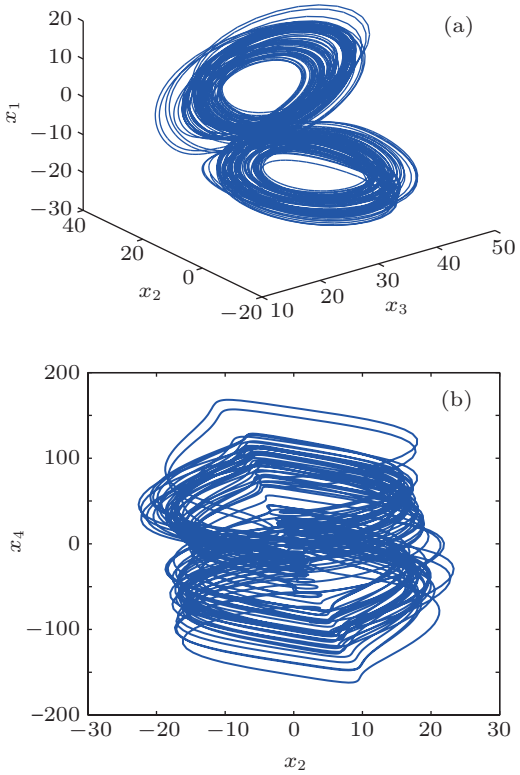


Fig. 4. (color online) The hyperchaotic attractor of the fractional-order Lorenz system (16).

Choose the 3D fractional-order Rössler system (14) as the response system, which is rewritten as

$$\begin{aligned} D^\beta y_1(t) &= -y_2(t) - y_3(t) + u_1(x_\tau, y(t)), \\ D^\beta y_2(t) &= y_1(t) + \theta y_2(t) + u_2(x_\tau, y(t)), \\ D^\beta y_3(t) &= \delta + y_1(t) y_3(t) - \gamma y_3(t) + u_3(x_\tau, y(t)), \end{aligned} \quad (17)$$

where $y(t) = (y_1(t), y_2(t), y_3(t))^{\mathrm{T}}$ is the state vector, θ, δ, γ are the real positive parameters, $U = (u_1, u_2, u_3)^{\mathrm{T}}$ is the controller to be designed later.

The error state vector is described by $e(t) = y(t) - K(x_\tau)x_\tau$, where

$$K(x_\tau) = \begin{pmatrix} M(x_\tau) & \mathbf{0} \\ \mathbf{0} & N(x_\tau) \end{pmatrix},$$

with $M(x_\tau) = \mathrm{diag}\{k_1(x_\tau), k_2(x_\tau)\}$, $N(x_\tau) = (k_3(x_\tau), k_4(x_\tau))$.

Hence, the error states can be written as

$$\begin{aligned} e_1(t) &= y_1(t) - k_1(x_\tau)x_{1\tau}, \\ e_2(t) &= y_2(t) - k_2(x_\tau)x_{2\tau}, \\ e_3(t) &= y_3(t) - (k_3(x_\tau)x_{3\tau} + k_4(x_\tau)x_{4\tau}), \end{aligned}$$

where $x_{i\tau} = x_i(t - \tau)$, $i = 1, 2, 3, 4$. With the suitable controller (5), the fractional-order error system can be obtained as

$$D^\beta e(t) = [A(x_\tau, y(t)) + B(x_\tau, y(t))]e(t),$$

where $A(x_\tau, y(t))e(t) = G(y(t)) - G(K(x_\tau)x_\tau)$ and the polynomial matrix

$$A(x_\tau, y(t)) = \begin{pmatrix} 0 & -1 & -1 \\ 1 & \theta & 0 \\ k_3(x_\tau)x_{3\tau} + k_4(x_\tau)x_{4\tau} & 0 & y_1(t) - \gamma \end{pmatrix}.$$

Selecting the polynomial matrix as

$$B(x_\tau, y(t)) = \begin{pmatrix} -d_1 & 0 & s \\ 0 & -\theta - d_2 & -y_2(t) \\ 0 & y_2(t) & -y_1(t) + \gamma - d_3 \end{pmatrix},$$

where $s = 1 - k_3(x_\tau)x_{3\tau} - k_4(x_\tau)x_{4\tau}$, $d_1 > 0, d_2 > 0, d_3 > 0$, we can derive

$$A(x_\tau, y(t)) + B(x_\tau, y(t)) = -C(x_\tau, y(t)),$$

where $C(x_\tau, y(t)) = [c_{ij}(x_\tau, y(t))]$, $i, j = 1, 2, 3$, $c_{ij} = -c_{ji}$, $i \neq j$, and $c_{ij} \in R^+$, $i = j$. Then, Proposition 1 assures that the fractional-order chaotic systems (16) and (17) realize the reduced dimensional FPLS.

For example, when $\alpha = 0.98$, $\beta = 0.95$, $(a, b, c, r) = (10, 28, 8/3, 1)$, $(\theta, \delta, \gamma) = (0.4, 0.2, 10)$, $x(0) = (-2.2, -6, 8.3, -9)^{\mathrm{T}}$, and $y(0) = (0.5, 0.5, 0.5)^{\mathrm{T}}$, the drive and response systems (16) and (17) are chaotic. Assuming $\tau = 0.2$, $d_1 = 1, d_2 = 6, d_3 = 3$, the reduced dimensional FPLS between systems (16) and (17) is achieved with a scaling function matrix

$$K(x_\tau) = \begin{pmatrix} -x_{2\tau}/2 & & \\ & x_{2\tau} + x_{3\tau} & \\ & & 1 \quad x_{1\tau}/2 \end{pmatrix}.$$

The corresponding error state curves are displayed in Fig. 5, which imply that the reduced dimensional FPLS for systems (16) and (17) is satisfactorily realized.

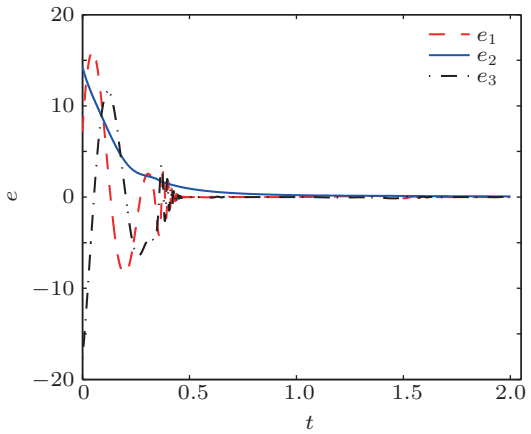


Fig. 5. (color online) The error state curves of the reduced dimensional FPLS between systems (16) and (17).

3.3. Increased dimensional FPLS with $n > m$

It is supposed that the fractional-order hyperchaotic Chen system is driven by the fractional-order chaotic Lü system (15). Then, the three-dimensional (3D) drive system is rewritten as

$$\begin{aligned} &D^{\alpha}x_1(t) = \delta(x_2(t) - x_1(t)),\\ &D^{\alpha}x_2(t) = -x_1(t)x_3(t) + \theta x_2(t),\\ &D^{\alpha}x_3(t) = x_1(t)x_2(t) - \gamma x_3(t),\\ &x(t) = x(0), \quad t \in [-\tau, 0], \end{aligned} \tag{18}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ is the state vector, δ, θ, γ are the real positive parameters. The four-dimensional (4D) response system is described by

$$\begin{aligned} &D^{\beta}y_1(t) = a(y_2(t) - y_1(t)) + y_4(t) + u_1(x_\tau, y(t)),\\ &D^{\beta}y_2(t) = by_1(t) - y_1(t)y_3(t) + cy_2(t) + u_2(x_\tau, y(t)),\\ &D^{\beta}y_3(t) = y_1(t)y_2(t) - dy_3(t) + u_3(x_\tau, y(t)),\\ &D^{\beta}y_4(t) = y_2(t)y_3(t) + ry_4(t) + u_4(x_\tau, y(t)), \end{aligned} \tag{19}$$

where $y(t) = (y_1(t), y_2(t), y_3(t), y_4(t))^{\mathrm{T}}$ is the state vector, a, b, c, d, r are the real positive parameters, $U = (u_1, u_2, u_3, u_4)^{\mathrm{T}}$ is the controller to be determined later. In Ref. [45], Hegazi and Matouk explained that the 4D fractional-order Chen system exhibited hyperchaotic behavior for $\beta \geq 0.94$. When $\beta = 0.96$ and $(a, b, c, d, r) = (35, 7, 12, 3, 0.5)$, the hyperchaotic attractor of the fractional-order Chen system (19) is shown in Fig. 6.

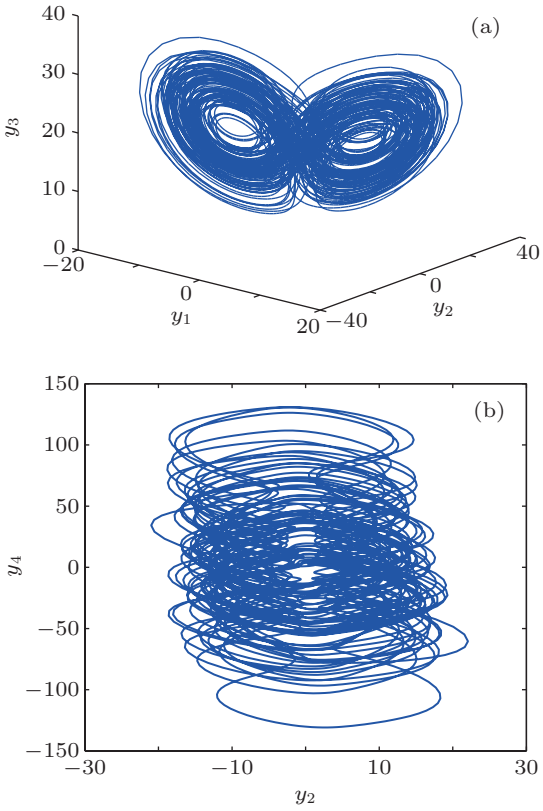


Fig. 6. (color online) The hyperchaotic attractor of the fractional-order Chen system (19).

The error state vector is defined as $e(t) = y(t) - K(x_\tau)x_\tau$, where

$$K(x_\tau) = \begin{pmatrix} M(x_\tau) & \mathbf{0} \\ \mathbf{0} & N^{\mathrm{T}}(x_\tau) \end{pmatrix},$$

with $M(x_\tau) = \mathrm{diag}\{k_1(x_\tau), k_2(x_\tau)\}$, $N(x_\tau) = (k_3(x_\tau), k_4(x_\tau))$.

Then, the error states are obtained as

$$\begin{aligned} e_1(t) &= y_1(t) - k_1(x_\tau)x_{1\tau},\\ e_2(t) &= y_2(t) - k_2(x_\tau)x_{2\tau},\\ e_3(t) &= y_3(t) - k_3(x_\tau)x_{3\tau},\\ e_4(t) &= y_4(t) - k_4(x_\tau)x_{3\tau}, \end{aligned}$$

where $x_{i\tau} = x_i(t - \tau)$, $i = 1, 2, 3$.

With the suitable controller (5), the fractional-order error system can be given as

$$D^{\beta}e(t) = [A(x_\tau, y(t)) + B(x_\tau, y(t))]e(t),$$

where $A(x_\tau, y(t))e(t) = G(y(t)) - G(K(x_\tau)x_\tau)$ and the polynomial matrix

$$A(x_\tau, y(t)) = \begin{pmatrix} -a & a & 0 & 1 \\ b - y_3(t) & c & -k_1(x_\tau)x_{1\tau} & 0 \\ k_2(x_\tau)x_{2\tau} & y_1(t) & -d & 0 \\ 0 & y_3(t) & k_2(x_\tau)x_{2\tau} & r \end{pmatrix}.$$

Supposing the polynomial matrix

$$B(x_\tau, y(t)) = \begin{pmatrix} d_1 & -a & -k_2(x_\tau)x_{2\tau} & 0 \\ s & -c - d_2 & k_1(x_\tau)x_{1\tau} & -y_3(t) \\ 0 & -y_1(t) & 0 & 0 \\ -1 & 0 & -k_2(x_\tau)x_{2\tau} & -r - d_3 \end{pmatrix},$$

where $s = -b + y_3(t)$, $0 < d_1 < a, d_2, d_3 > 0$, we can obtain

$$A(x_\tau, y(t)) + B(x_\tau, y(t)) = -C(x_\tau, y(t)),$$

in which $C(x_\tau, y(t)) = [c_{ij}(x_\tau, y(t))]$, $i, j = 1, 2, 3, 4$, $c_{ij} = -c_{ji}$, $i \neq j$, and $c_{ij} \in R^+$, $i = j$. Thus, the fractional-order chaotic systems (18) and (19) achieve the increased dimensional FPLS based on Proposition 1.

For example, when $\alpha = 0.9$, $\beta = 0.96$, $(\delta, \theta, \gamma) = (35, 28, 3)$, $(a, b, c, d, r) = (35, 7, 12, 3, 0.5)$, $x(0) = (4.2, 3.5, 11)^{\mathrm{T}}$, and $y(0) = (1.2, 2.1, 3.1, 0.1)^{\mathrm{T}}$, the drive and response systems (18) and (19) are chaotic. Selecting $\tau = 0.05$, $d_1 = 25$, $d_2 = 5$, $d_3 = 4$, the increased dimensional FPLS between systems (18) and (19) is realized with a scaling function matrix

$$K(x_\tau) = \begin{pmatrix} x_{2\tau} & & \\ & x_{1\tau} - x_{3\tau} & \\ & & x_{1\tau}/2 \\ & & 2.5 - x_{2\tau} \end{pmatrix}.$$

The corresponding error state curves are displayed in Fig. 7, which indicate the increased dimensional FPLS for systems (18) and (19) is successfully achieved.

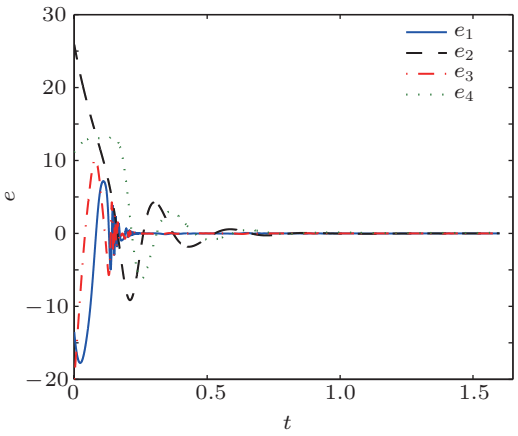


Fig. 7. (color online) The error state curves of the increased dimensional FPLS between systems (18) and (19).

4. Conclusions

The FPLS of different structural fractional-order chaotic systems is investigated based on the stability theory of the linear fractional-order systems. A nonlinear fractional-order controller is given for the response system to synchronize the past states of the drive system with respect to a scaling function matrix. In view of the proposed synchronization scheme, both of the same and different dimensional chaotic systems with the same and different fractional-orders can be applied to realize the synchronization. Finally, the same dimensional FPLS between the fractional-order chaotic Rössler and Lü systems, the reduced dimensional FPLS between the fractional-order hyperchaotic Lorenz and chaotic Rössler systems, the increased dimensional FPLS between the fractional-order chaotic Lü and hyperchaotic Chen systems are realized. The numerical results coincide with the theoretical analysis.

It is worth noting that the complete synchronization, anti-phase synchronization, and PLS belong to the FPLS. The obtained results are applicable and representative. However, the proposed FPLS is provided for the fractional-order chaotic systems without time-delay. The synchronization of the fractional-order chaotic systems with time-delay will be an interesting and important topic in future.

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