Solution of Dirac equation around a charged rotating black hole*

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The aim of this paper is to solve the radial parts of a Dirac equation in Kerr–Newman (KN) geometry. The potential is replaced by a collection of step functions, then the reflection and transmission coefficients as well as the solution of the wave equation are obtained by using a quantum mechanical method. The result shows that the waves with different values of mass will be scatted off very differently.

Keywords: Dirac equation, Kerr–Newman space, tortoise coordinate

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1. Introduction

The study of black hole physics has aroused much interest since the birth of black holes as the most striking predictions of general relativity. Particularly, more and more people have paid close attention to the quantum properties of black holes since the discovery of Hawking radiation.[1] By now, various methods and techniques have been used to study the radiation or scattering of different particles from different black hole spaces.[2–23] The scattering of Dirac particles has also been studied by many authors[24–33] since a Dirac equation governing massive spin-1/2 particles in Kerr geometry was separated by Chandrasekhar.[34,35] The behaviors of Dirac particles around Schwarzschild and Kerr black holes have been studied by many researchers[28–32] using a quantum mechanical method. They obtained the reflection and transmission coefficients as well as the solution of the incoming Dirac wave in the whole range. The present work has the advantage over the earlier work in which only the solutions close to the horizon(s) were presented. Following the studies in Refs. [28]–[32], we investigate the scattering of Dirac particles from Schwarzschild–de Sitter (SdS) space and Reissner–Nordström–de Sitter (RNdS) space,[36,37] which has seldom been studied in the aspect of the existence of the cosmological constant. While the SdS space and RNdS space are all spherical symmetric, in order to further study the scattering of Dirac particles from a general stationary black hole, we extend our work to a Kerr–Newman (KN) black hole in this paper. The KN black hole is in fact a charged, rotating black hole and it has two horizons. The region of physical interest is outside the outer horizon. We consider two cases, one is that the incoming waves do not hit the potential barrier (i.e. $\sigma^2 > V_+$), and the other is that the incoming waves hit the potential barrier (i.e. $\sigma^2 < V_+$).

The rest of this paper is organized as follows. In Section 2, we introduce the wave equation in KN space. In Section 3, we use a quantum mechanical method to solve the wave equation and the reflection and transmission coefficients. Then we compare the solutions for several different parameters. The conclusion is given in Section 4.

The sign $(-+++)$ is adopted, and $\hbar$, c, and G (Newton's gravitational constant) are all taken to be unity.

2. Wave equation

The line element of KN space time reads

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1-\frac{2Mr-Q^2}{\rho^2}\right)\mathrm{d}t^2+\frac{\rho^2}{\Delta}\mathrm{d}r^2+\rho^2\mathrm{d}\theta^2 \\ &+\left[(r^2+a^2)\sin^2\theta+\frac{(2Mr-Q^2)a^2\sin^4\theta}{\rho^2}\right]\mathrm{d}\phi^2 \\ &-\frac{2(2Mr-Q^2)a\sin^2\theta}{\rho^2}\mathrm{d}t\,\mathrm{d}\phi, \end{aligned} \tag{1}$$

where $\rho^2 = r^2+a^2\cos^2\theta$; $\Delta = r^2-2Mr+a^2+Q^2$, with M being the mass of the black hole, Q the charge of the black hole, and $a = J/M$; J is the angular momentum of the black hole.

Following the Chandrasekhar work,[34,35] we separate the Dirac equation in KN geometry into radial and angular equations. The equations governing the radial wave-functions $R_{\pm1/2}$ corresponding to spin $\pm1/2$ are given, respectively, by

$$\mathcal{D}_0 R_{-1/2} = (\lambda+\mathrm{i}mr)R_{+1/2}, \tag{2a}$$

$$\Delta\mathcal{D}_0^\dagger R_{+1/2} = (\lambda-\mathrm{i}mr)R_{-1/2}, \tag{2b}$$

where the operators $\mathcal{D}_n$ and $\mathcal{D}_n^\dagger$ are given by

$$\mathcal{D}_n = \partial_r+\frac{\mathrm{i}K}{\Delta}+2n\frac{r-M}{\Delta}, \tag{2c}$$

$$\mathcal{D}_n^\dagger = \partial_r-\frac{\mathrm{i}K}{\Delta}+2n\frac{r-M}{\Delta}, \tag{2d}$$

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where $K=(r^2+a^2)\sigma+am_{\rm a}$, n is an integer or half-integer, σ is the frequency of the incident wave, m is the rest mass of the Dirac particle, $m_{\rm a}$ is the azimuthal quantum number, and λ is the eigenvalue of the Dirac equation.

The equations governing the angular wave-functions $S_{\pm1/2}$ corresponding to spin $\pm1/2$ are given, respectively, by

$$L_{1/2}S_{+1/2}=-(\lambda-am\cos\theta)S_{-1/2}, \tag{3a}$$

$$L^{\dagger}_{1/2}S_{-1/2}=(\lambda+am\cos\theta)S_{+1/2}, \tag{3b}$$

where the operators L_n and $L^{\dagger}_n$ are given by

$$L_n=\partial_\theta+n\cot\theta+q \ \text{ and } \ L^{\dagger}_n=\partial_\theta+n\cot\theta-q, \tag{3c}$$

$$\text{with} \quad q=a\sigma\sin\theta+m_{\rm a}\csc\theta. \tag{3d}$$

The charge Q is included in the horizon function Δ, and it only affects the radial part of the Dirac equation. The angular equation has the same form as that in Kerr geometry. The eigenvalue λ satisfies the angular eigenvalue equation for the spin-$1/2$ particles as[38]

$$\left[L_{1/2}L^{\dagger}_{1/2}+\frac{am\sin\theta}{\lambda+am\cos\theta}L^{\dagger}_{1/2}+(\lambda^2-a^2m^2\cos^2\theta)\right]S_{-1/2}=0. \tag{4}$$

When $m=\sigma$, the eigenvalues and the eigenfunctions become exact, the detailed procedure to obtain eigenfunctions and eigenvalues can be found in Ref. [38], so it is not described here.

The radial equations (2a) and (2b) are in the coupled form. In order to decouple them and obtain a pair of independent one-dimensional (1D) wave equations, we define the tortoise coordinate

$$\begin{aligned} r_* &= r+\frac{2Mr_++am_{\rm a}/\sigma-Q^2}{r_+-r_-}\ln\left(\frac{r}{r_+}-1\right)\\ &\quad-\frac{2Mr_-+am_{\rm a}/\sigma-Q^2}{r_+-r_-}\ln\left(\frac{r}{r_-}-1\right), \end{aligned} \tag{5}$$

where

$$\begin{aligned} r_+&=M+\sqrt{M^2-a^2-Q^2},\\ r_-&=M-\sqrt{M^2-a^2-Q^2}, \end{aligned} \tag{6}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}r_*}=\frac{\Delta}{\omega^2}\frac{\mathrm{d}}{\mathrm{d}r}, \quad \omega^2=r^2+a^2+am_{\rm a}/\sigma. \tag{7}$$

In the above equations, $r_\pm$ are the radii of the two horizons.

In terms of r_*, the operators $\mathcal{D}_0$ and $\mathcal{D}^{\dagger}_0$ take the forms

$$\mathcal{D}_0=\frac{\omega^2}{\Delta}\left(\frac{\mathrm{d}}{\mathrm{d}r_*}+\mathrm{i}\sigma\right), \quad \mathcal{D}^{\dagger}_0=\frac{\omega^2}{\Delta}\left(\frac{\mathrm{d}}{\mathrm{d}r_*}-\mathrm{i}\sigma\right). \tag{8}$$

Choosing $\Delta^{1/2}R_{+1/2}=P_{+1/2}$, $R_{-1/2}=P_{-1/2}$, Eqs. (2a) and (2b) become

$$\left(\frac{\mathrm{d}}{\mathrm{d}r_*}+\mathrm{i}\sigma\right)P_{-1/2}=\frac{\Delta^{1/2}}{\omega^2}(\lambda+\mathrm{i}mr)P_{+1/2}, \tag{9a}$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}r_*}-\mathrm{i}\sigma\right)P_{+1/2}=\frac{\Delta^{1/2}}{\omega^2}(\lambda-\mathrm{i}mr)P_{-1/2}. \tag{9b}$$

Define a new variable

$$\theta=\tan^{-1}(mr/\lambda), \tag{10}$$

which yields $\cos\theta=\lambda/\sqrt{\lambda^2+m^2r^2}$, $\sin\theta=mr/\sqrt{\lambda^2+m^2r^2}$ and

$$(\lambda\pm\mathrm{i}mr)=\mathrm{e}^{\pm\mathrm{i}\theta}\sqrt{\lambda^2+m^2r^2}, \tag{11}$$

then the radial equations will be given by

$$\begin{aligned} &\left(\frac{\mathrm{d}}{\mathrm{d}r_*}+\mathrm{i}\sigma\right)P_{-1/2}\\ &=\frac{\Delta^{1/2}}{\omega^2}\sqrt{(\lambda^2+m^2r^2)}P_{+1/2}\exp[\mathrm{i}\tan^{-1}(mr/\lambda)], \end{aligned} \tag{12a}$$

$$\begin{aligned} &\left(\frac{\mathrm{d}}{\mathrm{d}r_*}-\mathrm{i}\sigma\right)P_{+1/2}\\ &=\frac{\Delta^{1/2}}{\omega^2}\sqrt{(\lambda^2+m^2r^2)}P_{-1/2}\exp[-\mathrm{i}\tan^{-1}(mr/\lambda)]. \end{aligned} \tag{12b}$$

Defining

$$P_{+1/2}=\psi_{+1/2}\exp\left[-\frac{1}{2}\mathrm{i}\tan^{-1}(mr/\lambda)\right] \tag{13a}$$

and

$$P_{-1/2}=\psi_{-1/2}\exp\left[+\frac{1}{2}\mathrm{i}\tan^{-1}(mr/\lambda)\right], \tag{13b}$$

we obtain

$$\begin{aligned} &\frac{\mathrm{d}\psi_{-1/2}}{\mathrm{d}r_*}+\mathrm{i}\sigma\left(1+\frac{\Delta}{\omega^2}\frac{\lambda m}{2\sigma}\frac{1}{\lambda^2+m^2r^2}\right)\psi_{-1/2}\\ &=\frac{\Delta^{1/2}}{\omega^2}(\lambda^2+m^2r^2)^{1/2}\psi_{+1/2} \end{aligned} \tag{14a}$$

and

$$\begin{aligned} &\frac{\mathrm{d}\psi_{+1/2}}{\mathrm{d}r_*}-\mathrm{i}\sigma(1+\frac{\Delta}{\omega^2}\frac{\lambda m}{2\sigma}\frac{1}{\lambda^2+m^2r^2})\psi_{+1/2}\\ &=\frac{\Delta^{1/2}}{\omega^2}(\lambda^2+m^2r^2)^{1/2}\psi_{-1/2}. \end{aligned} \tag{14b}$$

Further choosing

$$\hat{r}_*=r_*+\frac{1}{2\sigma}\tan^{-1}(mr/\lambda)$$

so that

$$\mathrm{d}\hat{r}_*=\left(1+\frac{\Delta}{\omega^2}\frac{\lambda m}{2\sigma}\frac{1}{\lambda^2+m^2r^2}\right)\mathrm{d}r_*,$$

the above equations become

$$\left(\frac{\mathrm{d}}{\mathrm{d}\hat{r}_*}+\mathrm{i}\sigma\right)\psi_{-1/2}=W\psi_{+1/2}, \tag{15a}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}\hat{r}_*} - \mathrm{i}\sigma\right)\psi_{+1/2} = W\psi_{-1/2}, \tag{15b}$$

where

$$W = \frac{\Delta^{1/2}(\lambda^2 + m^2r^2)^{3/2}}{\omega^2(\lambda^2 + m^2r^2) + \lambda m\Delta/2\sigma}. \tag{16}$$

Now letting $Z_\pm = \psi_{+1/2} \pm \psi_{-1/2}$, we can combine the differential equations to give

$$\left(\frac{\mathrm{d}}{\mathrm{d}\hat{r}_*} + W\right)Z_- = \mathrm{i}\sigma Z_+, \tag{17a}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}\hat{r}_*} - W\right)Z_+ = \mathrm{i}\sigma Z_-. \tag{17b}$$

From these equations given above, we readily obtain a pair of independent 1D wave equations

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\hat{r}_*^2} + \sigma^2\right)Z_\pm = V_\pm Z_\pm, \tag{18}$$

where

$$\begin{aligned} V_\pm &= W^2 \pm \frac{\mathrm{d}W}{\mathrm{d}\hat{r}_*} \\ &= \frac{\Delta^{1/2}(\lambda^2 + m^2r^2)^{3/2}}{[\omega^2(\lambda^2 + m^2r^2) + \lambda m\Delta/2\sigma]^2}\{\Delta^{1/2}(\lambda^2 + m^2r^2)^{3/2} \\ &\quad \pm [(r-M)(\lambda^2 + m^2r^2) + 3m^2r\Delta]\} \\ &\quad \mp \frac{\Delta^{3/2}(\lambda^2 + m^2r^2)^{5/2}}{[\omega^2(\lambda^2 + m^2r^2) + \lambda m\Delta/2\sigma]^3} \\ &\quad \times [2r(\lambda^2 + m^2r^2) + 2m^2\omega^2 r + \lambda m(r-M)/\sigma]. \end{aligned} \tag{19}$$

Equation (18) is nothing but the Schrödinger equation corresponding to the total energy σ^2 of the wave. By transforming r into $\hat{r}_*$, the horizon is shifted from $r = r_+$ to $\hat{r}_* = -\infty$.

3. Solutions of the wave equation

In the following calculations, we choose the parameters in such a way (see Refs. [28]–[30] for further details) that there is a significant interaction between the particle and the black hole, then the Compton wavelength of the incoming wave is of the same order as the KN radius. Similarly, the frequency of the incoming particle will be of the same order as the inverse of time, i.e.,

$$m \sim \sigma \sim \left[M + \sqrt{(M^2 - a^2 - Q^2)}\right]^{-1}.$$

For convenience, we choose $M = 1$ and $Q = 0.6$.

3.1. The case where waves do not hit the potential barrier

The variation of $V_+(\hat{r}_*)$ with $\hat{r}_*$ for $m = \sigma = 0.8$ and $a = 0.5$ is shown in Fig. 1.

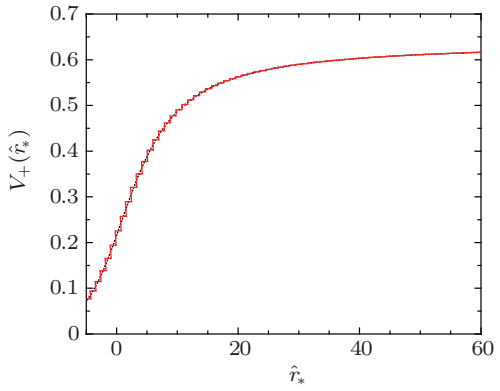


Fig. 1. (color online) Behavior of $V_+(\hat{r}_*)$ for $m = 0.8$, $\sigma = 0.8$, and $a = 0.5$. This is approximated by a collection of steps.

The energy σ^2 is always higher than the height of the potential barrier, and therefore the waves do not hit the barrier. The potential has been replaced by a collection of step functions. The step potential in Fig. 1 is only a diagram. In fact, we use as many as 10^4 steps to accurately follow the shape of the potential so that the steps become indistinguishable from the actual potential function. The wave function on each step is given by quantum mechanics as

$$Z_{+,n} = A_n \exp[\mathrm{i}k_n\hat{r}_{*,n}] + B_n \exp[-\mathrm{i}k_n\hat{r}_{*,n}], \tag{20}$$

where k is the wave number ($k = \sqrt{\sigma^2 - V_+}$) and k_n is its value at the n-th step. The following standard junction conditions are included to ensure that the function is smooth:

$$Z_{+,n} = Z_{+,n-1}, \tag{21}$$

$$\left.\frac{\mathrm{d}Z_+}{\mathrm{d}\hat{r}_*}\right|_n = \left.\frac{\mathrm{d}Z_+}{\mathrm{d}\hat{r}_*}\right|_{n-1}. \tag{22}$$

The local reflection and transmission coefficients at each step have the forms

$$R_n = \left|\frac{B_n}{A_n}\right|^2 = \left|\frac{\left(1 - \frac{k_{n-1}}{k_n}\right)A_{n-1}\exp[\mathrm{i}k_{n-1}\hat{r}_{*,n-1}] + \left(1 + \frac{k_{n-1}}{k_n}\right)B_{n-1}\exp[-\mathrm{i}k_{n-1}\hat{r}_{*,n-1}]}{\left(1 + \frac{k_{n-1}}{k_n}\right)A_{n-1}\exp[\mathrm{i}k_{n-1}\hat{r}_{*,n-1}] + \left(1 - \frac{k_{n-1}}{k_n}\right)B_{n-1}\exp[-\mathrm{i}k_{n-1}\hat{r}_{*,n-1}]}\right|^2, \tag{23}$$

and

$$T_n = 1 - R_n. \tag{24}$$

Here, the no-reflection inner boundary condition has been used: $R_n \to 0$ at $V_n \to 0$.

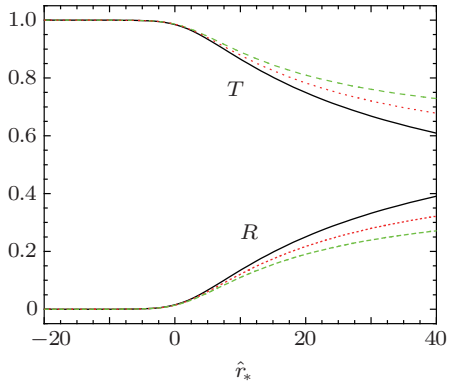


Fig. 2. (color online) Variations of reflection and transmission coefficients of waves with $\hat{r}_*$. The parameters are $\sigma = 0.8$, $m = 0.8$ (solid line), 0.79 (dotted line), and 0.78 (long-dashed line).

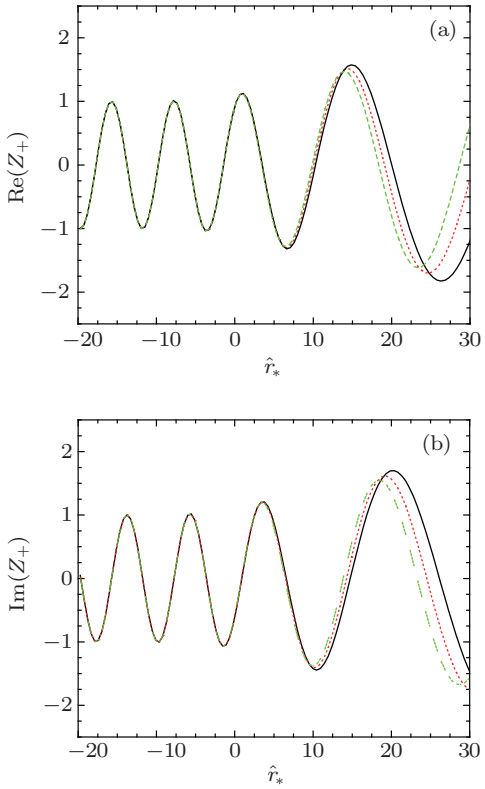


Fig. 3. (color online) Variations of amplitudes of (a) Re(Z_+) and (b) Im(Z_+) of waves with $\hat{r}_*$ for $\sigma = 0.8, m = 0.8$ (solid line), 0.79 (dotted line), and 0.78 (long-dashed line).

Figure 2 shows the instantaneous values of the reflection and transmission coefficients for parameters: $\sigma = 0.8$ and $m = 0.80$ (solid curve), 0.79 (dotted curve), and 0.78 (long-dashed curve). No matter what the value of the mass is, the reflection coefficient goes up and the transmission coefficient comes down with the increase of potential. As the particle mass decreases, R comes down and T goes up. While the reflection coefficient (or the transmission coefficient) seems to be independent of the mass of the particle when being close to the horizon where the potential is close to zero. Figure 3 shows three solutions [(a) amplitudes of Re(Z_+), (b) amplitudes of Im(Z_+)] for the same three parameters. It is obvious that the wave with the value of mass being 0.8 is scattered most strongly. As m comes down, the wavelength comes down and the amplitude also comes down. While the wavelength and the amplitude are all independent of the mass of the particle when being close to the horizon: different particles have the same wavelength and amplitude, and the waves are all almost harmonic.

3.2. The case where waves do hit the potential barrier

The potential is shown in Fig. 4 for $m = \sigma = 0.168$, $a = 0.79$. We replace the potential barrier by a series of step functions again and use the same quantum-mechanical method to solve the wave equation. Because the wave hits the potential barrier, inside the barrier we use the wave function in the form of

$$Z_{+,n} = A_n \exp[-\alpha_n \hat{r}_{*,n}] + B_n \exp[\alpha_n \hat{r}_{*,n}], \tag{25}$$

where $\alpha_n = \sqrt{V_+ - \sigma^2}$. Figure 5 shows the instantaneous values of the reflection coefficients for parameters: $\sigma = 0.168$ and $m = 0.168$ (solid curve), 0.166 (dotted curve), and 0.164 (long-dashed curve). R goes down as the particle mass decreases. When the particle is far away from the peak of the potential, the reflection coefficient R tends toward zero because the potential is so small that it can be ignored. The comparison among waves with different parameters in this case is shown in Fig. 6. The result turns out to be similar to that in Fig. 3. The variations of the amplitudes for different particles are not obvious because of the low potential barrier. In this case, the energy σ^2 is lower than the height of the potential barrier. According to classical mechanics, such a particle ($\sigma^2 < V_+$) cannot cross the barrier, while the same particles can cross the same barrier according to quantum mechanics. But inside the potential barrier, the wave will decay. So one can find that all the waves in Fig. 6 decay rapidly at $\hat{r}_* \to 0$, which is the location of the potential barrier. This result is also identical to that in Ref. [30].

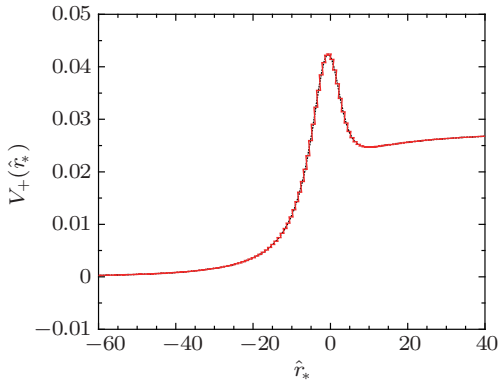


Fig. 4. (color online) Behavior of $V_+(\hat{r}_*)$ for $m = 0.168$, $\sigma = 0.168$, and $a = 0.79$. This is approximated by a collection of steps.

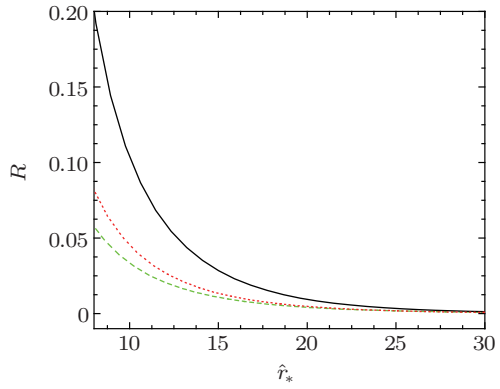


Fig. 5. (color online) Variations of reflection coefficients of waves with $\hat{r}_*$. The parameters are $\sigma = 0.168$, $m = 0.168$ (solid line), 0.166 (dotted line), and 0.164 (long-dashed line).

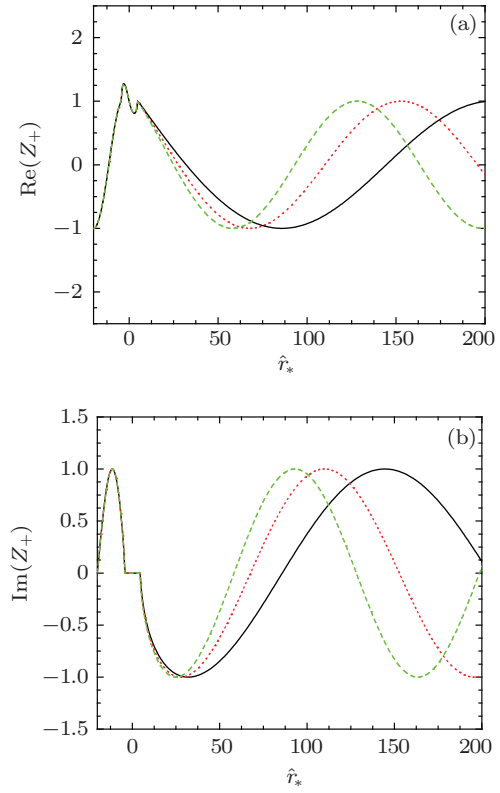


Fig. 6. (color online) Variations of amplitudes of (a) Re(Z_+) and (b) Im(Z_+) of waves with $\hat{r}_*$ for $\sigma = 0.168$, $m = 0.168$ (solid line), 0.166 (dotted line), and 0.164 (long-dashed line).

4. Conclusion

In this paper, we studied the scattering of massive, spin-half particles from a KN black hole, particularly the nature of the radial wave functions and the reflection and transmission coefficients. We used the step-potential method of quantum mechanics to obtain the spatial dependences of the reflection and transmission coefficients and the wave functions. Reflection and transmission coefficients are defined at a single point, they can be called "instantaneous" reflection and transmission coefficients. In any case, the waves scattered off are distinctly different: waves with a smaller mass are scattered off with a shorter wavelength. Therefore, a black hole can act as a mass spectrograph. A mixture of waves should be split into its components by the black hole. Unlike in the case of a Kerr black hole,[30] the wavelength is longer and the amplitude is smaller than those in Kerr geometry when the particle hits the potential.

There is another problem that needs to be explained. In the above discussion, we have considered the scattering of incident particles. If we consider the outgoing particles generated near the event horizon of the black hole according to Hawking radiation, the average number of particles at infinity should be the product of the Hawking radiation spectrum and the barrier penetration coefficient.

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