Classical interpretations of relativistic precessions

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Relativists have exposed various precessions and developed ingenious experiments to verify those phenomena with extreme precisions. The Gravity Probe B mission was designed to study the precessions of the gyroscopes rotating round the Earth in a nearly circular near-Earth polar orbit to demonstrate the geodetic effect and the Lense–Thirring effect as predicted by the general relativity theory. In this paper, we show in a very simple and novel analysis that the precession of the perihelion of Mercury, the Thomas precession, and the precession data (on the de Sitter and Lense–Thirring precessions) collected from the Gravity Probe B mission could easily be explained from classical physics, too.

Keywords: Thomas precession, precession of planetary orbits, de Sitter precession, Lense–Thirring precession

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1. Introduction

The precession of perihelion of Mercury has been a problem for physicists from the time of its recognition by Urbain Le Verrier in 1859. Albert Einstein solved the problem by the ingenious application of his General Relativity Theory in 1915. Schwarzschild went through the solution and found himself in a state of great confusion. However, after a friendly controversy, they reached an agreed happy solution.[1] The geodetic effect was first predicted by Willem de Sitter in 1916[2] and the Lense–Thirring effect was predicted in 1918.[3] Both of these effects were speculated from the consideration of general relativity theory. Thomas precession was predicted from the consideration of special relativity in 1926[4] to explain spectral structures and Zeeman effects quantitatively. In the first decade of this century, Gravity Probe B mission has demonstrated both the geodetic effect and the Lense–Thirring effect[5,6] in gyroscopes rotating round the Earth in a near-Earth nearly-circular polar orbit.

In the present paper, we shall first deduce the electromagnetic momentum of a steadily moving point charge from G. F. C. Searle (1897) and show that the $\gamma[=1/k, k=\sqrt{1-u^2/c^2}]$ coefficient of the electromagnetic momentum of a point charge moving with a velocity u (and c being the velocity of the electric/magnetic field) in the free space could explain the Thomas precession sharply. Then we shall show that the angular momentum and the linear momentum of a steadily moving material object (containing charges) should contain a γ^3 coefficient which could account for the precession of the perihelion of Mercury and the so-called de Sitter and Lense–Thirring precessions.

2. Two masses of a steadily moving point charge interacting with electromagnetic fields

Oliver Heaviside, the sage in solitude, first deduced the $\boldsymbol{E}$ field and the induced $\boldsymbol{B}^*$ field of a steadily moving point charge in free space in 1888–1889 as[7,8]

$$\boldsymbol{E} = \frac{Qk^2\boldsymbol{r}}{4\pi\varepsilon_0 r^3}\left(1-\frac{u^2}{c^2}\sin^2\varphi\right)^{-3/2}, \tag{1}$$

$$\boldsymbol{B}^* = \frac{\boldsymbol{u}\times\boldsymbol{E}}{c^2}, \tag{2}$$

where φ is the angle between $\boldsymbol{r}$ and $\boldsymbol{u}$, $\boldsymbol{r}$ is the instantaneous distance between the point charge and the point where the fields are to be determined, $\boldsymbol{u}$ is the velocity of the point charge in free space, and ε_0 is the permittivity of free space.

Electromagnetic momentum $\boldsymbol{P}$ and the magnetic energy T of a steadily moving point charge could be written from the consideration of classical electrodynamics as

$$\boldsymbol{P} = \int_{\text{all space}} (\boldsymbol{D}\times\boldsymbol{B}^*)\,\mathrm{d}\tau, \tag{3}$$

$$T = \frac{\varepsilon_0 c^2}{2}\int_{\text{all space}} B^{*2}\,\mathrm{d}\tau, \tag{4}$$

where $\boldsymbol{D}$ and $\boldsymbol{B}^*$ are the electric induction vector and the induced magnetic vector respectively, and $\mathrm{d}\tau$ is the infinitesimal volume element in the free space.

Using Eqs. (2)–(4), we have for the magnitude of this momentum

$$|\boldsymbol{P}| = \frac{2T}{u}, \tag{5}$$

$\boldsymbol{P}$ is a vector directed towards $\boldsymbol{u}$. Therefore, in vector notation we have

$$\boldsymbol{P} = \frac{2T\boldsymbol{u}}{u^2}. \tag{6}$$

Searle (1897)[9] has elegantly deduced the magnetic energy of a steadily moving point charge (a very small spherical

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charge with radius δR) from Heaviside's field equations

$$T = \frac{Q^2u^2}{12\pi\varepsilon_0 c^2 k\delta R}. \tag{7}$$

We could deduce the electromagnetic momentum of a steadily moving point charge from Eqs. (6) and (7) as

$$\boldsymbol{P} = \frac{Q^2\boldsymbol{u}}{6\pi\varepsilon_0 c^2 k\delta R} = \gamma m_0 \boldsymbol{u}, \tag{8}$$

where

$$m_0 = \frac{Q^2}{6\pi\varepsilon_0 c^2 \delta R} \tag{9}$$

is the classical electromagnetic mass of a point charge at rest.

An alternative deduction along with the classical interpretations of all special relativistic phenomena could be found in Refs. [10] and [11].

(i) When the point charge moves rectilinearly with a variable velocity u, the vector $\mathrm{d}\boldsymbol{P}/\mathrm{d}t$ is directed along the line of motion and using Eq. (8), its magnitude is given by

$$\frac{\mathrm{d}|\boldsymbol{P}|}{\mathrm{d}t} = \frac{\mathrm{d}|\boldsymbol{P}|}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}t} = \gamma^3 m_0 \frac{\mathrm{d}u}{\mathrm{d}t}, \tag{10}$$

(ii) When the charge is moving with a velocity $\boldsymbol{u}$ whose magnitude is constant but its direction is varying, the acceleration direction is then normal to the path and it is convenient to use vector equation. If $\boldsymbol{u}$ is the velocity and $\mathrm{d}\boldsymbol{u}/\mathrm{d}t$ the acceleration and it is taken into account that in this case there is a constant ratio between $|\boldsymbol{P}|$ and $|\boldsymbol{u}|$ then using Eq. (8) we have

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \frac{|\boldsymbol{P}|}{|\boldsymbol{u}|}\frac{\mathrm{d}u}{\mathrm{d}t} = \gamma m_0 \frac{\mathrm{d}u}{\mathrm{d}t}. \tag{11}$$

From Eq. (10), we have the longitudinal electromagnetic mass (LEM) of a moving point charge as[12]

$$\mathrm{LEM} = \gamma^3 m_0, \tag{12}$$

and from Eq. (11) we have the transverse electromagnetic mass (TEM) of a steadily moving point charge as[12]

$$\mathrm{TEM} = \gamma m_0. \tag{13}$$

Thus we see that there are two masses of charge viz., longitudinal and transverse electromagnetic masses that determine the motion of a point charge in electromagnetic fields.

3. Two masses of a material body interacting with gravitating fields

We know that electric charges possess momenta and energies just like all other physical objects. We could experience momenta and energies of these charges with our sense organs. Therefore, charges are real physical entities (objects). All objects are subjected to gravitation and they have the same acceleration towards the center of gravity in the same gravitating field. Therefore, charges should similarly be subjected to gravitation, and the acceleration of a point charge should similarly be the same as the acceleration of a point object and which should be directed towards the interacting gravitating field (unlike its acceleration during its interaction with the electric field). This implies that the gravitating mass of a point charge is proportional to its longitudinal electromagnetic mass $\gamma^3 m_0$.

Transverse electromagnetic mass γm_0 should play no role in this interaction. If it played any role in this interaction, material bodies (all of which contain charges) should have acceleration not always directed towards the center of gravity of the gravitating body and that acceleration too would differ in magnitude in different interactions. But this is not the case as observed in many precise experiments. Thus, in a gravitating field, a point charge acts as a mass point; the mass of the mass point is proportional to the longitudinal electromagnetic mass of the point charge.

Suppose that an object of material mass m_m[12] contains 'Q' amount of positive and negative charges with the rest electromagnetic mass m_0 in total [considering that electromagnetic mass is independent of the sign of charges as per Eq. (9)] and for simple calculation, assume that the positive and negative charges are concentrated at two points separately near the centre of mass of the object. Therefore, an object (containing charges) at rest in free space should have two masses viz., material mass m_m and electromagnetic mass m_0, both of which are subjected to gravitation.

But we know neither the material mass m_m nor the rest electromagnetic mass m_0 associated with the object. We know $(m_\mathrm{m} + m_0) = m$ which is a measurable quantity measured when the object is at rest. With a large velocity $\gamma^3 m_0$ could be much higher. But in any case, it could not be greater than $\gamma^3(m_\mathrm{m} + m_0) = \gamma^3 m$, which we could use as the limiting gravitating mass of an object moving with high velocity. This limiting value of the gravitating mass of an object will help us to understand the motion of objects in a gravitating field.

Thus, we see that the limiting gravitating mass of an object (LGM)

$$\mathrm{LGM} = \gamma^3(m_\mathrm{m} + m_0) = \gamma^3 m, \tag{14}$$

and if Galileo's experiment was exactly valid for all objects, this mass is proportional to the inertial mass of the object.

Therefore, considering the proportionality constant to be unity, we could write that the limiting inertial mass (LIM) of the object is

$$\mathrm{LIM} = \gamma^3(m_\mathrm{m} + m_0) = \gamma^3 m, \tag{15}$$

from which we get the limiting linear momentum $(\boldsymbol{P})$ of the object containing charges as

$$\boldsymbol{P} = \gamma^3(m_\mathrm{m} + m_0)\boldsymbol{u} = \gamma^3 m\boldsymbol{u}, \tag{16}$$

the rest mass of a moving object containing charges

$$m = m_\mathrm{m} + m_0. \tag{17}$$

4. Equation of motion of the planets of the sun

Suppose that a planet of material (non-electromagnetic) mass $m_{\rm m}$ contains 'Q' amount of positive and negative charges with electromagnetic rest mass m_0 in total [considering that electromagnetic mass is independent of the sign of charges as per Eq. (9)] and for simple calculation, let us assume that the positive and negative charges are concentrated separately near the center of mass of the planet. Let M be the non-electromagnetic mass and the mass originating from charges associated with the sun. The LGM of the planet is $\gamma^3(m_{\rm m}+m_0)=\gamma^3 m$ and the limiting angular momentum of the planet with a moderately high velocity is

$$\boldsymbol{A}=\gamma^3 m r^2\dot{\theta}, \tag{18}$$

when the mass of the sun is considered to be concentrated at the origin and the center of the mass of the planet passes through the point (r,θ) in a polar co-ordinate with a velocity $\boldsymbol{u}$. In such a situation the limiting radial force acting on the planet is

$$-GMm/r^2=m(\ddot{r}-r\dot{\theta}^2), \tag{19}$$

where G is the gravitational constant. Now, as per classical physics, cross-radial force, in a central force field, is always 0. Therefore, for the planet with charges, the limiting cross-radial force for the planet is as follows:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}[\gamma^3 m r^2\dot{\theta}]=0, \tag{20}$$

$$\gamma^3 r^2\dot{\theta}=H=\text{constant}. \tag{21}$$

Now let

$$U=1/r. \tag{22}$$

From Eqs. (21) and (22), we have

$$\dot{\theta}=HU^2k^3, \tag{23}$$

$$\dot{r}=-\frac{1}{U^2}\frac{\mathrm{d}U}{\mathrm{d}\theta}\dot{\theta}=-Hk^3\frac{\mathrm{d}U}{\mathrm{d}\theta}, \tag{24}$$

$$\ddot{r}=-Hk^3\Big(\frac{\mathrm{d}^2U}{\mathrm{d}\theta^2}\Big)\dot{\theta}=-H^2k^6U^2\frac{\mathrm{d}^2U}{\mathrm{d}\theta^2}. \tag{25}$$

Substituting Eqs. (23) and (25) into Eq. (19) (and when u is small compared with c), we have

$$\begin{aligned}\frac{\mathrm{d}^2U}{\mathrm{d}\theta^2}+U &\approx \frac{GM}{H^2}\gamma^6\approx\frac{GM}{H^2}+\frac{3GM}{H^2}\frac{u^2}{c^2}\\ &\approx\frac{GM}{H^2}+\frac{3GM}{r^2u_\theta^2}\frac{(u_\theta^2+u_{\rm r}^2)}{c^2}\Big(1-3\frac{u^2}{c^2}\Big)\\ &\approx\frac{GM}{H^2}+\frac{3GM}{r^2u_\theta^2}\frac{(u_\theta^2+u_{\rm r}^2)}{c^2}\\ &\approx\frac{GM}{H^2}+\frac{3GM}{c^2r^2}\Big(1+\frac{u_{\rm r}^2}{u_\theta^2}\Big)\\ &\approx\frac{GM}{H^2}+\frac{3GM}{c^2r^2}+\frac{3GM}{c^2r^2}\frac{e^2(\sin\theta)^2}{(1+e\cos\theta)^2},\end{aligned} \tag{26}$$

where $u_{\rm r}$ is the radial velocity, u_θ is the cross-radial velocity of the planet, and e is the eccentricity of the orbit of the planet.

The solution of this general equation will describe the general motion of all planets and asteroids with high eccentricities in the gravitating field of the sun.

Now excepting Mercury and Mars, all planets have negligibly small eccentricities for their orbits for which the last term on the right-hand side (RHS) of Eq. (26) could be omitted. The eccentricity of Mercury is 0.2056 and that of Mars is 0.0934. Therefore, for Mercury, the last term on the RHS should be less than a 0.043th part of the penultimate term, which is itself a very small quantity, and for Mars the last term should be less than a 0.009th part of the penultimate term which is too negligibly small.

Therefore, for all the planets of the sun (including Mercury and Mars) we could write

$$\frac{\mathrm{d}^2U}{\mathrm{d}\theta^2}+U\approx\frac{GM}{H^2}+\frac{3GM}{c^2r^2}. \tag{27}$$

The deduction is concerned with Kepler's Problem, which nowadays is variously treated in more sophisticated manners as in Refs. [13]–[16].

Equation (27) contains an additional term $3GM/(c^2r^2)$ in the well-known equation of planetary motion as per classical mechanics. This arises from our consideration of mass originating from charges associated with the planets. Therefore, we may call the above term the mass effect due to the charges inside the planets.

This classical equation (27) will at once explain the advance of the perihelia of Mercury, Venus, and Earth.

Calculation of the advance of the perihelia of planets by our Eq. (27) is as follows:

Equation (27) could be written in the following form:

$$\frac{\mathrm{d}^2U}{\mathrm{d}\theta^2}+U\approx\alpha+3\beta U^2, \tag{i}$$

where

$$\alpha=GM/H^2, \tag{ii}$$

$$\beta=GM/c^2. \tag{iii}$$

As a first approximation, the last term of Eq. (i) viz., $3\beta U^2$ may be neglected for the time being and in that case we have

$$\frac{\mathrm{d}^2U}{\mathrm{d}\theta^2}+U\approx\alpha, \tag{iv}$$

which corresponds to the well-known Newtonian equation of motion of the planets.

The solution of the above equation

$$U=\alpha[1+e\cos(\theta-w)], \tag{v}$$

where e is the eccentricity of the orbit and w is the initial longitude of the perihelion.

To obtain the second approximation for Eq. (i), we replace U of the RHS of Eq. (i) by Eq. (v) and write

$$\frac{\mathrm{d}^2U}{\mathrm{d}\theta^2}+U=\alpha+3\beta\times\alpha^2[1+e\cos(\theta-w)]^2$$
$$=\alpha+3\beta\alpha^2+6\beta\alpha^2e\cos(\theta-w)$$
$$+3\beta\alpha^2e^2\cos^2(\theta-w). \quad \text{(vi)}$$

The only term which could take effect in our study is the term containing $\cos(\theta-w)$. Now for the particular integral of this equation we could write

$$\frac{\mathrm{d}^2U}{\mathrm{d}\theta^2}+U=A\cos\theta, \quad \text{(vii)}$$

$$U_1=\frac{1}{2}A\theta\sin\theta. \quad \text{(viii)}$$

The additional term $6\beta\alpha^2e\cos(\theta-w)$ gives a part of U as expressed below

$$U=3\beta\alpha^2e\theta\sin(\theta-w). \quad \text{(ix)}$$

Therefore, the solution of Eq. (i) to the second order of approximation is

$$U=\alpha[1+e\cos(\theta-w)]+3\beta\alpha^2e\theta\sin(\theta-w)$$
$$=\alpha+\alpha e[\cos(\theta-w)+3\alpha\beta\theta\sin(\theta-w)]. \quad \text{(x)}$$

Let

$$3\alpha\beta\theta=\delta w, \quad \text{(xi)}$$

and note $3\alpha\beta\theta=\mathrm{d}w$ is very small, then we will have

$$\sin\delta w=\delta w \quad \text{and} \quad \cos\delta w=1. \quad \text{(xii)}$$

Then we may write,

$$U=\alpha+\alpha e[\cos(\theta-w)+3\alpha\beta\theta\sin(\theta-w)]$$
$$=\alpha+\alpha e[\cos(\theta-w)+\delta w\sin(\theta-w)]$$
$$=\alpha[1+e\{\cos(\theta-w)\cos\delta w+\sin(\theta-w)\sin\delta w\}]$$
$$=\alpha[1+e\cos(\theta-w-\delta w)]. \quad \text{(xiii)}$$

The last equation is the solution of Eq. (27), where δw represents an advance of the perihelion of the orbit of the planet. When the planet completes one revolution, that is, when $\theta=2\pi$ the planet will not return to its initial position, it will advance by an angle

$$\delta w=3\alpha\beta2\pi=6\pi\alpha\beta=6\pi\frac{GM}{H^2}\frac{GM}{c^2}. \quad \text{(xiv)}$$

Using the standard equations of planetary motion viz.,

$$\frac{GM}{H^2}=\frac{1}{a(1-e^2)}, \quad \text{(xv)}$$

$$T^2=\frac{4\pi^2a^3}{GM}, \quad \text{(xvi)}$$

where a is the semi-major axis of the orbit of the planet and T is the time needed to complete one revolution of the planet. Therefore,

$$\delta w=\frac{24\pi^3a^2}{(1-e^2)c^2T^2}. \quad \text{(xvii)}$$

As per "Relativistic Mechanics" by Satya Prakash [Pragati Prakashan, Begum Bridge, Meerut-25001, UP, India, Tenth Edition, 2000) pp. 438–441]. Putting for Mercury as per NASA data,

$$e=0.2056,\ \ a=0.5791\times10^{11}\ \text{m},\ \ T=87.969\ \text{days},$$

($\pi=3.141592$; $c=299792458$ m/s; $1-e^2=0.958$; $\pi^3=31.006$; $a^2=0.3354\times10^{22}$ m^2; $c^2=8.99\times10^{16}$ m^2/s^2; $T^2=57.76\times10^{12}$ s^2; 1 radian $=0.2062\times10^6$ arc second; 1 century $=36524$ days). We obtain this angle to 42.97 seconds of an arc per century for Mercury.

Perihelia advances as calculated (in arc sec per century) from the above formula (xvii) are as follows:

Mercury 42.97; Venus 8.6186; Earth 3.8346; Mars 1.3502; Jupiter 0.0623; Saturn 0.0137; Uranus 0.0024; Neptune 0.0008.

The perturbing effects on Mercury are well determined (531.63 arc seconds per century) and when it is subtracted from the observed precession (574.10 arc seconds per century) we have 42.47 arc seconds per century. The result calculated from Eq. (xvii) is 42.97 arc seconds per century, which upholds Eq. (27) as the equation of motion of the planet Mercury. Recent radar, laser, and satellite studies disclose many interesting data[17,18], which should be studied from the consideration of the limiting gravitating mass of objects as presented in this paper.

In recent years, the perturbing effects on Venus and Earth have been separated using recent astronomical and astrophysical data and when these data are subtracted from the observed precessions of those planets, the analyzers have claimed that they have obtained the values 8.6247 arc sec per century for Venus and 3.8387 arc per century for Earth, which confirms Eq. (27).[19] The advances of the perihelia of the planets like Mars, Jupiter, Saturn, Uranus, and Neptune as per Eq. (27) are too small to be verified at the present moment with reliable astronomical and astrophysical data.

5. Precession of a moving spherical rigid electron in a circular orbit

When a spinning spherical rigid electron rotates in its orbit, we shall show as per classical physics that its axis of symmetry should precess and this is outlined below.

The direction of motion of the center of the spinning electron and the direction of motion of the infinitesimal charge elements of the spinning electron body are different.

We now study this precession of the axis of a spherical rigid spinning electron rotating in its orbit from the consideration of classical physics. This precession of the axis of the electron arises from the difference in γ coefficient between the x component and y component of the electromagnetic momen-

tum [Eq. (8)] of the infinitesimal charge element around any point O_1 on the body of the moving electron.

The center of the electron rotates in its orbit in a fixed path. But the infinitesimal body elements of the electron (due to the difference in γ coefficient between the x component and the y component of the electromagnetic momentum Eq. (8)) will not follow the same direction of motion as that of the center of the electron, which will cause a torque on the spinning electron body.

To obtain a rough view of this interesting precession phenomenon, we shall consider that this paper plane represents the orbital plane of the electron while it rotates around the nucleus (Fig. 1). Let the center of the electron be pivoted in its orbit such that the center could slide in its orbit and the electron could spin around its axis. We shall now construct a Cartesian coordinate system OX and OY at any point O in this paper plane (Fig. 1). Now let the center of the electron be moved from O to P in the curved path OP during a time period $\mathrm{d}t$. Let us, for simplicity, consider that the center of curvature of the curved path OP lies on the line OY.

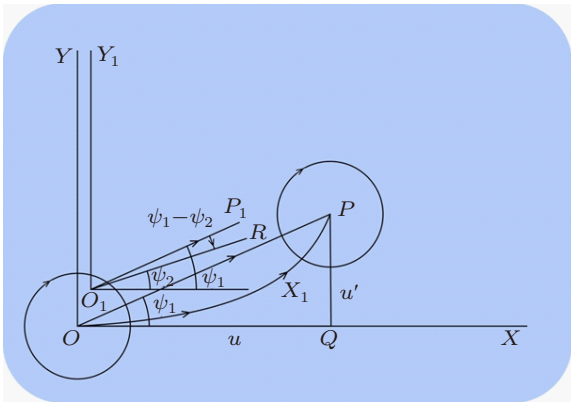


Fig. 1. (color online) Orbital precession of an orbiting spherical rigid spinning electron.

In this set-up, we shall view first the curved motion of the center and the body of electron, ignoring the spinning of the electron for the time being.

Consider that the center of the electron starts its motion from O with a velocity u. After a small time interval $\mathrm{d}t$, following a continuous curved path OP, it reaches P. Now, this continuous curved path OP is very small and therefore we may consider it to be a straight path. Therefore, in the orbital plane, the velocity $\boldsymbol{u}$ of the center of the electron at the point O could be decomposed into two components u_x and u_y, such that

$$\tan XOP = \tan\psi_1 = \frac{u'}{u}, \tag{28}$$

where $u_y = u'$, and u_y is small, $u_x = u$ (roughly). The angle ψ_1 is very small, so we have

$$\psi_1 = \frac{u'}{u}. \tag{29}$$

Therefore, we may conclude that when the center of the spherical rigid electron travels along the small curved path OP, each and every point of the body of the electron has always an instantaneous direction of motion which makes a small angle ψ_1 (roughly equal to u'/u) with respect to the lines parallel to OX axis of the coordinate system constructed at the center of the electron in the orbital plane.

Now the x component and the y component of the electromagnetic momentum of an infinitesimal charge element (moving with the electron) around any point O_1 on the circular electron strip in the same orbital plane could be written as [cf. Eq. (8)]

$$P_x = \gamma m_0 u, \tag{30}$$

$$P_y = m_0 u', \tag{31}$$

where u is large and u' is very small, m_0 is the electromagnetic mass of that infinitesimal charge element at rest in free space, as defined in Eq. (9).

Therefore, the resultant direction of momentum (O_1R) of that infinitesimal charge element at O_1 will make an angle ψ_2 with respect to O_1X_1 axis drawn parallel to OX at O_1, such that

$$\tan X_1O_1R = \tan\psi_2 = \frac{u'}{\gamma u} = k\frac{u'}{u}, \tag{32}$$

where ψ_2 is small, we have

$$\psi_2 = k\frac{u'}{u}. \tag{33}$$

But, the charge element at O_1 is constrained to move with the center of the electron towards O_1P_1 parallel to OP, such that

$$\angle X_1O_1P_1 = \psi_1 = \frac{u'}{u}. \tag{34}$$

Under this situation, the direction of motion of the point O_1 of the electron body should be deflected towards O_1X_1 axis (parallel to OX axis) in the orbital plane.

The angle of deflection of the direction of motion of the moving infinitesimal charge element around O_1 towards O_1X_1 axis (parallel to OX axis) in the orbital plane is

$$\begin{aligned}\mathrm{d}\psi &= \angle X_1O_1R - \angle X_1O_1P_1 = \psi_2 - \psi_1 \\ &= (k-1)\frac{u'}{u} = -\frac{1}{2}\frac{u^2}{c^2}\frac{u'}{u}.\end{aligned} \tag{35}$$

Thus during the travel of the center of the electron in its orbit for the time period $\mathrm{d}t$, every infinitesimal charge element of the electron body will be deflected in the orbital plane by a very small angle $\mathrm{d}\psi$ towards a direction parallel to OX from its instantaneous direction of motion O_1P_1 parallel to OP (which is the direction of motion of the center).

Now in the case where OP is extremely small, we may consider that the magnitudes of both the velocity components of the instantaneous motion of the electron at the points O and

P are roughly the same. Therefore, at the point P, the velocity components of the motion of the center of the electron read

$$u_x = u = \omega_e r \cos\alpha, \tag{36}$$

$$u_y = u' = \omega_e r \sin\alpha, \tag{37}$$

where α is the angle that the arc OP subtends at the center (not shown in Fig. 1) of curvature of OP and ω_e is the angular velocity of the electron in its orbit.

From Eqs. (36) and (37), we have

$$\tan\alpha \approx \frac{u'}{u} \approx \frac{u\,\mathrm{d}t}{r}. \tag{38}$$

Using Eq. (38), we obtain from Eq. (35)

$$\mathrm{d}\psi = \psi_2 - \psi_1 = (k-1)\frac{u\,\mathrm{d}t}{r}. \tag{39}$$

Thus we see that the direction of movement of each and every point of the body of the electron always makes an angle $\mathrm{d}\psi$ [as given in Eq. (39)] in the orbital plane with respect to the instantaneous direction of motion of the center of the electron while the electron moves with its center.

Now we construct a vertical (normal) of the orbital plane from the center of the electron and let the axis of the spinning electron be directed towards ($\boldsymbol{n}_0$) at some angle with respect to this vertical. Therefore, we may say that during the time $\mathrm{d}t$ of travel of the electron, the center of the electron being fixed in its orbit, the orbital plane of the electron body will rotate through an angle $\mathrm{d}\psi$ (and continues this rotation if the electron rotates in its orbit). This should cause a torque on the spinning electron body. Now suppose that this torque does not change the magnitude of the spin angular momentum of the spinning electron but changes only its direction. In that case, the spin axis of the electron will rotate through an angle $\mathrm{d}\psi$ (in the time $\mathrm{d}t$) around the vertical as per the usual torque rule:

$$\frac{\mathrm{d}\boldsymbol{n}_0}{\mathrm{d}t} = (\boldsymbol{\Omega}_{\mathrm{pr}}) \times \boldsymbol{n}_0, \tag{40}$$

with $\boldsymbol{\Omega}_{\mathrm{pr}}$ being the precession of the axis of the electron.

Therefore, we could conclude from Eqs. (39) and (40) that when the center of an electron moves in a circular orbit, its spin axis will precess in the orbital plane around the vertical with the angular velocity

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \Omega_{\mathrm{pr}} = (\Omega_e)_O = -\frac{1}{2}\frac{u^2}{c^2}\frac{u}{r} = -\frac{1}{2}\frac{u^2}{c^2}\omega_e, \tag{41}$$

where $(\boldsymbol{\Omega}_e)_O$ may be called the orbital precession of the electron.

Now the direction of this precession is towards the normal (vertical) to the orbital plane. Therefore, we could write

$$(\boldsymbol{\Omega}_e)_O = \frac{1}{2c^2}(\boldsymbol{f} \times \boldsymbol{u}) \tag{42}$$

with $(-u^2/r = f)$, as could be seen in many textbooks where f is the acceleration of the electron.

Relativists call this precession Thomas p recession.

It should be noted that when the spinning spherical rigid electron rotates in the anticlockwise direction in its orbit, the orbital precession (Thomas precession) is in the clockwise direction $(-)$.

6. Precession of a spinning rigid spherical gyroscope rotating in a near-Earth nearly circular orbit

6.1. Orbital precession

In Fig. 1, instead of the electron, we consider a spinning spherical rigid gyroscope rotating with a velocity u in a near-Earth nearly circular polar orbit in the anticlockwise direction, and the distance of the gyroscope from the center of the Earth is r.

Let the center of the gyroscope be pivoted in its orbit such that the center of the gyroscope could slide in its orbit and the gyroscope spins around its axis. We shall now construct a Cartesian coordinate system OX and OY at any point O in this paper plane (Fig. 1) that represents the orbital plane of the gyroscope too.

The direction of motion of the center of the gyroscope and the direction of motion of the infinitesimal mass elements of the gyroscope are different.

Now let the center of the gyroscope be moved from O to P along the curved path OP during a time period $\mathrm{d}t$. Let us, for simplicity, consider that the center of curvature of the curved path OP lies on the line OY.

Just like the electron, the velocity $\boldsymbol{u}$ of the center of the gyroscope at the point O could be decomposed into two components u_x and u_y, such that

$$\tan XOP = \tan\psi_1 = \frac{u'}{u}, \tag{43}$$

where $u_y = u'$, and u_y is small, $u_x = u$ (roughly). The angle ψ_1 is very small, so we have

$$\psi_1 = \frac{u'}{u}. \tag{44}$$

Therefore, we may conclude that when the center of the spherical rigid gyroscope travels along the small curved path OP, each and every point of the body of the gyroscope has always an instantaneous direction of motion which makes a small angle ψ_1 (roughly equal to u'/u) with respect to the lines parallel to the OX axis of the coordinate system constructed at O in the orbital plane of the gyroscope, as in the case of the electron.

Now the x component and the y component of the linear momentum of infinitesimal mass element of the moving gyroscope body around any point O_1 on the circular gyroscope strip in the same orbital plane could be written as per Eq. (16) [instead of Eq. (8) as has been considered for the electron]

$$P_x = \gamma^3 mu, \tag{45}$$

$$P_y = mu', \tag{46}$$

where u is large, u' is very small, m is the gravitating (or inertial) mass element of the infinitesimal mass at rest in free space, as defined in Eq. (17).

Therefore, the resultant direction of momentum (O_1R) of that infinitesimal mass element at O_1 will make an angle ψ_2 with respect to the O_1X_1 axis drawn parallel to OX at O_1, such that

$$\tan X_1O_1R = \tan\psi_2 = \frac{u'}{\gamma^3 u} = k^3\frac{u'}{u}, \tag{47}$$

where ψ_2 is small, so we have

$$\psi_2 = k^3\frac{u'}{u}. \tag{48}$$

But, the infinitesimal mass element at O_1 is constrained to move with the center of the gyroscope towards O_1P_1 parallel to OP, such that

$$\angle X_1O_1P_1 = \psi_1 = \frac{u'}{u}. \tag{49}$$

Under this situation, the direction of motion of the mass element of the point O_1 should be deflected towards the O_1X_1 axis (parallel to the OX axis) in the orbital plane.

The angle of deflection of the direction of motion of the moving infinitesimal mass element around O_1 towards O_1X_1 axis (parallel to the OX axis) in the orbital plane is

$$\mathrm{d}\psi = \psi_2 - \psi_1 = (k^3-1)\frac{u'}{u}. \tag{50}$$

Thus during the travel of the center of the spherical gyroscope in its orbit for the time period $\mathrm{d}t$, every infinitesimal mass element of the gyroscope body will be deflected by a very small angle $\mathrm{d}\psi$ towards a direction parallel to OX from its instantaneous direction of motion (O_1P_1 parallel to OP) with the center of the gyroscope in the orbital plane.

Now in the case where OP is extremely small, we may consider that the magnitudes of both the velocity components of the instantaneous motion of the center of the gyroscope at the points O and P are roughly the same.

Therefore, at the point P, just like the electron, the velocity components of the motion of the center of the gyroscope read

$$u_x = u = \omega_{\mathrm{g}} r\cos\alpha, \tag{51}$$

$$u_y = u' = \omega_{\mathrm{g}} r\sin\alpha, \tag{52}$$

where α is the angle that the arc OP subtends at the center (not shown in Fig. 1) of the curvature of OP and ω_{g} is the angular velocity of the gyroscope in its orbit.

From Eqs. (51) and (52), we have

$$\tan\alpha \approx \frac{u'}{u} \approx \frac{u\,\mathrm{d}t}{r}. \tag{53}$$

Using Eq. (53), we have from Eq. (50)

$$\mathrm{d}\psi = \angle X_1O_1R - \angle X_1O_1P_1 = \psi_2 - \psi_1 = (k^3-1)\frac{u\,\mathrm{d}t}{r}. \tag{54}$$

Thus we see that the direction of movement of each and every point of the body of the gyroscope will rotate through an angle $\mathrm{d}\psi$ [as given in Eq. (54)] in the orbital plane with respect to the instantaneous direction of motion of the center of the gyroscope while the gyroscope moves with its center.

Now we come to construct a vertical (normal) of the orbital plane from the center of the gyroscope and let the axis of the spinning gyroscope be at some angle with this vertical. Therefore, we may say that during the time $\mathrm{d}t$ of travel of the gyroscope, the orbital planes of the gyroscope body, the center of the gyroscope being fixed in its orbit, will rotate through an angle $\mathrm{d}\psi$ (and continues this rotation if the gyroscope rotates in its orbit). This should cause a torque on the spinning gyroscope body. Now suppose that this torque does not change the angular momentum of the spinning gyroscope but changes only its direction. In that case, the spin axis of the gyroscope will rotate through an angle $\mathrm{d}\psi$ (in the time $\mathrm{d}t$) around the vertical. This means that when the center of the gyroscope moves in a circular orbit, its spin axis will precess in the orbital plane around its vertical with the angular velocity

$$\begin{aligned}\frac{\mathrm{d}\psi}{\mathrm{d}t} &= (\boldsymbol{\Omega}_{\mathrm{g}})_O = -\frac{3}{2}\frac{u^2}{c^2}\frac{u}{r} = -\frac{3}{2}\frac{u^2}{c^2}\boldsymbol{\omega}_{\mathrm{g}}\\ &= -\frac{3}{2c^2 r}\left(\frac{GM}{r}\right)^{3/2} = \frac{3}{2c^2}(\boldsymbol{g}\times\boldsymbol{u}),\end{aligned} \tag{55}$$

where $(\boldsymbol{\Omega}_{\mathrm{g}})_O$ may be called the orbital precession of the gyroscope and

$$\left[\frac{GM}{r} = u^2, -\frac{u^2}{r} = g\right]$$

with ω_{g} being the angular velocity of the gyroscope, G the gravitational constant, M the mass of the Earth, and g the gravitational acceleration due to Earth at the position of the gyroscope. This orbital precession is directed towards the normal to the orbital plane and from the Vandenberg air base, this precession will be seen to be in the clockwise direction.

This orbital precession of a gyroscope given in Eq. (55) is called by the relativists geodetic or de Sitter precession.[20] When the center of the spinning spherical rigid gyroscope rotates in the anticlockwise direction in its orbit, this orbital precession takes place in the clockwise direction ($-$).

In the Gravity Probe B experiment, the velocity of the satellite in its orbit is well determined by direct measurement. Therefore, the practical result should not differ much from the theoretical calculation, provided that the experiment is carried out ideally. Gravity Probe B results[5] for orbital precession equally upholds classical physics.

6.2. Tangential (Coriolis) precession

Suppose that the Earth is rotating with an angular velocity ω_{E} and the center of a spherical rigid spinning gyroscope

is moving in the anticlockwise direction with a velocity u in a constrained near-Earth polar near-circular orbit.

Let us concentrate on the motion of the gyroscope when the center of the gyroscope passes a point O at the latitude φ radian north of the Earth. Construct a tangential plane of the spherical Earth at O (Fig. 2). In this situation, a Coriolis force will act on the gyroscope at O towards the west while the gyroscope moves towards the north.

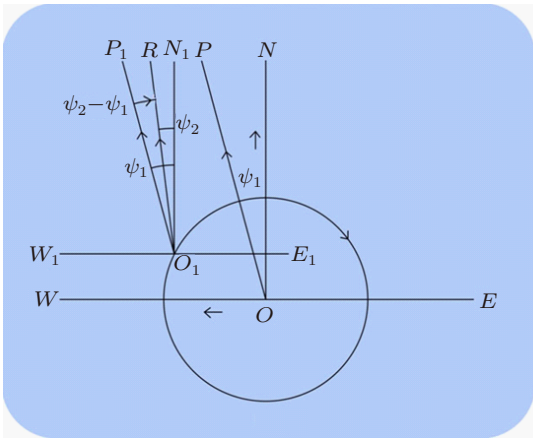


Fig. 2. (color online) Tangential (Coriolis) precession of an orbiting spherical rigid spinning gyroscope.

Direction of motion of the center of the gyroscope and that of the infinitesimal mass elements of the gyroscope body are different.

The magnitude of this Coriolis acceleration of the spherical gyroscope towards the west is $2\omega_{\rm E}u\sin\phi$. After a time $\mathrm{d}t$, the velocity of the gyroscope towards the west will be $2\omega_{\rm E}u(\sin\phi)\mathrm{d}t$, the average velocity being $u' = \omega_{\rm E}u\sin\phi\,\mathrm{d}t$ in the same direction. Now in case the center of the gyroscope moves towards north with a velocity u, the Coriolis velocity acts towards the west and the resultant direction of instantaneous motion of the center of the gyroscope will be towards OP (Fig. 2).

This direction will make an angle ψ_1 with the direction of motion of the gyroscope in the tangential plane (the radius vector of the Earth is normal to this plane) such that

$$\tan NOP = \tan\psi_1 = \frac{u'}{u} = \frac{\omega_{\rm E}u(\sin\phi)\,\mathrm{d}t}{u} = \omega_{\rm E}(\sin\phi)\,\mathrm{d}t. \tag{56}$$

The center of the gyroscope moves towards OP due to the effects of the motion of the center of the gyroscope with the velocity u towards ON and Coriolis average velocity towards the west. But the infinitesimal body elements of the gyroscope (due to the difference of the γ^3 coefficient of the x component and the y component of linear momentum [vide Eq. (16)] will move in a different direction.

The west-bound component and the north-bound components of the linear momentum of infinitesimal mass element of the moving gyroscope around any point O_1 on the circular gyroscope strip in the same tangential plane could be written as per Eq. (16)

$$P_{W_1} = \gamma^3 mu, \tag{57}$$

$$P_{N_1} = mu', \tag{58}$$

(u being large and u' being very small) where m is the limiting gravitating (or inertial) mass of the infinitesimal mass element at rest in free space, as defined in Eq. (17).

Therefore, the resultant direction of momentum (O_1R) of that infinitesimal mass element of the gyroscope at O_1 will make an angle ψ_2 with O_1N_1 axis drawn parallel to ON at O_1, such that

$$\tan N_1O_1R = \tan\psi_2 = \frac{u'}{\gamma^3 u} = \frac{\omega_{\rm E}u(\sin\phi)\,\mathrm{d}t}{\gamma^3 u} == k^3\omega_{\rm E}(\sin\phi)\,\mathrm{d}t. \tag{59}$$

But, the mass element at O_1 is constrained to move with the center of the gyroscope towards O_1P_1 parallel to OP, such that

$$\tan N_1O_1P_1 = \tan\psi_1 = \frac{u'}{u} = \omega_{\rm E}(\sin\phi)\,\mathrm{d}t. \tag{60}$$

Thus the motion of each and every element of the gyroscope body at each instant will not follow the same direction of motion of the center of the gyroscope. The direction of motion of the mass element of the point O_1 should be deflected towards O_1N_1 axis (parallel to the OX axis) in the tangential plane.

The angle of deflection of the direction of motion of the moving infinitesimal mass element around O_1 towards O_1N_1 axis (parallel to the ON axis) in the tangential plane could be calculated as follows:

The angles being very small,

$$\tan\psi_1 \simeq \psi_1,\ \ \tan\psi_2 \simeq \psi_2, \tag{61}$$

$$\mathrm{d}\psi = \angle N_1O_1R - \angle N_1O_1P_1 = \psi_2 - \psi_1 = (k^3-1)\omega_{\rm E}(\sin\phi)\,\mathrm{d}t, \tag{62}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = (k^3-1)\omega_{\rm E}(\sin\phi) = -\frac{3}{2}\frac{u^2}{c^2}\omega_{\rm E}\cos\left(\frac{\pi}{2}-\phi\right) = -\frac{3}{2}\frac{u^2}{c^2}\omega_{\rm E}\cos\left(\frac{\pi}{2}-\phi\right), \tag{63}$$

[ϕ is the latitude of the point O, the instantaneous position of the center of the gyroscope and $(\pi/2-\phi)$ is the angle between the direction of rotation of the Earth and its radius vector at O, the instantaneous position of the center of the gyroscope]. The result will be the same even if the calculation of average velocity for Coriolis force is done with the formula $u' = u_0 + \omega_{\rm E}u\sin\phi\,\mathrm{d}t$ where u_0 is the initial velocity of the

gyroscope towards the west at any time.

Thus we see that the direction of movement of each and every point of the body of the spherical rigid gyroscope will make an angle $\mathrm{d}\psi$ [as given in Eq. (62)] in the tangential plane with the instantaneous direction of motion of the center of the gyroscope while the gyroscope moves with its center.

Let us construct a vertical (normal) of the tangential plane from the center of the gyroscope and let the axis of the spinning gyroscope be at some angle with this vertical. Therefore, we may say that during the time $\mathrm{d}t$ of travel of the gyroscope, the center of the gyroscope being fixed in its orbit, the tangential planes of the gyroscope body will rotate through an angle $\mathrm{d}\psi$ (and continues this rotation if the gyroscope rotates in its orbit). This should cause a torque on the spinning gyroscope body. Now suppose that this torque does not change the angular momentum of the spinning gyroscope, but changes only its direction. In that case, the spin axis of the gyroscope will rotate through an angle $\mathrm{d}\psi$ (in the time $\mathrm{d}t$) around the vertical. This means that when the center of the gyroscope moves in a circular orbit, its spin axis will precess at its tangential plane around the vertical with the angular velocity as given in Eq. (63) as per similar analyses given in Sections 5, Subsections 6.1 and 6.2.

The radius vector of the point of consideration is the direction of this precession. Therefore, in vectorial form, this Coriolis precession could be written as

$$(\boldsymbol{\Omega}_{\mathrm{g}})_{\mathrm{T}} = -\frac{3}{2}\frac{u^2}{c^2}[\hat{\boldsymbol{r}}(\boldsymbol{\omega}_{\mathrm{E}}\cdot\hat{\boldsymbol{r}})], \tag{64}$$

$(\boldsymbol{\Omega}_{\mathrm{g}})_{\mathrm{T}}$ may be called the tangential precession of the gyroscope.

Now

$$\begin{aligned}\langle\hat{\boldsymbol{r}}(\boldsymbol{\omega}_{\mathrm{E}}\cdot\hat{\boldsymbol{r}})\rangle &= (\cos\phi\hat{\boldsymbol{i}}+\sin\phi\hat{\boldsymbol{j}})\omega_{\mathrm{E}}\cos(90^\circ-\phi)\\ &= \omega_{\mathrm{E}}(\cos\hat{\boldsymbol{i}}+\sin\phi\hat{\boldsymbol{j}})\sin\phi.\end{aligned} \tag{65}$$

In case of averaging when integrated over one revolution we have,

$$\begin{aligned}&\langle\hat{\boldsymbol{r}}(\boldsymbol{\omega}_{\mathrm{E}}\cdot\hat{\boldsymbol{r}})\rangle = \omega_{\mathrm{E}}/2,\\ &\Big[\frac{1}{2\pi}\int_0^{2\pi}\omega_{\mathrm{E}}(\cos\hat{\boldsymbol{i}}+\sin\phi\hat{\boldsymbol{j}})\sin\phi\,\mathrm{d}\phi = \omega_{\mathrm{E}}/2\Big].\end{aligned} \tag{66}$$

Therefore, the average value of the tangential precession due to Coriolis acceleration is

$$\begin{aligned}\langle(\boldsymbol{\Omega}_{\mathrm{g}})_{\mathrm{T}}\rangle &= -\frac{3}{4}\frac{u^2}{c^2}\boldsymbol{\omega}_{\mathrm{E}} = -\frac{3}{4}\frac{GM}{c^2r}\boldsymbol{\omega}_{\mathrm{E}}\\ &= -\frac{15}{8}\frac{G}{c^2r^3}\frac{2}{5}Mr^2\boldsymbol{\omega}_{\mathrm{E}} = -\frac{15}{8}\frac{G}{c^2r^3}\boldsymbol{J}\\ &= (\boldsymbol{\Omega}_{\mathrm{g}})_{\mathrm{Eq}},\end{aligned} \tag{67}$$

with $\boldsymbol{J}$ being the spinning angular momentum of the Earth.

Thus we see that this tangential precession of the gyroscope when averaged over the orbit should be an equatorial precession directed towards the axis of rotation of the Earth and therefore should be designated as $(\boldsymbol{\Omega}_{\mathrm{g}})_{\mathrm{Eq}}$.

When the center of the spinning spherical rigid gyroscope rotates in the anticlockwise direction in its orbit, Coriolis precession takes place in the clockwise direction $(-)$.

Equation (67) represents an equatorial precession when summed up around its orbit while the center of the gyroscope is moving in a nearly circular near-Earth polar orbit.

The magnitude of Coriolis acceleration may decrease by some factor κ (averaged over the orbit) which should originate from the environmental condition that the moving gyroscope faces around its orbit. The rotating Earth may be considered as an ideal magnetic dipole and in that case, its magnetic field when averaged over the orbit is proportional to ω_{E} and directed to its direction. This field should have some contribution to the precession of the gyroscopes as the gyroscopes contain charges and currents and these in spite of shielding are likely to be affected by the magnetic field of the Earth. Therefore, the actual formula for equatorial precession should read,

$$\begin{aligned}\langle(\boldsymbol{\Omega}_{\mathrm{g}})_{\mathrm{T}}\rangle &= (\boldsymbol{\Omega}_{\mathrm{g}})_{\mathrm{Eq}} = -\frac{3}{4}\kappa\frac{u^2}{c^2}\boldsymbol{\omega}_{\mathrm{E}}+\lambda\boldsymbol{\omega}_{\mathrm{E}}\\ &= -\frac{15}{8}\frac{G}{c^2r^3}\kappa\boldsymbol{J}+\frac{5}{2Mr^2}\lambda\boldsymbol{J}\\ &= \Big(-\frac{15}{8}\frac{G}{c^2r^3}\kappa+\frac{5}{2Mr^2}\lambda\Big)\boldsymbol{J}\\ &= \Big(-\frac{15}{8}\kappa+\frac{5rc^2}{2GM}\lambda\Big)\frac{G}{c^2r^3}\boldsymbol{J},\end{aligned} \tag{68}$$

where λ is constant.

Relativists call this precession given in Eq. (68) Lense–Thirring precession.[20]

In the Gravity Probe B results[5] for equatorial precession, the coefficient of $[G/(c^2r^3)]\boldsymbol{J}$ in Eq. (68) has been measured to be $-1/2$ (with a wide range of error), which in no way contradicts the result from the classical physics.

7. Conclusion

In the present study, we see that the precession of the perihelion of Mercury, the so-called Thomas precession, the de Sitter precession, and the Lense–Thirring precession could readily be accounted as classical phenomena. In the measurements of the so-called Lense–Thirring effects, Gravity Probe B organizers have not measured the actual Coriolis acceleration of the satellite nor the contribution of the Earth's magnetic field to precession of the gyroscope by some independent experiments. In the absence of these measurements, it seems that the main objective of the Gravity Probe B mission has not been fulfilled.

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