Robust H_∞ cluster synchronization analysis of Lurie dynamical networks*

Guo Ling(郭 凌)[a)†], Nian Xiao-Hong(年晓红)[b)], Pan Huan(潘 欢)[c)], and Bing Zhi-Tong(邴志桐)[d)]

[a)] College of Electrical Engineering, Northwest University for Nationalities, Lanzhou 730030, China
[b)] College of Information Science and Engineering, Central South University, Changsha 410075, China
[c)] College of Physics Electrical Information Engineering, Ningxia University, Yinchuan 750021, China
[d)] Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou 730000, China

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The cluster synchronization problem of complex dynamical networks with each node being a Lurie system with external disturbances and time-varying delay is investigated in this paper. Some criteria for cluster synchronization with desired H_∞ performance are presented by using a local linear control scheme. Firstly, sufficient conditions are established to realize cluster synchronization of the Lurie dynamical networks without time delay. Then, the notion of the cluster synchronized region is introduced, and some conditions guaranteeing the cluster synchronized region and unbounded cluster synchronized region are derived. Furthermore, the cluster synchronization and cluster synchronized region in the Lurie dynamical networks with time-varying delay are considered. Numerical examples are finally provided to verify and illustrate the theoretical results.

Keywords: cluster synchronization, Lurie dynamics network, H_∞ performance, time-varying delay

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1. Introduction

During the past decades, the research on the dynamical behavior of complex networks has received increasing attention in various fields of science and engineering. A typical complex network consists of interacting nodes connected by links, in which every node represents an individual of the system and each edge represents the interaction between them. Control and synchronization of complex dynamical networks have been extensively investigated, and many efforts have been devoted to studying different synchronization patterns, such as complete synchronization, phase synchronization, anti-synchronization, impulsive synchronization, lag synchronization, projective synchronization, generalized synchronization, cluster synchronization, and so on. Roughly speaking, synchronization of complex dynamical networks[1–6] used to study all nodes approaching to a common behavior. In particular, consensus of multi-agent systems with first-order dynamics can be regarded as a special case of it.[7–10] Cluster synchronization[11–20] focuses on when the dynamical individuals of the network are divided into several groups (called clusters), all individuals in the same group realize complete synchronization, but there is no synchronization among different groups. Obviously, cluster synchronization reduces to global synchronization if there is only one cluster in the network. It is a significant and interesting problem in brain science,[21] engineering control,[22] ecological science,[23,24] social science,[25] and so on.

Recently, the synchronization problem for complex dynamical networks composed of Lurie subsystems have been widely studied. The Lurie system[26] is represented as a linear dynamical system, feedback interconnected to a nonlinearity satisfying a sector condition, and many nonlinear systems can be categorized as it.[27–31] For example, the global synchronization problem for Lurie dynamical networks under the pinning controllers was addressed in Ref. [6]. Some sufficient conditions were presented to guarantee global synchronization of Lurie dynamical networks with complex topology, directed and weighted couplings in Ref. [2].

In practical applications, some variables of the nodes in a complex dynamical network might be caused by measurement or calculation errors due to time delay, external disturbances, etc. They are inevitable and play key roles in influencing the stability of complex networks. It is thus very important to consider synchronization of general complex dynamical networks and consensus of first-order multi-agent systems with time delay and external disturbances. The adaptive feedback synchronization in a general complex dynamical network with delayed nodes was studied in Ref. [5]. A robust consensus tracking problem of a multi-agent system with first-order and unmodelled dynamics in the presence of disturbances was discussed in Ref. [8]. The authors in Ref. [9] investigated the consensus problem of multi-agent systems with nonlinear dynamics and external disturbances on a fixed directed information network and intermittent information transmissions. In Ref. [7], the authors studied consensus problems for first-order multi-agent systems with external disturbances

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†Corresponding author. E-mail: gling0826@126.com

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and model uncertainty on directed networks with and without time-delay. Moreover, the master-slave synchronization of chaotic Lurie systems with time delay or external disturbances has been well studied.[28,29] The cluster synchronization for both continuous-time and discrete-time Lurie dynamical networks with interval time-varying delay was studied in Ref. [15]. In Ref. [16], the cluster synchronization problem and cluster synchronized region of Lurie dynamical networks without disturbance were explored by using a local linear control strategy. However, to the best of our knowledge, the cluster synchronization problem of complex dynamical networks with external disturbances has not been considered in detail.

The purpose of this paper is to investigate the global cluster synchronization of complex dynamical networks composed of identical Lurie subsystems with external disturbances and time delay. Inspired by the previous works,[3,4,6,17] the notion of a cluster synchronized region is developed to demonstrate the relationship between the cluster synchronization and the structure parameters of the networks. Criteria in terms of linear matrix inequalities (LMIs) for the cluster synchronization and cluster synchronized region with desired H_∞ performance are derived for both the Lurie subsystems with and without time delay. Noted that the cluster synchronization of the high-dimensional Lurie dynamical networks is converted into the effectiveness of a set of much lower-dimensional LMIs, and it may help to reduce the computation complexity significantly for large scale networks. Different from the existing works,[11–13,16,18] the assumption of each block in the outer coupling matrix is eliminated. Namely, there is not an inhibitory coupling between the nodes in different clusters. In addition, it is worth noticing that the cluster synchronization criteria obtained in this paper do not require the inner-linking matrix to be diagonalizable, and some previous results[11–16] are just special cases of ours.

The following standard notations are used throughout this paper. $\mathscr{L}_2[0,\infty)$ denotes the space of square integrable vector functions over $[0,\infty)$. Let $\mathscr{R}^n$ and $\mathscr{R}^{m\times n}$ be the n-dimensional Euclidean space and the set of $m\times n$ real matrices, respectively. $I_n\in\mathscr{R}^{n\times n}$ denotes the identity matrix. 0 stands for zero value or zero matrix with compatible dimensions. For a symmetric matrix P, the notation $P>0$ (<0) means that the matrix P is positive-definite (negative-definite). The superscript T means transpose for real matrices. Symbol $\otimes$ denotes the Kronecker product.

2. Problem formulation and preliminaries

Consider a network consisting of N coupled delayed Lurie systems with external disturbances described by

$$\begin{aligned}\dot{x}_i(t) &= Ax_i(t)+A_dx_i(t-\tau(t))+B\varphi(Cx_i(t))\\ &\quad +D\omega_i(t)+\epsilon\sum_{j=1}^{N}g_{ij}\Gamma x_j(t)+u_i(t),\\ y_i(t) &= Cx_i(t),\\ x_i(t) &= \phi_i(t),\ \ t\in[-h,0],\ \ i=1,2,\ldots,N,\end{aligned}\tag{1}$$

where $x_i(t)\in\mathscr{R}^n$ and $y_i(t)=[y_{i1}(t),y_{i2}(t),\ldots,y_{im}(t)]^{\mathrm{T}}\in\mathscr{R}^m$ are the state and the output of node i, respectively. $\omega_i(t)\in\mathscr{L}_2[0,\infty)$ denotes the external disturbance. $u_i(t)\in\mathscr{R}^n$ is a controller to be designed later. A, $A_d\in\mathscr{R}^{n\times n}$, $B\in\mathscr{R}^{n\times m}$, $C\in\mathscr{R}^{m\times n}$, and $D\in\mathscr{R}^{n\times p}$ are constant matrices. The constant $\epsilon>0$ denotes the coupling strength and $\Gamma\in\mathscr{R}^{n\times n}$ is the inner-linking matrix. The nonlinear function $\varphi(y_i)=[\varphi_1(y_{i1}),\varphi_2(y_{i2}),\ldots,\varphi_m(y_{im})]^{\mathrm{T}}$ satisfies a sector condition with $\varphi_i(\cdot)$ $(i=1,2,\ldots,m)$ belonging to the sector $[0,K]$, where $K=\mathrm{diag}\{k_1,k_2,\ldots,k_m\}$, i.e.,

$$\begin{aligned}&0\le\frac{\varphi_i(\xi_1)-\varphi_i(\xi_2)}{\xi_1-\xi_2}\le k_i\ \text{ and }\ \varphi_i(0)=0,\\ &\forall\ \xi_1\ne\xi_2\in\mathscr{R},\ \ i=1,2,\ldots,m.\end{aligned}\tag{2}$$

$\phi_i(t)$ is a given continuous vector-valued initial function. $\tau(t)$ is the time-varying delay satisfying the following assumption

$$0\le\tau(t)\le h<\infty\ \text{ and }\ \dot{\tau}(t)\le\nu<\infty\tag{3}$$

with constants $h\ge0$ and ν.

$G=[g_{ij}]\in\mathscr{R}^{N\times N}$ is the outer coupling matrix describing the coupling configuration of the network. If there is a connection between node i and node j $(i\ne j)$, then $g_{ij}=g_{ji}>0$; otherwise, $g_{ij}=g_{ji}=0$. Assume that the Lurie network (1) is diffusively coupled, i.e., the diagonal elements of matrix G are defined by

$$g_{ii}=-\sum_{j=1,j\ne i}^{N}g_{ij},\ \ i=1,2,\ldots,N.$$

The network (1) is divided into d communities, where $2\le d\le N$. Let $\{\mathscr{C}_1,\mathscr{C}_2,\ldots,\mathscr{C}_d\}$ denote the d communities of the network satisfying (i) $\bigcup_{i=1}^{d}\mathscr{C}_i=\{1,2,\ldots,N\}$; (ii) $\mathscr{C}_i\cap\mathscr{C}_j=\emptyset$ for $i\ne j$. The function μ_i denotes the index of the community in which node i lies, that is, $\mu_i=j$ if node i belongs to the j-th community for $i=1,2,\ldots,N$, $j=1,2,\ldots,d$. For any pair of indices i and j, node i and node j belong to different communities if and only if $\mu_i\ne\mu_j$.

Definition 1 The Lurie network (1) is said to realize cluster synchronization with communities $\{\mathscr{C}_1,\mathscr{C}_2,\ldots,\mathscr{C}_d\}$, if and only if, for any initial values, $\lim_{t\to\infty}\|x_i(t)-x_j(t)\|=0$ for $\mu_i=\mu_j$ and $\lim_{t\to\infty}\|x_i(t)-x_j(t)\|\ne0$ for $\mu_i\ne\mu_j$.

Let $s_i=[s_{i1},s_{i2},\ldots,s_{in}]^{\mathrm{T}}\in\mathscr{R}^n$ be the solution of an isolated node in the i-th community satisfying

$$\begin{aligned}\dot{s}_i(t) &= As_i(t)+A_ds_i(t-\tau(t))+B\varphi(Cs_i(t)),\\ y_{s_i}(t) &= Cs_i(t),\ \ i=1,2,\ldots,d,\end{aligned}$$

where $y_{s_i}(t)=[y_{s_i1}(t),y_{s_i2}(t),\ldots,y_{s_im}(t)]^{\mathrm{T}}\in\mathscr{R}^m$ and $\lim_{t\to\infty}\|s_i(t)-s_j(t)\|\ne0$ for $i\ne j$.

Define the performance variable $z_i(t)=C_z(x_i-s_{\mu_i})\in\mathscr{R}^q$, where $C_z\in\mathscr{R}^{q\times n}$ is a constant matrix. Let

$$\begin{aligned}z(t)&=[z_1^{\mathrm{T}}(t),z_2^{\mathrm{T}}(t),\ldots,z_N^{\mathrm{T}}(t)]^{\mathrm{T}},\\ \omega(t)&=[\omega_1^{\mathrm{T}}(t),\omega_2^{\mathrm{T}}(t),\ldots,\omega_N^{\mathrm{T}}(t)]^{\mathrm{T}}.\end{aligned}$$

Definition 2 The Lurie network (1) is said to realize cluster synchronization with H_∞ performance index $\rho > 0$, if

i) the Lurie network (1) with zero disturbance can realize cluster synchronization;

ii) under zero initial conditions $x_i(0) = x_j(0) = 0, \quad i = 1,2,\ldots,N$, one has

$$J = \int_0^\infty (z^{\mathrm{T}}(t)z(t) - \rho^2 \omega^{\mathrm{T}}(t)\omega(t))\,\mathrm{d}t \le 0.$$

The local control scheme is designed as follows:

$$u_i(t) = \begin{cases} -\epsilon \sum_{j=1}^N g_{ij}\Gamma s_{\mu_j}(t) \\ \quad -\epsilon\theta_i\Gamma(x_i(t) - s_{\mu_i}(t)), & i \in \widetilde{U}_{\mu_i}, \\ 0, & i \in U_{\mu_i} - \widetilde{U}_{\mu_i}, \end{cases} \tag{4}$$

where U_{μ_i} denotes all nodes in the μ_i-th community, $\widetilde{U}_{\mu_i}$ represents the nodes in the μ_i-th community which has direct links with the nodes in other communities, $\theta_i > 0$ is the feedback control gain for $i \in \widetilde{U}_{\mu_i}$ and $\theta_i = 0$ otherwise.

Let $e_i(t) = x_i(t) - s_{\mu_i}(t)$, $e_i(t-\tau(t)) = x_i(t-\tau(t)) - s_{\mu_i}(t-\tau(t))$, $e_{\varphi_{ij}} = \varphi_j(y_{ij}) - \varphi_j(y_{s_{\mu_i}j})$, and $e_{yi} = y_i - y_{s_{\mu_i}}$. Then the controlled error dynamical system can be written as

$$\begin{aligned} \dot{e}_i(t) &= Ae_i(t) + A_d e_i(t-\tau(t)) + Be_{\varphi_{ij}} \\ &\quad + D\omega_i(t) + \epsilon \sum_{j=1}^N g_{ij}\Gamma e_j(t) - \epsilon\theta_i\Gamma e_i(t), \\ e_{yi} &= Ce_i, \quad i = 1,2,\ldots,N. \end{aligned} \tag{5}$$

Setting $\Theta = \mathrm{diag}\{\theta_1,\theta_2,\ldots,\theta_N\}$, $e = [e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, \ldots, e_N^{\mathrm{T}}]^{\mathrm{T}}$, $e_y = [e_{y1}^{\mathrm{T}}, e_{y2}^{\mathrm{T}}, \ldots, e_{yN}^{\mathrm{T}}]^{\mathrm{T}}$, and $E_\varphi = [e_{\varphi_1}^{\mathrm{T}}, e_{\varphi_2}^{\mathrm{T}}, \ldots, e_{\varphi_N}^{\mathrm{T}}]^{\mathrm{T}}$ with $e_{\varphi_i} = [e_{\varphi_{i1}}, \ldots, e_{\varphi_{im}}]^{\mathrm{T}}$, system (5) takes a compact form as follows:

$$\begin{aligned} \dot{e} &= \widetilde{A}e + \widetilde{A}_d e(t-\tau(t)) + \widetilde{B}E_\varphi + \widetilde{D}\omega, \\ e_y &= \widetilde{C}e, \end{aligned} \tag{6}$$

where $\widetilde{A} = I_N \otimes A + \epsilon G_0 \otimes \Gamma$, $G_0 = G - \Theta$, $\widetilde{A}_d = I_N \otimes A_d$, $\widetilde{B} = I_N \otimes B$, $\widetilde{D} = I_N \otimes D$, and $\widetilde{C} = I_N \otimes C$. According to inequality (2), E_φ belongs to the sector $[0, I_N \otimes K]$, i.e.,

$$\begin{aligned} &0 \le \frac{e_{\varphi_{ij}}}{e_{y_{ij}}} = \frac{\varphi_j(y_{ij}) - \varphi_j(y_{s_{\mu_i}j})}{y_{ij} - y_{\mu_i j}} \le k_j \ \text{ and } \ \varphi_j(0) = 0, \\ &i = 1,2,\ldots,N, \quad j = 1,2,\ldots,m. \end{aligned} \tag{7}$$

Note that the Lurie network (1) realizes cluster synchronization if the error system (6) is absolutely stable for any nonlinearity satisfying the sector condition (7). The following lemma will be useful in this paper.

Lemma 1[1] Suppose that $\hat{R} = [\hat{r}_{ij}] \in \mathscr{R}^{N\times N}$ is an irreducible and symmetric matrix satisfying $\hat{r}_{ij} \ge 0$ for $i \ne j$ and $\sum_{j=1}^N \hat{r}_{ij} = 0$ for $i = 1,2,\ldots,N$. If $\hat{M} = \mathrm{diag}\{\hat{m}_1,\hat{m}_2,\ldots,\hat{m}_N\} > 0$ with at least one $\hat{m}_i > 0$. Then all eigenvalues of the matrix $\hat{R} - \hat{M}$ are negative.

3. Main results

In this section, under the local control scheme (4), the cluster synchronization problem of the Lurie network (1) will be investigated for two cases: Lurie subsystems without delay and Lurie subsystems with time-varying delay. Denote λ_i $(i = 1,2,\ldots,N)$ as the eigenvalues of the matrix G_0. Assume that the network is connected, it can be easily derived from Lemma 1 that λ_i is negative, i.e., $\lambda_N \le \cdots \le \lambda_2 \le \lambda_1 < 0$.

3.1. Lurie subsystems without delay

Firstly, consider the Lurie network (1) with each node being a Lurie system without delay, i.e., $A_d = 0$.

Theorem 1 The Lurie network (1) without delay can realize cluster synchronization with H_∞ performance index $\rho > 0$ under the control scheme (4), if a matrix $P > 0$ and a diagonal matrix $Q > 0$ exist, such that

$$\begin{bmatrix} P(A+\epsilon\lambda_i\Gamma) + (A+\epsilon\lambda_i\Gamma)^{\mathrm{T}}P & PB + C^{\mathrm{T}}Q & PD & C_z^{\mathrm{T}} \\ B^{\mathrm{T}}P + QC & -2QK^{-1} & 0 & 0 \\ D^{\mathrm{T}}P & 0 & -\rho^2 I_p & 0 \\ C_z & 0 & 0 & -I_q \end{bmatrix} < 0, \quad i = 1,2,\ldots,N. \tag{8}$$

Proof It follows from Theorem 1 in Ref. [17] and inequality (8) that the first condition in Definition 2 is satisfied.

Choose a Lyapunov–Krasovskii candidate as

$$V_1(t) = \sum_{i=1}^N e_i^{\mathrm{T}}(t)Pe_i(t). \tag{9}$$

Taking the derivative of V_1 along the trajectories of Eq. (5) gives

$$\begin{aligned} \dot{V}_1 &= \sum_{i=1}^N (e_i^{\mathrm{T}}P\dot{e}_i + \dot{e}_i^{\mathrm{T}}Pe_i) \\ &= \sum_{i=1}^N \Big[e_i^{\mathrm{T}}P\Big(Ae_i + Be_{\varphi_i} + D\omega_i + \epsilon\sum_{j=1}^N g_{ij}\Gamma e_j - \epsilon\theta_i\Gamma e_i\Big) \\ &\quad + \Big(Ae_i + Be_{\varphi_i} + D\omega_i + \epsilon\sum_{j=1}^N g_{ij}\Gamma e_j - \epsilon\theta_i\Gamma e_i\Big)^{\mathrm{T}}Pe_i \Big]. \end{aligned} \tag{10}$$

Define the following cost performance index

$$J_T = \int_0^T (z^{\mathrm{T}}(t)z(t) - \rho^2\omega^{\mathrm{T}}(t)\omega(t))\,\mathrm{d}t, \tag{11}$$

where $T > 0$ and $J = \lim_{T\to\infty} J_T$.

Then

$$\begin{aligned} &J_T \\ &= \int_0^T (z^{\mathrm{T}}(t)z(t) - \rho^2\omega^{\mathrm{T}}(t)\omega(t) + \dot{V}(t))\,\mathrm{d}t - (V(T) - V(0)) \end{aligned}$$

$$= \int_0^{\mathrm{T}} (z^{\mathrm{T}}(t)z(t) - \rho^2\omega^{\mathrm{T}}(t)\omega(t) + \dot{V}(t))\,\mathrm{d}t - V(T). \quad (12)$$

It can be derived from inequality (7) that $e_{\varphi_{ij}} e_{y_{ij}} \geq 0$, and

$$k_j^{-1} e_{\varphi_{ij}} e_{y_{ij}}^{-1} - 1 \leq 0. \quad (13)$$

Thus, multiplying expression (13) by $e_{\varphi_{ij}} e_{y_{ij}}$ obtains

$$e_{\varphi_{ij}}(k_j^{-1} e_{\varphi_{ij}} - e_{y_{ij}}) \leq 0,$$
$$i = 1,2,\ldots,N, \quad j = 1,2,\ldots,m. \quad (14)$$

By making use of $\mathscr{S}$-procedure[32] and Eq. (12), $J_T \leq 0$ for all nonzero $e_{\varphi_{ij}}$, if there exist scalars $r_{ij} > 0$, $i = 1,2,\ldots,N$, $j = 1,2,\ldots,m$, such that

$$J_T - 2\sum_{i=1}^{N}\sum_{j=1}^{m} r_{ij} e_{\varphi_{ij}}(k_j^{-1} e_{\varphi_{ij}} - e_{y_{ij}})$$
$$= \int_0^T \eta_1^{\mathrm{T}}(t)\Xi\eta_1(t)\,\mathrm{d}t - V(T) < 0, \quad (15)$$

where $\eta_1 = [e, E_\varphi, \omega]^{\mathrm{T}}$, $R = \mathrm{diag}\{r_{11},\ldots,r_{1m};\ldots;r_{N1},\ldots,r_{Nm}\}$, $\widetilde{P} = I_N \otimes P$ and

$$\Xi = \begin{bmatrix} \tilde{P}\tilde{A} + \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{C}_z^{\mathrm{T}}\tilde{C}_z & \tilde{P}\tilde{B} + \tilde{C}^{\mathrm{T}}R & \tilde{P}\tilde{D} \\ \tilde{P}\tilde{B} + \tilde{C}^{\mathrm{T}}R & -2R(I_N \otimes K^{-1}) & 0 \\ \tilde{D}^{\mathrm{T}}\tilde{P} & 0 & -\rho^2 I_{Np} \end{bmatrix}.$$

It follows from Schur complement that $\Xi < 0$ if and only if

$$\begin{bmatrix} \tilde{P}\tilde{A} + \tilde{A}^{\mathrm{T}}\tilde{P} & \tilde{P}\tilde{B} + \tilde{C}^{\mathrm{T}}R & \tilde{P}\tilde{D} & \tilde{C}_z^{\mathrm{T}} \\ \tilde{P}\tilde{B} + \tilde{C}^{\mathrm{T}}R & -2R(I_N \otimes K^{-1}) & 0 & 0 \\ \tilde{D}^{\mathrm{T}}\tilde{P} & 0 & -\rho^2 I_{Np} & 0 \\ \tilde{C}_z & 0 & 0 & -I_{Nq} \end{bmatrix} < 0.$$

Defining $U \in \mathscr{R}^{N\times N}$ as such a unitary matrix that $U^{\mathrm{T}}DU = \Lambda$, where the real-diagonal matrix $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. Set $\tilde{U}_1 = \mathrm{diag}\{U \otimes I_n, U \otimes I_m, U \otimes I_p, U \otimes I_q\}$. Then, one has

$$\tilde{U}_1^{\mathrm{T}}\Xi\tilde{U}_1 = \Xi_0 < 0, \quad (16)$$

where

$$\Xi_0 = \begin{bmatrix} A_0^{\mathrm{T}}\widetilde{P} + \widetilde{P}A_0 & \tilde{P}\tilde{B} + \tilde{C}^{\mathrm{T}}R & \tilde{P}\tilde{D} & \tilde{C}_z^{\mathrm{T}} \\ \tilde{P}\tilde{B} + \tilde{C}^{\mathrm{T}}R & -2R(I_N \otimes K^{-1}) & 0 & 0 \\ \tilde{D}^{\mathrm{T}}\tilde{P} & 0 & -\rho^2 I_{Np} & 0 \\ \tilde{C}_z & 0 & 0 & -I_{Nq} \end{bmatrix}$$

and $A_0 = I_N \otimes A + \epsilon\Lambda \otimes \Gamma$. Define $R = I_N \otimes Q$ and all the matrices in Ξ_0 are block-diagonal. It is clear from inequality (8) that inequality (16) holds. From expressions (15), (16), and $V(T) \geq 0$, we can obtain that $J_T \leq 0$. Let $T \to \infty$. It is easy to see that $J \leq 0$.

Furthermore, the cluster synchronization region of Lurie network (1) without delay will be studied. It can demonstrate the relationship between cluster synchronization and structure parameters of the networks.

Setting $e = (U \otimes I_n)\hat{e}$, $E_\varphi = (U \otimes I_m)\hat{E}_\varphi$, $\omega = (U \otimes I_p)\hat{\omega}$, and $e_y = (U \otimes I_m)\hat{e}_y$, where $\hat{e} = [\hat{e}_1^{\mathrm{T}}, \hat{e}_2^{\mathrm{T}}, \ldots, \hat{e}_N^{\mathrm{T}}]^{\mathrm{T}}$, $\hat{E}_\varphi = [\hat{e}_{\varphi_1}^{\mathrm{T}}, \hat{e}_{\varphi_2}^{\mathrm{T}}, \ldots, \hat{e}_{\varphi_N}^{\mathrm{T}}]^{\mathrm{T}}$, $\hat{\omega} = [\hat{\omega}_1^{\mathrm{T}}, \hat{\omega}_2^{\mathrm{T}}, \ldots, \hat{\omega}_N^{\mathrm{T}}]^{\mathrm{T}}$, and $\hat{e}_y = [\hat{e}_{y1}^{\mathrm{T}}, \hat{e}_{y2}^{\mathrm{T}}, \ldots, \hat{e}_{yN}^{\mathrm{T}}]^{\mathrm{T}}$, system (6) without delay is equivalent to

$$\dot{\hat{e}}_i = (A + \epsilon\lambda_i\Gamma)\hat{e}_i + B\hat{e}_{\varphi_i} + D\hat{\omega}_i,$$
$$\hat{e}_{yi} = C\hat{e}_i, \quad i = 1,2,\ldots,N.$$

Therefore, the cluster synchronization of the Lurie network (1) without delay can be recast into analyzing the following system:

$$\dot{\xi} = (A + \delta\Gamma)\xi + B\hat{e}_{\varphi_i} + D\hat{\omega}_i,$$
$$\hat{e}_{yi} = C\xi, \quad i = 1,2,\ldots,N. \quad (17)$$

where $\xi \in \mathscr{R}^n$ and $\delta \in \mathscr{R}$ depending on ϵ and λ_i.

Definition 3 The region $\Delta \subseteq R$ of parameter δ is called the cluster synchronized region of Lurie network (1) without delay under the control scheme (4), if system (17) is asymptotically stable.

The following corollary is derived from Theorem 1.

Corollary 1 The region $\Delta \subseteq R$ of parameter δ is the cluster-synchronized region of Lurie network (1) with no delay and H_∞ performance index $\rho > 0$ under the control scheme (4), if a matrix $P > 0$ and a diagonal matrix $Q > 0$ exist satisfying

$$\begin{bmatrix} PA + A^{\mathrm{T}}P + \delta(P\Gamma + \Gamma^{\mathrm{T}}P) & PB + C^{\mathrm{T}}Q & PD & C_z^{\mathrm{T}} \\ B^{\mathrm{T}}P + QC & -2QK^{-1} & 0 & 0 \\ D^{\mathrm{T}}P & 0 & -\rho^2 I_p & 0 \\ C_z & 0 & 0 & -I_q \end{bmatrix} < 0. \quad (18)$$

Remark 1 Due to the coupling strength $\epsilon > 0$ and $\lambda_i < 0$, the cluster synchronized region Δ is an interval or the union of several intervals on the left real axis. Note that it may be bounded or unbounded.

Next, we focus on how to choose the inner-coupling matrix Γ to yield an unbounded cluster-synchronized region.

Theorem 2 For a given matrix Γ_1, there exists an inner-coupling matrix $\Gamma = \Gamma_2\Gamma_1$ such that Lurie network (1) with no delay and H_∞ performance index $\rho > 0$ under the control scheme (4) yields an unbounded cluster-synchronized region $\Delta = (-\infty, \delta_0]$ for $\delta_0 < 0$, if and only if a matrix $P > 0$ and a diagonal matrix $Q > 0$ exist, such that

$$\begin{bmatrix} PA + A^{\mathrm{T}}P + \delta_0\Gamma_1^{\mathrm{T}}\Gamma_1 & PB + C^{\mathrm{T}}Q & PD & C_z^{\mathrm{T}} \\ B^{\mathrm{T}}P + QC & -2QK^{-1} & 0 & 0 \\ D^{\mathrm{T}}P & 0 & -\rho^2 I_p & 0 \\ C_z & 0 & 0 & -I_q \end{bmatrix} < 0. \quad (19)$$

Proof It follows from Schur complement that inequality (19) is equivalent to

$$A^{\mathrm{T}}P + PA + \delta_0\Gamma_1^{\mathrm{T}}\Gamma_1 + \frac{1}{2}(PB + C^{\mathrm{T}}Q)KQ^{-1}(B^{\mathrm{T}}P + QC)$$
$$+ \frac{1}{\rho^2}PD^{\mathrm{T}}DP + C_z^{\mathrm{T}}C_z < 0. \quad (20)$$

Necessity 1 According to Definition 3, if the cluster-synchronized region of network (1) without delay under the control scheme (4) is unbounded, inequality (18) holds for

some matrix Γ, matrix $P>0$, diagonal matrix $Q>0$, and parameter δ. By means of Schur complement, inequality (18) with $\delta=-1$ holds, if and only if a matrix $P>0$ and a diagonal matrix $Q>0$ exist, such that

$$\begin{aligned}&A^{\mathrm{T}}P+PA-\Gamma_1^{\mathrm{T}}\Gamma_2^{\mathrm{T}}P-P\Gamma_2\Gamma_1\\&\quad+\frac{1}{2}(PB+C^{\mathrm{T}}Q)KQ^{-1}(B^{\mathrm{T}}P+QC)+\frac{1}{\rho^2}PD^{\mathrm{T}}DP\\&\quad+C_z^{\mathrm{T}}C_z<0.\end{aligned}\tag{21}$$

By introducing a new variable $\Gamma_0=P\Gamma_2$, inequality (21) becomes

$$\begin{aligned}&A^{\mathrm{T}}P+PA-\Gamma_1^{\mathrm{T}}\Gamma_0^{\mathrm{T}}-\Gamma_0\Gamma_1\\&\quad+\frac{1}{2}(PB+C^{\mathrm{T}}Q)KQ^{-1}(B^{\mathrm{T}}P+QC)\\&\quad+\frac{1}{\rho^2}PD^{\mathrm{T}}DP+C_z^{\mathrm{T}}C_z<0.\end{aligned}\tag{22}$$

According to Finsler's Lemma,[33] there exists a matrix Γ_0 satisfying inequality (22), if and only if there exists a scalar $\delta_0<0$, such that inequality (20) holds.

Sufficiency 1 If $\delta_0\geq\epsilon\lambda_i$, $i=1,2,\ldots,N$, and take $\Gamma_2=P^{-1}\Gamma_1^{\mathrm{T}}/2$, one has

$$\begin{aligned}&(A+\epsilon\lambda_i\Gamma)^{\mathrm{T}}P+P(A+\epsilon\lambda_i\Gamma)\\&+\frac{1}{2}(PB+C^{\mathrm{T}}Q)KQ^{-1}(B^{\mathrm{T}}P+QC)\\&+\frac{1}{\rho^2}PD^{\mathrm{T}}DP+C_z^{\mathrm{T}}C_z\\&=A^{\mathrm{T}}P+PA+\epsilon\lambda_i\Gamma_1^{\mathrm{T}}\Gamma_1\\&+\frac{1}{2}(PB+C^{\mathrm{T}}Q)KQ^{-1}(B^{\mathrm{T}}P+QC)\\&+\frac{1}{\rho^2}PD^{\mathrm{T}}DP+C_z^{\mathrm{T}}C_z<0,\end{aligned}\tag{23}$$

which is equivalent to inequality (8) that holds by means of Schur complement. According to Theorem 1, the Lurie network (1) without delay under the controller (4) with $\Gamma=P^{-1}\Gamma_1^{\mathrm{T}}\Gamma_1/2$ has an unbounded cluster-synchronized region $(-\infty,\delta_0]$. This completes the proof.

According to Theorem 2, the largest unbounded cluster-synchronized region will be discussed. Let the largest unbounded cluster-synchronized region $\Delta_m=(-\infty,\delta_m]$, and it can be obtained in the following corollary.

Corollary 2 For a given matrix Γ_1, δ_m can be obtained by solving the following optimization problem

$$\begin{aligned}&\text{maximize}\ \ \delta_0,\\&\text{subject to LMI (19) with a matrix}\\&P>0\ \text{and a diagonal matrix}\ Q>0.\end{aligned}$$

Moreover, the inner-linking matrix $\Gamma=P^{-1}\Gamma_1^{\mathrm{T}}\Gamma_1/2$, and the coupling strength ϵ is not less than $\delta_m\lambda_1^{-1}$.

3.2. Lurie subsystems with time-varying delays

Theorem 3 The Lurie network (1) with a time-varying delay $\tau(t)$ satisfying expression (3) can realize cluster synchronization with H_∞ performance index $\rho>0$ under the control scheme (4), if there exist matrices $P>0$, $Q>0$, $S>0$, W, and a diagonal matrix $M>0$ such that

$$\begin{bmatrix}\overline{A}_0-W^{\mathrm{T}}-W+S & PA_d+W^{\mathrm{T}}+W & PB+C^{\mathrm{T}}M & hW^{\mathrm{T}} & PD & h\overline{A}^{\mathrm{T}}Q & C_z^{\mathrm{T}}\\ A_d^{\mathrm{T}}P+W^{\mathrm{T}}+W & -(1-\upsilon)S-W^{\mathrm{T}}-W & 0 & -hW^{\mathrm{T}} & 0 & hA_d^{\mathrm{T}}Q & 0\\ B^{\mathrm{T}}P+MC & 0 & -2MK^{-1} & 0 & 0 & hB^{\mathrm{T}}Q & 0\\ hW & -hW & 0 & -hQ & 0 & 0 & 0\\ D^{\mathrm{T}}P & 0 & 0 & 0 & -\rho^2I_p & 0 & 0\\ hQ\overline{A} & hQA_d & hQB & 0 & 0 & -hQ & 0\\ C_z & 0 & 0 & 0 & 0 & 0 & -I_q\end{bmatrix}<0,\tag{24}$$

where $\overline{A}_0=(A+\epsilon\lambda_i\Gamma)^{\mathrm{T}}P+P(A+\epsilon\lambda_i\Gamma)$, $i=1,2,\ldots,N$.

Proof It follows from Theorem 4 in Ref. [17] and inequality (24) that the first condition in Definition 2 is satisfied. Let $e_{it}=e_i(t+\widetilde{\theta}),-2h\leq\widetilde{\theta}\leq0$, $i=1,2,3$. Choose a Lyapunov–Krasovskii function as

$$V_2(e_{it})=V_{21}(e_{it})+V_{22}(e_{it})+V_{23}(e_{it}),\tag{25}$$

where

$$\begin{aligned}V_{21}(e_{it})&=\sum_{i=1}^{N}e_i^{\mathrm{T}}(t)Pe_i(t),\\V_{22}(e_{it})&=\sum_{i=1}^{N}\int_{-h}^{0}\int_{t+\beta}^{t}\dot{e}_i^{\mathrm{T}}(\alpha)Q\dot{e}_i(\alpha)\mathrm{d}\alpha\mathrm{d}\beta,\\V_{23}(e_{it})&=\sum_{i=1}^{N}\int_{t-\tau(t)}^{t}e_i^{\mathrm{T}}(\alpha)Se_i(\alpha)\mathrm{d}\alpha,\end{aligned}$$

equation (5) can be rewritten as

$$\begin{aligned}\dot{e}_i(t)&=(A-J)e_i(t)+(A_d+J)e_i(t-\tau(t))\\&\quad+Be_{\varphi_i}(t)+D\omega_i(t)+\epsilon\sum_{j=1}^{N}g_{ij}\Gamma e_j(t)\\&\quad-\epsilon\theta_i\Gamma e_i(t)+Je_i(t)-Je_i(t-\tau(t)),\\&\quad i=1,2,\ldots,N,\end{aligned}\tag{26}$$

where J is a constant matrix. Set $W=PJ$ and use the Newton–Leibniz formula

$$e_i(t)=e_i(t-\tau(t))+\int_{t-\tau(t)}^{t}\dot{e}_i(\alpha)\mathrm{d}\alpha.$$

Calculating $\dot{V}_2$ $(i=1,2,3)$ along the solution of Eq. (26), it follows that

$$\dot{V}_2 = \dot{V}_{21}(e_{it}) + \dot{V}_{22}(e_{it}) + \dot{V}_{23}(e_{it}), \tag{27}$$

$$\begin{aligned}
\dot{V}_{21}(e_{it}) &= e_i^{\mathrm{T}}(t)(A^{\mathrm{T}}P + PA - W^{\mathrm{T}} - W)e_i(t) \\
&\quad + e_i^{\mathrm{T}}(t)(PA_d + W)e_i(t-\tau(t)) + e_i^{\mathrm{T}}(t)PD\omega_i(t) \\
&\quad + \omega_i^{\mathrm{T}}(t)D^{\mathrm{T}}Pe_i(t) + e_i(t-\tau(t))(A_d^{\mathrm{T}}P + W^{\mathrm{T}})e_i(t) \\
&\quad + e_{\varphi_i}^{\mathrm{T}}B^{\mathrm{T}}Pe_i(t) + e_i^{\mathrm{T}}(t)PBe_{\varphi_i} \\
&\quad + (e_i(t) - e_i(t-\tau(t)))^{\mathrm{T}}W^{\mathrm{T}} \\
&\quad \times \left(e_i(t-\tau(t)) + \int_{t-\tau(t)}^{t} \dot{e}_i(\alpha)\,\mathrm{d}\alpha\right) \\
&\quad + \left(e_i(t-\tau(t)) + \int_{t-\tau(t)}^{t} \dot{e}_i(\alpha)\,\mathrm{d}\alpha\right)^{\mathrm{T}} \\
&\quad \times W(e_i(t) - e_i(t-\tau(t))), \\
\dot{V}_{22}(e_{it}) &= \int_{-h}^{0} (\dot{e}_i^{\mathrm{T}}(t)Q\dot{e}_i(t) - \dot{e}_i^{\mathrm{T}}(t+\beta)Q\dot{e}_i(t+\beta))^{\mathrm{T}}\,\mathrm{d}\beta \\
&= \int_{t-h}^{t} (\dot{e}_i^{\mathrm{T}}(t)Q\dot{e}_i(t) - \dot{e}_i^{\mathrm{T}}(\alpha)Q\dot{e}_i(\alpha))^{\mathrm{T}}\,\mathrm{d}\alpha, \\
\dot{V}_{23}(e_{it}) &= e_i^{\mathrm{T}}(t)Se_i(t) - (1-\dot{\tau}(t))e_i^{\mathrm{T}}(t-\tau(t))Se_i(t-\tau(t)), \\
&\quad i = 1,2,\ldots,N.
\end{aligned}$$

As similar as the proof of Theorem 1, by making use of $\mathscr{S}$-procedure,[33] expressions (11), (14), and (25)–(27), $J_T \le 0$ for all nonzero $e_{\varphi_{ij}}$, if there exist scalars $\widetilde{r}_{ij} > 0$, $i = 1,2,\ldots,N$, $j = 1,2,\ldots,m$, such that

$$\begin{aligned}
&J_T - 2\sum_{i=1}^{N}\sum_{j=1}^{m} \widetilde{r}_{ij} e_{\varphi_{ij}}(k_i^{-1}e_{\varphi_{ij}} - e_{y_{ij}}) \\
&\le \frac{1}{h}\int_{t-h}^{t} \eta_2^{\mathrm{T}}(\Xi_1 + h\Xi_2)\eta_2\,\mathrm{d}\alpha - V(T) < 0,
\end{aligned} \tag{28}$$

where $\eta_2 = [e(t), e(t-\tau(t)), \dot{e}(\alpha), E_\varphi(t), \omega]^{\mathrm{T}}$, $e_t = [e_{1t}^{\mathrm{T}}, \ldots, e_{Nt}^{\mathrm{T}}]^{\mathrm{T}}$, $\widetilde{A}_d = I_N \otimes A_d$, $\widetilde{P} = I_N \otimes P$, $\widetilde{Q} = I_N \otimes Q$, $\widetilde{S} = I_N \otimes S$, $\widetilde{W} = I_N \otimes W$, $\widetilde{R} = \mathrm{diag}\{\widetilde{r}_{11}, \ldots, \widetilde{r}_{1m}; \ldots; \widetilde{r}_{N1}, \ldots, \widetilde{r}_{Nm}\}$,

$$\Xi_1 = \begin{bmatrix}
\Xi_{11} & \widetilde{P}\widetilde{A}_d + \widetilde{W}^{\mathrm{T}} + \widetilde{W} & \widetilde{P}\widetilde{B} + \widetilde{C}^{\mathrm{T}}\widetilde{R} & h\widetilde{W}^{\mathrm{T}} & \widetilde{P}\widetilde{D} \\
\widetilde{A}_d^{\mathrm{T}}\widetilde{P} + \widetilde{W}^{\mathrm{T}} + \widetilde{W} & -(1-\upsilon)\widetilde{S} - \widetilde{W}^{\mathrm{T}} - \widetilde{W} & 0 & -h\widetilde{W}^{\mathrm{T}} & 0 \\
\widetilde{B}^{\mathrm{T}}\widetilde{P} + \widetilde{R}\widetilde{C} & 0 & -2\widetilde{Q}(I_N \otimes K^{-1}) & 0 & 0 \\
h\widetilde{W} & -h\widetilde{W} & 0 & -h\widetilde{Q} & 0 \\
\widetilde{D}^{\mathrm{T}}\widetilde{P} & 0 & 0 & 0 & -\rho^2 I_{Np}
\end{bmatrix},$$

$\Xi_{11} = \widetilde{A}^{\mathrm{T}}\widetilde{P} + \widetilde{P}\widetilde{A} - \widetilde{W}^{\mathrm{T}} - \widetilde{W} + \widetilde{S} + \widetilde{C}_z^{\mathrm{T}}\widetilde{C}_z$, and

$$\Xi_2 = \begin{bmatrix} \widetilde{A}^{\mathrm{T}}\widetilde{Q} \\ \widetilde{A}_d^{\mathrm{T}}\widetilde{Q} \\ \widetilde{B}^{\mathrm{T}}\widetilde{Q} \\ 0 \\ 0 \end{bmatrix} \widetilde{Q}^{-1} \begin{bmatrix} \widetilde{A}^{\mathrm{T}}\widetilde{Q} \\ \widetilde{A}_d^{\mathrm{T}}\widetilde{Q} \\ \widetilde{B}^{\mathrm{T}}\widetilde{Q} \\ 0 \\ 0 \end{bmatrix}^{\mathrm{T}}.$$

It follows from Schur complement that $\Xi_1 + h\Xi_2 < 0$ is equivalent to the following inequality

$$\begin{bmatrix}
\widetilde{A}^{\mathrm{T}}\widetilde{P} + \widetilde{P}\widetilde{A} - \widetilde{W}^{\mathrm{T}} - \widetilde{W} + \widetilde{S} & \widetilde{P}\widetilde{A}_d + \widetilde{W}^{\mathrm{T}} + \widetilde{W} & \widetilde{P}\widetilde{B} + \widetilde{C}^{\mathrm{T}}\widetilde{R} & h\widetilde{W}^{\mathrm{T}} & \widetilde{P}\widetilde{D} & h\widetilde{A}^{\mathrm{T}}\widetilde{Q} & \widetilde{C}_z^{\mathrm{T}} \\
\widetilde{A}_d^{\mathrm{T}}\widetilde{P} + \widetilde{W}^{\mathrm{T}} + \widetilde{W} & -(1-\upsilon)\widetilde{S} - \widetilde{W}^{\mathrm{T}} - \widetilde{W} & 0 & -h\widetilde{W}^{\mathrm{T}} & 0 & h\widetilde{A}_d^{\mathrm{T}}\widetilde{Q} & 0 \\
\widetilde{B}^{\mathrm{T}}\widetilde{P} + \widetilde{R}\widetilde{C} & 0 & -2\widetilde{R}(I_N \otimes K^{-1}) & 0 & 0 & h\widetilde{B}^{\mathrm{T}}\widetilde{Q} & 0 \\
h\widetilde{W} & -h\widetilde{W} & 0 & -h\widetilde{Q} & 0 & 0 & 0 \\
\widetilde{D}^{\mathrm{T}}\widetilde{P} & 0 & 0 & 0 & -\rho^2 I_{Np} & 0 & 0 \\
h\widetilde{Q}\widetilde{A} & h\widetilde{Q}\widetilde{A}_d & h\widetilde{Q}\widetilde{B} & 0 & 0 & -h\widetilde{Q} & 0 \\
\widetilde{C}_z & 0 & 0 & 0 & 0 & 0 & -I_{Nq}
\end{bmatrix} < 0.$$

Defining $\widetilde{U}_2 = \mathrm{diag}\{U \otimes I_n, U \otimes I_n, U \otimes I_m, U \otimes I_n, U \otimes I_p, U \otimes I_n, U \otimes I_q\}$, where $U \in \mathscr{R}^{N\times N}$ is a unitary matrix satisfying $U^{\mathrm{T}}DU = \Lambda$, one has

$$\widetilde{U}_2^{\mathrm{T}}(\Xi_1 + h\Xi_2)\widetilde{U}_2 = \widetilde{\Xi}_0 < 0,$$

where

$$\begin{bmatrix}
A_0 - \widetilde{W}^{\mathrm{T}} - \widetilde{W} + \widetilde{S} & \widetilde{P}\widetilde{A}_d + \widetilde{W}^{\mathrm{T}} + \widetilde{W} & \widetilde{P}\widetilde{B} + \widetilde{C}^{\mathrm{T}}\widetilde{R} & h\widetilde{W}^{\mathrm{T}} & \widetilde{P}\widetilde{D} & h\widetilde{A}^{\mathrm{T}}\widetilde{Q} & \widetilde{C}_z^{\mathrm{T}} \\
\widetilde{A}_d^{\mathrm{T}}\widetilde{P} + \widetilde{W}^{\mathrm{T}} + \widetilde{W} & -(1-\upsilon)\widetilde{S} - \widetilde{W}^{\mathrm{T}} - \widetilde{W} & 0 & -h\widetilde{W}^{\mathrm{T}} & 0 & h\widetilde{A}_d^{\mathrm{T}}\widetilde{Q} & 0 \\
\widetilde{B}^{\mathrm{T}}\widetilde{P} + \widetilde{R}\widetilde{C} & 0 & -2\widetilde{R}(I_N \otimes K^{-1}) & 0 & 0 & h\widetilde{B}^{\mathrm{T}}\widetilde{Q} & 0 \\
h\widetilde{W} & -h\widetilde{W} & 0 & -h\widetilde{Q} & 0 & 0 & 0 \\
\widetilde{D}^{\mathrm{T}}\widetilde{P} & 0 & 0 & 0 & -\rho^2 I_p & 0 & 0 \\
h\widetilde{Q}\widetilde{A} & h\widetilde{Q}\widetilde{A}_d & h\widetilde{Q}\widetilde{B} & 0 & 0 & -h\widetilde{Q} & 0 \\
\widetilde{C}_z & 0 & 0 & 0 & 0 & 0 & -I_q
\end{bmatrix},$$

and

$$A_0 = (I_N \otimes A + \epsilon\Lambda \otimes \Gamma)^{\mathrm{T}}\widetilde{P} + \widetilde{P}(I_N \otimes A + \epsilon\Lambda \otimes \Gamma).$$

Defining $\widetilde{R} = I_N \otimes M$, this result can be obtained by following the proof in Theorem 1. This completes the proof.

Similar to Definition 3, cluster-synchronized region of Lurie network (1) under the control scheme (4) is developed.

Definition 4 The region $\Delta \subseteq R$ of parameter δ is called the cluster-synchronized region of the Lurie network (1) under the controllers (4), if the following system

$$\dot{\hat{\xi}}(t) = (A + \delta\Gamma)\hat{\xi}(t) + A_d\hat{\xi}(t - \tau(t)) + B\hat{e}_{\varphi_i(t)}(t) + D\hat{\omega}_i(t),$$
$$\hat{e}_{yi}(t) = C\hat{\xi}(t),$$

is asymptotically stable, where $\hat{\xi}(t), \hat{\xi}(t - \tau(t)) \in \mathscr{R}^n$ and $\delta \in \mathscr{R}$ depending on ϵ and λ_i.

According to Theorem 3, the following corollary can be obtained:

Corollary 3 The region $\Delta \subseteq R$ of parameter δ is the cluster-synchronized region of the Lurie delayed network (1) with H_∞ performance index $\rho > 0$ under the control scheme (4), if there exist matrices $P > 0$, $Q > 0$, $S > 0$, W, and a diagonal matrix $M > 0$, such that

$$\begin{bmatrix}
\hat{A}_0 - W^{\mathrm{T}} - W + S & PA_d + W^{\mathrm{T}} + W & PB + C^{\mathrm{T}}M & hW^{\mathrm{T}} & PD & h\bar{A}^{\mathrm{T}}Q & C_z^{\mathrm{T}} \\
A_d^{\mathrm{T}}P + W^{\mathrm{T}} + W & -(1-\upsilon)S - W^{\mathrm{T}} - W & 0 & -hW^{\mathrm{T}} & 0 & hA_d^{\mathrm{T}}Q & 0 \\
B^{\mathrm{T}}P + MC & 0 & -2MK^{-1} & 0 & 0 & hB^{\mathrm{T}}Q & 0 \\
hW & -hW & 0 & -hQ & 0 & 0 & 0 \\
D^{\mathrm{T}}P & 0 & 0 & 0 & -\rho^2 I_p & 0 & 0 \\
hQ\bar{A} & hQA_d & hQB & 0 & 0 & -hQ & 0 \\
C_z & 0 & 0 & 0 & 0 & 0 & -I_q
\end{bmatrix} < 0, \tag{29}$$

where $\hat{A}_0 = (A + \delta\Gamma)^{\mathrm{T}}P + P(A + \delta\Gamma)$.

Remark 2 In most existing works about the cluster synchronization of networks,[11–16] the inner-linking matrix describing the interaction of variables was assumed to be a diagonal constant matrix. In this paper, cluster synchronization can be achieved with a general inner-linking matrix. It may be a symmetric or nonsymmetric matrix. Particularly, in view of Theorem 2, for giving the nodes dynamics in Lurie subsystem and matrix Γ_1, matrix Γ will be designed to yield an unbounded cluster synchronization region of Lurie network (1) under the control scheme (4).

Remark 3 In view of expressions (8), (19), (24), the cluster synchronization problem of high-dimensional Lurie dynamical networks can be converted into solving a series of much lower-dimensional LMIs, which is numerically easy to verify and may help to reduce the computation complexity significantly if the number of nodes in networks is large. Particularly, according to Corollaries 1 and 3, the cluster-synchronized region of this network can be introduced by making use of LMIs feasible problems.

4. Simulation examples

In this section, two numerical examples will be provided to verify the theoretical analyses obtained in the previous sections. For simplicity, assume that the matrices G and Θ have 0∼1 elements.

Example 1 Consider a network composed of seventeen identical nodes without delay as shown in Fig. 1, and it is divided into three clusters: $\mathscr{C}_1 = \{1,2,3,4,5,6\}$, $\mathscr{C}_2 = \{7,8,9,10,11,12\}$, and $\mathscr{C}_3 = \{13,14,15,16,17\}$. Only the nodes 2, 6, 7, 9, 13, 15 are controlled. The uncoupled subsystem is described by the following Chua circuit

$$\begin{aligned}
\dot{x}_1 &= \alpha(-x_2(t) + x_1(t) - f(x_1(t))), \\
\dot{x}_2 &= x_1(t) - x_2(t) + x_3(t), \\
\dot{x}_3 &= -\beta x_2(t) - \gamma x_3(t),
\end{aligned} \tag{30}$$

where $f(x_1(t)) = m_1x_1(t) + (m_0 - m_1)(|x_1(t) + 1| - |x_1(t) - 1|)/2$, $\alpha = 10$, $\beta = 15$, $\gamma = 0.0385$, $m_0 = 1.27$, and $m_1 = 0.68$.

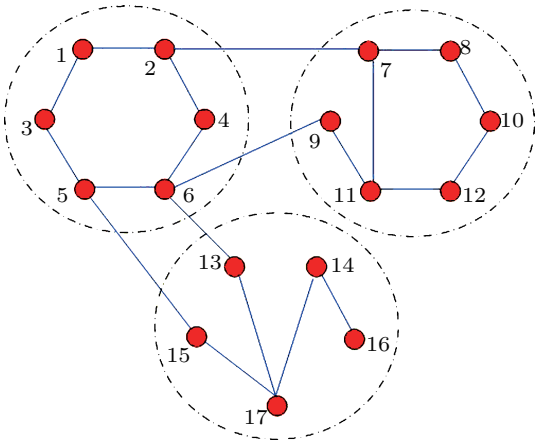


Fig. 1. (color online) The topology of a network in Example 1.

Let

$$B = \begin{bmatrix} 5.9 \\ 0 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}, \quad \omega = \begin{bmatrix} 10 \\ 1 \\ 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}, \quad C_z = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix},$$

and $\varphi(Cx) = \frac{1}{2}(|x_1(t) + 1| - |x_1(t) - 1|)$ belonging to a slope $[0,1]$ in Lurie network (1). Choose the H_∞ performance index $\rho = 2$.

It can be solved that inequality (8) is feasible for nonsymmetric inner-linking matrix

$$\Gamma = \begin{bmatrix} 7 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{31}$$

and coupling strength $\epsilon = 58$. According to Theorem 1, cluster synchronization can be realized in the two cases, and the state trajectories of all nodes are shown in Fig. 2. Next, consider the unbounded cluster-synchronized region with $\Gamma_1 = [-1,-1,0.5]$. Solve LMI (19) for feasible solutions

$$P = \begin{bmatrix} 1.1316 & -1.4999 & -0.3723 \\ -1.4999 & 9.9371 & -0.9018 \\ -0.3723 & -0.9018 & 0.7196 \end{bmatrix}.$$

Then for the above matrix P, the network with

$$\Gamma = \begin{bmatrix} 0.8249 & 0.8249 & -0.4124 \\ 0.2054 & 0.2054 & -0.1027 \\ 0.3367 & 0.3367 & -0.1684 \end{bmatrix} \tag{32}$$

has an unbounded cluster-synchronized region $(-\infty,-20.4]$ and the coupling strength $\epsilon \geq \delta_m \lambda_1^{-1} \approx 60.7324$. The state trajectories of each node with $\epsilon = 61$ are shown in Fig. 3.

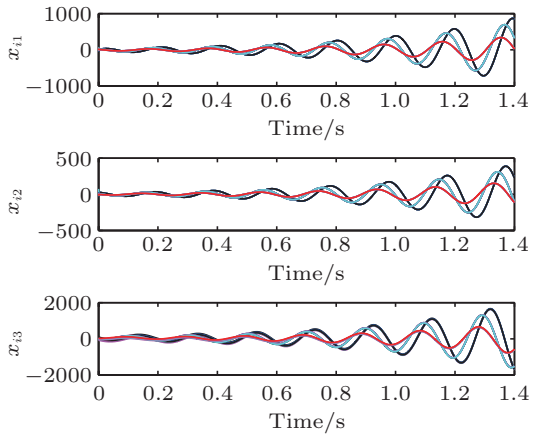


Fig. 2. (color online) States trajectories of the Lurie network in Example 1 with Γ in Eq. (31).

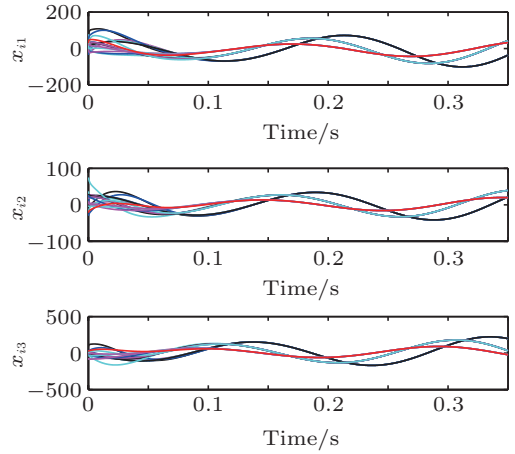


Fig. 3. (color online) States trajectories of the Lurie network in Example 1 with Γ in Eq. (32) and $\epsilon = 61$.

Example 2 Consider a network composed of six identical nodes with delays as shown in Fig. 4, and it is divided into two clusters: $\mathscr{C}_1 = \{1,2\}$, and $\mathscr{C}_2 = \{3,4,5,6\}$. Only the nodes 2 and 4 are controlled. The uncoupled subsystem is described by the following two-dimensional Lurie networks (1) with the following parameters

$$A = \begin{bmatrix} -8 & 1 \\ 0 & -7.5 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.4 & 1 \\ -1 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.2 \\ -0.3 \end{bmatrix}, \quad D = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $C_z = \begin{bmatrix} 1 & 5.5 \end{bmatrix}$, $\varphi(Cx_i) = (|x_{i1}(t)+1| - |x_{i1}(t)-1|)/2$ $(i = 1,2,3,4,5)$ belonging to the slope $[0,1]$ and $\tau(t) = 1.2$. Choose the H_∞ performance index $\rho = 2$. It can be verified that LMI (24) with inner-linking matrix $\Gamma = I_2$ and the coupling strength $\epsilon = 8$ is feasible. According to Theorem 3, cluster synchronization can be realized and the state trajectories of all nodes are shown in Fig. 5.

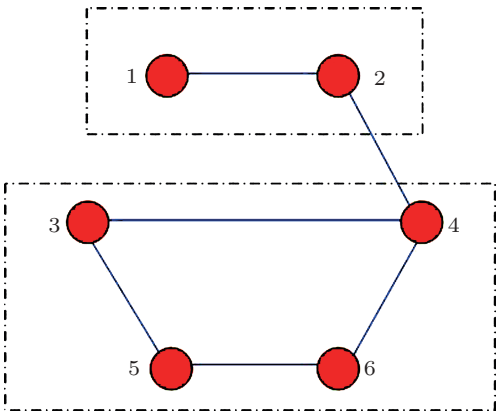


Fig. 4. (color online) The topology of a network in Example 2.

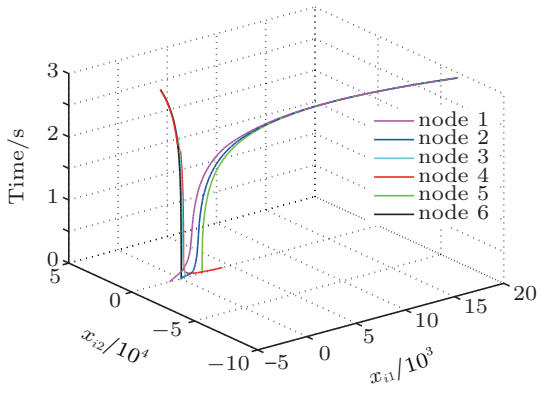


Fig. 5. (color online) States trajectories of the Lurie network in Example 2.

5. Conclusions

In this paper, we focus on the cluster synchronization problem of complex dynamical networks with each node being a Lurie system with external disturbances and time-varying delay. Under a local linear control strategy, sufficient conditions for achieving cluster synchronization with desired H_∞ performance are considered for two cases: Lurie subsystem with no delay and with time-varying delay. The notion of the cluster-synchronized region and some conditions guaranteeing the cluster-synchronized region are derived. It is noted that the cluster synchronization in the Lurie dynamical networks can be converted into solving a series of LMIs, and the dimension of LMIs is the same as that of a single node dynamical system.

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