Supergravity-induced interactions on thick branes

Nejat T. Yılmaz[†]

Department of Electrical and Electronics Engineering, Yaşar University, Selçuk Yaşar Kampüsü Üniversite Caddesi, No:35-37, Ağaçlı Yol, 35100, Bornova, İzmir, Turkey

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The gravity coupling of the symmetric space sigma model is studied in the solvable Lie algebra parametrization. The corresponding Einstein equations are derived and the energy–momentum tensor is calculated. The results are used to derive the dynamical equations of the warped five-dimensional (5D) geometry for localized bulk scalar interactions in the framework of thick brane world models. The Einstein and scalar field equations are derived for flat brane geometry in the context of minimal and non-minimal gravity-bulk scalar couplings.

Keywords: supergravity, branes, 5D warped geometry, symmetric spaces

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1. Introduction

Brane world scenarios, which have their roots in open string theory, have ingredients, D-branes, that are alternative approaches to relate higher dimensions to the standard model of fundamental interactions. Instead of considering diminished and compact extra dimensions, they provide a sensible resolution especially to the hierarchy problem with non-compact extra dimensions. Although there were pioneering ideas in the 1980's,[1,2] the string theory inspired brane world cosmological scenarios were proposed and constructed much later.[3–10] These theories have flat or non-flat (warped), compact or non-compact bulk geometries. However, they share the common feature that the four-dimensional (4D) spacetime is an embedded solution in the bulk in such a way that the standard model fields are localized on it, whereas gravity can probe the extra dimension(s). The foundation of this picture lies in the dynamics of the various superstring theories in which the charge carrier open strings may be described by their end point dynamics of D-branes, whereas the closed strings, which accommodate gravitons, are not constrained to live on the brane like open string boundaries. Unlike the pioneering models, which ignore the brane size by explicitly containing Dirac-delta functions, more realistic brane models that have a finite brane size have also been constructed and are currently being studied.[11–28] These models fulfill the requirement of a fundamental length scale due to string theory. An updated review of these so-called thick branes can be found in Ref. [29].

In Ref. [28], it is discussed that the localized interactions of bulk scalars on a thick brane solution of a warped five-dimensional (5D) bulk can be described by a sigma-model in which the brane forming scalar is non-linearly coupled to the bulk scalar fields via its derivatives. In this manner, the kinetic term of the brane forming scalar is a function of the bulk scalars.

On the other hand, the nonlinear sigma model[30–32] governs the scalar sector of the supergravity theories.[33–36] In particular, if the target space (the scalar manifold) is a symmetric space,[37] then we have a special case of the general sigma model which can be named as the symmetric space sigma model.[38–40] The compactifications of the $D = 11$ supergravity (with its S^1-compactified IIA supergravity redundant) as well as the $D = 10$ IIB supergravity and the $D = 10$ type-I supergravity, which is coupled to the Yang–Mills theory, in majority produce scalar sectors in lower dimensions in the form of symmetric space sigma models. Moreover, the IIA and the IIB supergravities describe the tree-level low energy dynamics of the type-IIA and the type-IIB superstring theories whereas the type-I Yang–Mills supergravity is the effective low energy limit of the heterotic superstring theory. With this motivation in the above mentioned context of Ref. [28], in this work, we study and derive the thick brane dynamical equations in the presence of a generic symmetric space sigma model coupling of the 5D bulk gravity. We will focus on the flat Minkowski brane dynamics and derive the corresponding Einstein and scalar field equations for the warped geometry of the bulk. We will consider both cases of the minimal and the non-minimal gravity–scalar couplings. Our formulation will be for localized bulk scalars coupling to the brane solution-generating scalar via a generic symmetric space sigma model Lagrangian constructed in the solvable Lie algebra gauge. The outline of the construction of symmetric space sigma models by means of the solvable Lie algebra or the axion–dilaton parametrization can be referred to in Refs. [41]–[46]. Furthermore in Ref. [47], an explicit construction of the symmetric space sigma model Lagrangian in terms of the solvable Lie algebra parameters of the target space is presented for arbitrary

[†]Corresponding author. E-mail: nejat.yilmaz@yasar.edu.tr

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trace conventions. The most general form of the field equations are also derived for the target space coordinates in the same work within a general algebraic formalism.

The rest of this paper is organized as follows. In Section 2, we consider the coupling of the gravity to the symmetric space sigma model in a dimension-free and general framework. After defining the action, we derive the field equations and identify the energy–momentum tensor, which we will adopt in the following section. In Section 3, we turn our attention to the dynamics which give rise to thick branes with localized bulk scalar interactions on them. We will first discuss the ingredients of the warped 5D geometry, then we will derive the corresponding dynamical equations for the minimal and non-minimal gravity–scalar couplings. We will also express these equations in their appropriate form for a first-order formalism. Finally, we will obtain the scalar field equations in the 5D warped geometry context. For completeness, we collect some of the variational steps followed in deriving field equations of gravity–scalar couplings in Appendix A.

2. Coupling gravity to the symmetric space σ-model

In this section, we will focus on the gravity coupling of the scalar sectors of the dimensionally-reduced supergravity theories, which can in general be modelled as symmetric space or reductive coset sigma models.[38–40] Our construction will be for a general G/K coset scalar manifold where the scalars live in D dimensions. In the next section, we will use our general results in the context of thick brane world scenarios in which the scalar configuration will correspond to the interactions on the brane.

2.1. The action

The general nonlinear sigma model[30–32] action can be given as

$$S_{\rm NLSM} = \int {\rm d}^{(D)}\sigma\sqrt{-h}\,h^{AB}g_{ab}\partial_A\varphi^a\partial_B\varphi^b. \tag{1}$$

To clarify Eq. (1), one has to consider an immersion map

$$f : N \longrightarrow M, \tag{2}$$

of a smooth D-dimensional manifold N into another smooth manifold M. In supergravity, N is the spacetime and M is the scalar manifold; on the other hand, in p-brane dynamics, N is the world volume of the p-brane and M is the background space. The action (1) is defined on N. If one considers the metric h_{AB} on N as an independent field and varies the action (1) with respect to it, one finds the corresponding field equations

$$h_{AB} = g_{ab}\partial_A\varphi^a\partial_B\varphi^b, \tag{3}$$

which denote that h_{AB} comes out to be the pullback of the metric g_{ab} that is defined on M onto N through the immersion map (2) of N. In Eq. (1), h is the determinant of the metric h_{AB}. If on a chart of M the coordinates of the target space M are taken to be φ^b, then they can be considered to be scalar fields on N via the immersion map (2). For a local coordinate chart of N with coordinates x^A, the composition of the coordinate charts with Eq. (2) also gives the fields $\varphi^b(x^A)$ that appear in Eq. (1). In particular, the target manifold M can be chosen to be a coset space G/K. Furthermore, if G is a non-compact real form of any other semi-simple Lie group and if K is a maximal compact subgroup of G, also if the Lie algebra of K is a maximal compactly imbedded Lie subalgebra of the Lie algebra of G, then G/K is a Riemannian globally symmetric space for all the G-invariant Riemannian structures on it.[37] In this case, the sigma model is called the symmetric space sigma model (SSSM).[38–40] The action of the SSSM can be given as[41–47]

$$S_{\rm SSSM} = \frac{1}{4}\int {\rm tr}(*{\rm d}\mathcal{M}^{-1}\wedge{\rm d}\mathcal{M}), \tag{4}$$

where $\mathcal{M}$ is a map from N into $M = G/K$, which is based on the global solvable Lie algebra parametrization of G/K and the trace is over the matrix representation of G.[1)] Equation (4) is invariant under the global action of G from the right and the local action of K from the left. In Ref. [47], equation (4) is explicitly derived in terms of the global solvable Lie algebra parameters of G/K,[41–47] namely in terms of the fields

$$\{\phi^1,\phi^2,\ldots,\phi^r,\chi^1,\chi^2,\ldots,\chi^n\}, \tag{5}$$

which can be considered to be independent scalar fields on N and for which $r+n = \dim(G/K)$[2)]. We may further assume that the global parametrization of Eq. (5) coincides with a local coordinate chart of M. Also, if we take a local chart for N, then the fields in Eq. (5) can be considered to be coinciding with the ones in Eq. (1). In terms of these scalar fields from Ref. [47], the action reads

$$\begin{aligned} S_{\rm SSSM} = \int\Big(&-\frac{1}{8}\mathcal{A}_{ij}*{\rm d}\phi^i\wedge{\rm d}\phi^j \\ &-\frac{1}{4}\mathcal{B}_{i\alpha}*{\rm d}\phi^i\wedge\exp\Big(\frac{1}{2}\alpha_j\phi^j\Big)\Omega^\alpha_\gamma{\rm d}\chi^\gamma \\ &-\frac{1}{2}\mathcal{C}_{\alpha\beta}\exp\Big(\frac{1}{2}\alpha_j\phi^j\Big) \\ &*\Omega^\alpha_\gamma{\rm d}\chi^\gamma\wedge\exp\Big(\frac{1}{2}\beta_i\phi^i\Big)\Omega^\beta_\tau{\rm d}\chi^\tau\Big), \end{aligned} \tag{6}$$

where the coefficients $\mathcal{A}_{ij}$, $\mathcal{B}_{i\alpha}$, $\mathcal{C}_{\alpha\beta}$ are normalization constants that originate from the choice of the matrix representation of G; their exact definitions may be found in Ref. [47].

[1)] We refer the reader to the references given above for a detailed construction of Eq. (4).

[2)] In the following, we will use the indices $i,j,\ldots$ for the fields $\{\phi^i\}$, the indices $\alpha,\beta,\gamma,\ldots$ for the fields $\{\chi^\beta\}$, and we will use $A,B,C,\ldots$ as the indices of the manifold N.

α_j and β_i are the root vector components of the Cartan generators coupled to the fields ϕ^i in the solvable Lie algebra parametrization of G/K. We will not explicitly construct the solvable Lie algebra parametrization of the symmetric space G/K here which is extensive in its own right; therefore, we again refer the reader to Refs. [43]–[46] for further definitions where the solvable Lie algebra gauge is studied in detail. For our purposes, we will also not need to know the explicit form of the $n \times n$ matrix functions

$$\Omega = \Omega(\chi^\beta), \tag{7}$$

that appear in Eq. (6) and which will solely appear in our further formulation as coefficient functions. For this reason, we also refer the reader to Refs. [43]–[46] for their rigorous derivation and the involved definitions. Next we will introduce the elements of the gravity sector, which we will couple to Eq. (4). Since brane world scenarios are open to non-standard and modified gravity formalisms, we will try to be specific and transparent in our definitions. The gravity coupling will be by means of the metric h_{AB} on N. Thus, we will consider the unique metric-compatible, torsion-free Levi–Civita connection of the pseudo-Riemannian structure h_{AB} on N. The gravitational sector on N reads

$$S_{\mathrm{GRAV}} = \int R_{AB} \wedge *e^{AB}, \tag{8}$$

where we have chosen an orthogonal moving co-frame $\{e^A\}$ with $A = 1, \ldots, D$ on N. The curvature two-forms R_{AB} of the unique Levi–Civita connection of h_{AB} can be written as[3)]

$$R_{AB} = \mathrm{d}\omega_{AB} + \omega_{AC} \wedge \omega^C{}_B, \tag{9}$$

where $\{\omega^A{}_B\}$ are the connection one-forms of the Levi–Civita connection of h_{AB}, generated by the moving co-frame $\{e^A\}$. Since the moving co-frame is an orthogonal one, the metric-compatibility reads

$$\omega_{AB} = -\omega_{BA}. \tag{10}$$

The torsion-free condition can be written as[48–53]

$$\mathrm{d}e^A = -\omega^A{}_B \wedge e^B. \tag{11}$$

We may also introduce a potential term for the scalar fields as

$$S_{\mathrm{POT}} = \int *V, \tag{12}$$

where $V = V(\phi^i, \chi^\beta)$ is the potential that usually arises as a result of gauging away some of the symmetry in the scalar sector. Now we can write the total action as

$$S = \int \Big(R_{AB} \wedge *e^{AB} - *V - \frac{1}{4} \mathrm{tr}(*\mathrm{d}\mathcal{M}^{-1} \wedge \mathrm{d}\mathcal{M}) \Big), \tag{13}$$

where we have chosen the negative sign in front of the sigma model term to achieve positive kinetic terms for the scalars. Explicitly, we have

$$\begin{aligned} S = \int \Big(& R_{AB} \wedge *e^{AB} - *V + \frac{1}{8} \mathcal{A}_{ij} * \mathrm{d}\phi^i \wedge \mathrm{d}\phi^j \\ & + \frac{1}{4} \mathcal{B}_{i\alpha} * \mathrm{d}\phi^i \wedge \exp\Big(\frac{1}{2}\alpha_j \phi^j\Big) \Omega^\alpha_\gamma \, \mathrm{d}\chi^\gamma \\ & + \frac{1}{2} \mathcal{C}_{\alpha\beta} \exp\Big(\frac{1}{2}\alpha_j \phi^j\Big) \\ & * \Omega^\alpha_\gamma \, \mathrm{d}\chi^\gamma \wedge \exp\Big(\frac{1}{2}\beta_i \phi^i\Big) \Omega^\beta_\tau \, \mathrm{d}\chi^\tau \Big). \end{aligned} \tag{14}$$

The variation of this action results in the same field equations for the scalar fields with the ones already derived in Ref. [47] for the pure symmetric space sigma model. Essentially, the metric g on M is responsible for the global and the local symmetry of the sigma model action (4). There is a remarkable difference between the pure and the gravity-coupled cases. In the pure sigma model, the field equation of the the metric h is simply Eq. (3), which denotes that the metric h on N is the one induced by g via Eq. (2). Thus since h is considered to be an independent field, to generate the solution space of the model, one may choose an arbitrary h and then solve the sigma model field equations, which will minimize the pure sigma model action and which will also indirectly determine the metric g. One may repeat this procedure by changing h to generate the entire solution space. Coupling the sigma model to the gravity on the other hand abolishes this methodology. Now the coupling occurs by means of h and h cannot be chosen arbitrarily since when we introduce gravity on N, we solve for the class of pseudo-Riemannian metrics h whose curvatures minimize the action (14). The field equations of the scalar fields contain h, whereas the Einstein's equations, as it will be clear in the next subsection, contain the scalar fields as sources. Thus these two sets of equations must be solved simultaneously. Another divergence between the two cases is that the energy–momentum tensor associated with the fields Eq. (5) and the potential will not be null in the gravity-coupled case, thus equation (3) is no longer valid and the metric h on N is not induced by the metric g on M via Eq. (2) in this case.

2.2. The field equations

In this subsection, we will vary the action (14) to obtain the corresponding field equations of the SSSM that is coupled to gravity with also a potential term. We will not lay out the detailed steps of this variation, and we have collected some of the results in Appendix A. We refer the reader to Ref. [51] for the standard elements of the variational calculus of differential forms. We will divide the variation of the action into two parts; the first will contain the gravitational variation terms whereas

[3)] We should state that we raise and lower indices by using the metric h_{AB}. Since we choose an orthogonal moving co-frame, the metric components $\{h_{AB}\}$ are constant, thus one can freely raise and lower indices on both sides of Eq. (9). Also, in particular, we assume that N is connected, which guarantees that the signature of the pseudo-Riemannian structure h_{AB} is constant on N.

the second wil contain the variation of the scalars introduced in Eq. (5), which is independent of the variation of the metric. Thus we have

$$\delta S = \delta S_1 + \delta S_2. \tag{15}$$

For the Levi–Civita connection and for the choice of an orthogonal moving co-frame $\{e^A\}$, the explicit form of the first term, which is induced by the variation of the orthogonal co-frame, can be seen in the Appendix. By imposing the least action principle in Eq. (15) explicitly from Eq. (A1), we can obtain the Einstein equations, which read

$$\begin{aligned} *e_{ABC}\wedge R^{AB} &= V * e_C + (-1)^{D-1}\frac{1}{8}\mathcal{A}_{ij} \\ &\quad \times (\mathrm{d}\phi^i \wedge i_C * \mathrm{d}\phi^j + i_C \mathrm{d}\phi^j \wedge * \mathrm{d}\phi^i) \\ &\quad + (-1)^{D-1}\frac{1}{4}\mathcal{B}_{i\alpha}\exp\Big(\frac{1}{2}\alpha_j\phi^j\Big)\Omega^\alpha_\beta \\ &\quad \times (\mathrm{d}\phi^i \wedge i_C * \mathrm{d}\chi^\beta + i_C \mathrm{d}\chi^\beta \wedge * \mathrm{d}\phi^i) \\ &\quad + (-1)^{D-1}\frac{1}{2}\mathcal{C}_{\alpha\beta}\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\exp\Big(\frac{1}{2}\beta_j\phi^j\Big) \\ &\quad \times \Omega^\alpha_\tau \Omega^\beta_\rho (\mathrm{d}\chi^\tau \wedge i_C * \mathrm{d}\chi^\rho \\ &\quad + i_C \mathrm{d}\chi^\rho \wedge * \mathrm{d}\chi^\tau), \end{aligned} \tag{16}$$

where we have introduced the interior derivative $i_{\omega_1}\omega_2$[51]

$$i : (\omega_1, \omega_2) \in E_p \times E_q \longrightarrow i_{\omega_1}\omega_2 \in E_{(q-p)}, \tag{17}$$

which takes a p-form and a q-form and maps them to a $(q-p)$-form. We use the notation $i_C \equiv i_{e^C}$. The second term in Eq. (15), which contains the non-gravitational variation of the fields, is also derived in the Appendix. Equating it separately to zero leads us to the scalar field equations. From Eq. (A3), we read the dilatonic scalar field equations as

$$\begin{aligned} &(-1)^{(D-1)}\mathrm{d}\Big(\frac{1}{2}(\mathcal{A}_{ik}+\mathcal{A}_{ki}) * \mathrm{d}\phi^i + \mathcal{B}_{k\alpha}\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\Omega^\alpha_\beta * \mathrm{d}\chi^\beta\Big) \\ &= \frac{1}{2}\mathcal{B}_{i\alpha}\alpha_k * \mathrm{d}\phi^i \wedge \exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\Omega^\alpha_\beta \mathrm{d}\chi^\beta \\ &\quad + \mathcal{C}_{\alpha\beta}(\alpha_k+\beta_k)\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\Omega^\alpha_\tau \\ &\quad * \mathrm{d}\chi^\tau \wedge \exp\Big(\frac{1}{2}\beta_j\phi^j\Big)\Omega^\beta_\gamma \mathrm{d}\chi^\gamma - 4\partial_k V * 1. \end{aligned} \tag{18}$$

Similarly again via Eq. (A3), we obtain the axionic scalar field equations as

$$\begin{aligned} &(-1)^{(D-1)}\mathrm{d}\Big(\frac{1}{2}\mathcal{B}_{i\alpha}\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\Omega^\alpha_\theta * \mathrm{d}\phi^i \\ &+ \mathcal{C}_{\alpha\beta}\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\exp\Big(\frac{1}{2}\beta_j\phi^j\Big)(\Omega^\alpha_\gamma\Omega^\beta_\theta + \Omega^\alpha_\theta\Omega^\beta_\gamma) * \mathrm{d}\chi^\gamma\Big) \\ &= \frac{1}{2}\mathcal{B}_{i\alpha}\mathcal{D}^\alpha_{\theta\beta}\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big) * \mathrm{d}\phi^i \wedge \mathrm{d}\chi^\beta \\ &\quad + \mathcal{C}_{\alpha\beta}\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\exp\Big(\frac{1}{2}\beta_j\phi^j\Big) \\ &\quad \times (\mathcal{D}^\alpha_{\theta\tau}\Omega^\beta_\gamma + \Omega^\alpha_\tau\mathcal{D}^\beta_{\theta\gamma}) * \mathrm{d}\chi^\tau \wedge \mathrm{d}\chi^\gamma - 2\partial_\theta V * 1, \end{aligned} \tag{19}$$

where, as we have introduced in Appendix A, the matrix functions $\mathcal{D}_\theta \equiv \partial\Omega/(\partial\chi^\theta)$ in Eqs. (18) and (19) are already derived in Ref. [47] by a direct application of the Euler–Lagrange equations to the pure scalar action. However, in the present work, in Appendix A, we have preferred to obtain a complete formulation for the variation of the total action.

2.3. The energy–momentum tensor

Now following the identification of the energy–momentum one-forms associated with the symmetric space sigma model and the corresponding scalar potential coupling of the pure gravity, we will derive the component expression of the energy–momentum tensor resulting from these sources in Eq. (14). Starting with the Einstein equations (16), we immediately see that the energy–momentum one-forms t_C satisfy[51]

$$\begin{aligned} *t_C &= -V * e_C - (-1)^{D-1}\frac{1}{8}\mathcal{A}_{ij}(\mathrm{d}\phi^i \wedge i_C * \mathrm{d}\phi^j + i_C \mathrm{d}\phi^j \wedge * \mathrm{d}\phi^i) \\ &\quad - (-1)^{D-1}\frac{1}{4}\mathcal{B}_{i\alpha}\exp\Big(\frac{1}{2}\alpha_j\phi^j\Big)\Omega^\alpha_\beta \\ &\quad \times (\mathrm{d}\phi^i \wedge i_C * \mathrm{d}\chi^\beta + i_C \mathrm{d}\chi^\beta \wedge * \mathrm{d}\phi^i) \\ &\quad - (-1)^{D-1}\frac{1}{2}\mathcal{C}_{\alpha\beta}\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\exp\Big(\frac{1}{2}\beta_j\phi^j\Big) \\ &\quad \times \Omega^\alpha_\tau\Omega^\beta_\rho(\mathrm{d}\chi^\tau \wedge i_C * \mathrm{d}\chi^\rho + i_C \mathrm{d}\chi^\rho \wedge * \mathrm{d}\chi^\tau). \end{aligned} \tag{20}$$

Now we may express the field strengths of the scalar fields $\{\phi^i, \chi^\beta\}$ in terms of their components with respect to the orthogonal moving co-frame $\{e^C\}$ on N as

$$\begin{aligned} \mathrm{d}\phi^i &= \mathcal{F}^i_{\ C} e^C, \\ \mathrm{d}\chi^\beta &= \mathcal{H}^\beta_{\ C} e^C. \end{aligned} \tag{21}$$

If the orthogonal moving co-frame $\{e^C\}$ is taken to coincide with a coordinate basis $\{\mathrm{d}x^C\}$, then we have

$$\begin{aligned} \mathrm{d}\phi^i &= \frac{\partial\phi^i(x^A)}{\partial x^C}\mathrm{d}x^C \equiv \partial_C\phi^i \mathrm{d}x^C, \\ \mathrm{d}\chi^\beta &= \frac{\partial\chi^\beta(x^A)}{\partial x^C}\mathrm{d}x^C \equiv \partial_C\chi^\beta \mathrm{d}x^C, \end{aligned} \tag{22}$$

which implies that

$$\begin{aligned} \mathcal{F}^i_{\ C} &= \frac{\partial\phi^i(x^A)}{\partial x^C} \equiv \partial_C\phi^i(x^A), \\ \mathcal{H}^\beta_{\ C} &= \frac{\partial\chi^\beta(x^A)}{\partial x^C} \equiv \partial_C\chi^\beta(x^A). \end{aligned} \tag{23}$$

By taking the Hodge-dual of both sides of Eq. (20), also by inserting the component expansions Eq. (21) and by further simplifying, we can obtain the energy–momentum one-forms as

$$\begin{aligned} t_C = \Big(&-V h_{CA} - (-1)^{D-1}\Big(\frac{1}{8}\mathcal{A}_{ij}(-\mathcal{F}^{iB}\mathcal{F}^j_{\ B}h_{CA} + \mathcal{F}^i_{\ C}\mathcal{F}^j_{\ A} + \mathcal{F}^j_{\ C}\mathcal{F}^i_{\ A}) \\ &+ \frac{1}{4}\mathcal{B}_{i\alpha}\exp\Big(\frac{1}{2}\alpha_j\phi^j\Big)\Omega^\alpha_\beta(-\mathcal{F}^{iB}\mathcal{H}^\beta_{\ B}h_{CA} + \mathcal{F}^i_{\ C}\mathcal{H}^\beta_{\ A} + \mathcal{H}^\beta_{\ C}\mathcal{F}^i_{\ A}) \\ &+ \frac{1}{2}\mathcal{C}_{\alpha\beta}\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\exp\Big(\frac{1}{2}\beta_j\phi^j\Big)\Omega^\alpha_\tau\Omega^\beta_\rho \\ &\times (-\mathcal{H}^{\tau B}\mathcal{H}^\rho_{\ B}h_{CA} + \mathcal{H}^\tau_{\ C}\mathcal{H}^\rho_{\ A} + \mathcal{H}^\rho_{\ C}\mathcal{H}^\tau_{\ A})\Big)\Big)e^A. \end{aligned} \tag{24}$$

Since the energy–momentum tensor components $\{T_{CA}\}$ are defined through

$$t_C = T_{CA} e^A, \tag{25}$$

we can finally write the energy–momentum tensor in terms of the components of the scalar field strengths, which are expanded in the chosen orthogonal moving co-frame as

$$T_{CA} = -V h_{CA} + \mathcal{T}_{CA}, \tag{26}$$

where

$$\begin{aligned}\mathcal{T}_{CA} = &-(-1)^{D-1}\Big(\frac{1}{8}\mathcal{A}_{ij}(-\mathcal{F}^{iB}\mathcal{F}^j_{\ B}h_{CA} + \mathcal{F}^i_{\ C}\mathcal{F}^j_{\ A} + \mathcal{F}^j_{\ C}\mathcal{F}^i_{\ A}) \\ &+ \frac{1}{4}\mathcal{B}_{i\alpha}\exp\Big(\frac{1}{2}\alpha_j\phi^j\Big) \\ &\times \Omega^\alpha_\beta(-\mathcal{F}^{iB}\mathcal{H}^\beta_{\ B}h_{CA} + \mathcal{F}^i_{\ C}\mathcal{H}^\beta_{\ A} + \mathcal{H}^\beta_{\ C}\mathcal{F}^i_{\ A}) \\ &+ \frac{1}{2}\mathcal{C}_{\alpha\beta}\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\exp\Big(\frac{1}{2}\beta_j\phi^j\Big) \\ &\times \Omega^\alpha_\tau\Omega^\beta_\rho(-\mathcal{H}^{\tau B}\mathcal{H}^\rho_{\ B}h_{CA} + \mathcal{H}^\tau_{\ C}\mathcal{H}^\rho_{\ A} + \mathcal{H}^\rho_{\ C}\mathcal{H}^\tau_{\ A})\Big)\end{aligned} \tag{27}$$

is the energy–momentum tensor contribution of the symmetric space sigma model. In the next section, when we consider the thick brane world scenario, we will adopt especially the sigma model part of the energy–momentum tensor Eq. (26). However, after renaming the coefficients, it will be reduced to a simpler form by choosing the co-frame as a coordinate one and, in particular, by assigning a metric ansatz for the manifold N.

3. Bulk scalar interactions on thick brane worlds

After the detailed analysis of the symmetric space sigma model-gravity coupling in the previous section, we now turn our attention to the main objective of the present work, which aims at the derivation of the dynamical equations describing the bulk scalar interactions in the context of the smoothed brane world scenario. As discussed in Ref. [28], the bulk scalars interact with the brane-forming scalar within the framework of a sigma-model in which the bulk moduli scalars couple to the kinetic term of the brane-forming one. As we have already discussed in the Introduction, we will focus on the symmetric space sigma model form of interactions, which is the most realistic one, as it appears as a result of the dimensional reduction of the higher dimensional supergravity theories.

The link between the physical scenarios discussed in this section and the formal gravity-symmetric space sigma model coupling of the previous section will be apparent when we consider the minimal coupling in Subsection 3.2. In summary, in the gravity-sigma model coupling context, we will assume a 5D warped geometry of the bulk, which accommodates a Minkovski brane with a warp factor and derive the Einstein and scalar field equations for the 5D bulk in component form (rather than their global form of the previous section) so that they will lead us to the modified 4D brane spacetime dynamics, which is implicit in the warped-geometric cosmological scenario. The formalism of minimal coupling in Subsection 3.2 is almost the same with Section 2 and in the non-minimal coupling of Subsection 3.3, we will modify this formalism by deforming the Einstein–Hilbert term. The reason why we have studied the sigma model-gravity coupling separately in Section 2 is basically to derive the right-hand side of the Einstein equations, namely the matter energy–momentum tensor arising from the scalars of the sigma model in a neat and a general framework. On the other hand, in this section, we will work out the left-hand side of the Einstein equations, namely the Einstein tensor for the special geometry mentioned above. For both of the minimal and the non-minimal couplings, we will adopt the energy–momentum tensor of the sigma model generically derived in Section 2.

3.1. 5D geometry

Before we discuss the actions which define the brane dynamics in the presence of bulk scalars, let us set the back-scene geometry. First of all, we fix the bulk dimension to five so that $\dim N = 5$ for the notation we have introduced in the previous section. In this work, we will consider the flat (Minkovski) geometry for the brane.[28,29,54,55] Therefore, our 5D metric ansatz will be

$$h = \mathrm{e}^{2A}\eta_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \mathrm{d}y \otimes \mathrm{d}y, \tag{28}$$

where $x^4 = y$ is the fifth dimension coordinate, $A = A(y)$, and e^{2A} is the warp-factor for the brane geometry. Our conventions are such that: $A,B,C = 0,1,2,3,4$; $\mu,\nu = 0,1,2,3$; and $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. Now considering the 5D metric h_{AB} and its inverse h^{AB}, since via Eq. (28) we have

$$\begin{aligned}h &= \mathrm{diag}(\mathrm{e}^{2A},-\mathrm{e}^{2A},-\mathrm{e}^{2A},-\mathrm{e}^{2A},-1), \\ h^{-1} &= \mathrm{diag}(\mathrm{e}^{-2A},-\mathrm{e}^{-2A},-\mathrm{e}^{-2A},-\mathrm{e}^{-2A},-1),\end{aligned} \tag{29}$$

from

$$\Gamma^A_{BC} = \frac{1}{2}h^{DA}(h_{DB,C} + h_{DC,B} - h_{BC,D}), \tag{30}$$

one can calculate the Levi–Civita connection coefficients for the coordinate frame suggested in Eq. (28). They read

$$\begin{aligned}&\Gamma^4_{44} = \Gamma^4_{A4} = \Gamma^A_{44} = \Gamma^\mu_{44} = \Gamma^\gamma_{\mu\nu} = 0, \\ &\Gamma^\mu_{\nu 4} = A'\delta^\mu_\nu, \quad \Gamma^4_{\mu\nu} = e^{2A}A'\eta_{\mu\nu},\end{aligned} \tag{31}$$

where the prime denotes derivation with respect to the coordinate y. Via the Riemann tensor

$$R^A_{\ BCD} = \Gamma^A_{BD,C} - \Gamma^A_{BC,D} + \Gamma^A_{EC}\Gamma^E_{BD} - \Gamma^A_{ED}\Gamma^E_{BC}, \tag{32}$$

we can calculate the Ricci tensor $R_{AB} = R^{C}{}_{ACB}$ components and the Ricci scalar $R = h^{AB}R_{AB}$ as

$$R_{4\mu} = 0, \quad R_{44} = -4(A')^2 - 4A'',$$
$$R_{\mu\nu} = e^{2A}(4(A')^2 + A'')\eta_{\mu\nu}, \quad R = 20(A')^2 + 8A''. \tag{33}$$

Now we can calculate the corresponding Einstein tensor components $G_{AB} = R_{AB} - \frac{1}{2}h_{AB}R$ as

$$G_{4\mu} = 0, \quad G_{44} = 6(A')^2,$$
$$G_{\mu\nu} = e^{2A}(-6(A')^2 - 3A'')\eta_{\mu\nu}. \tag{34}$$

Furthermore, for a generic scalar field $f = f(y)$, by considering the double action of the covariant derivative

$$\nabla_A \nabla_B f = \partial_A \partial_B f - \partial_C f \Gamma^{C}_{BA}, \tag{35}$$

via Eq. (31), one can show that

$$\nabla_\mu \nabla_4 f = \nabla_4 \nabla_\mu f = 0, \quad \nabla_4 \nabla_4 f = f'',$$
$$\nabla_\mu \nabla_\nu f = -e^{2A} A' f' \eta_{\mu\nu}. \tag{36}$$

Also

$$\nabla^2 f = \nabla^C \nabla_C f = (\partial_B \partial_A f - \partial_C f \Gamma^{C}_{AB}) h^{AB} = -f'' - 4A' f'. \tag{37}$$

3.2. Minimal coupling

Now in this subsection we will derive the component form of the dynamical equations for the minimal coupling of the bulk scalars to the gravity sector and the brane forming scalar. The relative 5D-action is[28,54]

$$S = \int \Big(-\frac{1}{4} R_{AB} \wedge *e^{AB} - *V - \frac{1}{4}\mathrm{tr}(*\mathrm{d}\mathcal{M}^{-1} \wedge \mathrm{d}\mathcal{M}) \Big), \tag{38}$$

which differs from Eq. (13) by a gravitational coefficient. We may adopt the general results of the previous section for the gravity-σ-model coupling, bearing in mind that now we restrict ourselves to the 5D geometry we have introduced above. As usual, we will assume that $\phi^i = \phi^i(y)$ and $\chi^\beta = \chi^\beta(y)$. To derive the component form of the energy–momentum tensor, let us first introduce

$$g_{i\theta} = \mathcal{B}_{i\alpha} \exp\Big(\frac{1}{2}\alpha_j \phi^j\Big) \Omega^\alpha_\theta,$$
$$g_{\tau\gamma} = \mathcal{C}_{\alpha\beta} \exp\Big(\frac{1}{2}\alpha_i \phi^i\Big) \exp\Big(\frac{1}{2}\beta_j \phi^j\Big) \Omega^\alpha_\tau \Omega^\beta_\gamma,$$
$$h_{i\theta\beta} = \mathcal{B}_{i\alpha} \exp\Big(\frac{1}{2}\alpha_j \phi^j\Big) \mathcal{D}^\alpha_{\theta\beta},$$
$$h_{\theta\tau\gamma} = \mathcal{C}_{\alpha\beta} \exp\Big(\frac{1}{2}\alpha_i \phi^i\Big) \exp\Big(\frac{1}{2}\beta_j \phi^j\Big) \Omega^\beta_\gamma \mathcal{D}^\alpha_{\theta\tau}, \tag{39}$$

where, since $\mathcal{C}_{\alpha\beta} = \mathcal{C}_{\beta\alpha}$, we have $g_{\tau\gamma} = g_{\gamma\tau}$. In order to express the components of the scalar field equations in a compact form in a later subsection, we also define

$$g_{ki\theta} = \alpha_k \mathcal{B}_{i\alpha} \exp\Big(\frac{1}{2}\alpha_j \phi^j\Big) \Omega^\alpha_\theta,$$

$$\tilde{g}_{k\tau\gamma} = (\alpha_k + \beta_k)\mathcal{C}_{\alpha\beta} \exp\Big(\frac{1}{2}\alpha_i \phi^i\Big) \exp\Big(\frac{1}{2}\beta_j \phi^j\Big) \Omega^\alpha_\tau \Omega^\beta_\gamma. \tag{40}$$

With these definitions, further remembering that $D = 5$, $\mathcal{A}_{ij} = \mathcal{A}_{ji}$, and as the scalars are only functions of y $\partial^C(\cdot)\partial_C(\cdot) = -(\cdot)'(\cdot)'$ we can write down the components of the symmetric space sigma model energy–momentum tensor Eq. (27) as

$$\mathcal{T}_{44} = -\frac{1}{8}\mathcal{A}_{ij}\phi^{i\prime}\phi^{j\prime} - \frac{1}{4}g_{i\beta}\phi^{i\prime}\chi^{\beta\prime} - \frac{1}{2}g_{\tau\rho}\chi^{\tau\prime}\chi^{\rho\prime},$$
$$\mathcal{T}_{\mu\nu} = e^{2A}\eta_{\mu\nu}\Big(-\frac{1}{8}\mathcal{A}_{ij}\phi^{i\prime}\phi^{j\prime} - \frac{1}{4}g_{i\beta}\phi^{i\prime}\chi^{\beta\prime} - \frac{1}{2}g_{\tau\rho}\chi^{\tau\prime}\chi^{\rho\prime}\Big),$$
$$\mathcal{T}_{4\mu} = \mathcal{T}_{\mu 4} = 0. \tag{41}$$

In order to obtain the component form of the Einstein equations, we may make use of the relation

$$-*e_{ABC} \wedge R^{BC} = 2G_{AB} * e^B, \tag{42}$$

in Eq. (16). Now, by also performing the appropriate normalization arising from the coefficient of the gravity sector in Eq. (38), we can read the component form of the Einstein equations from Eq. (16) as

$$G_{CB} = 2(Vh_{CB} - \mathcal{T}_{CB}), \tag{43}$$

where, like in Ref. [54], we use the convention $4\pi G = 1$ for the 5D bulk. By reading the appropriate metric components from Eq. (28), the Einstein tensor components from Eq. (34), also by using Eq. (41) in Eq. (43), we can derive the dynamical equations which will shape the finite brane solutions in the 5D bulk. We easily see that the $G_{4\mu}$ and the $G_{\mu 4}$ components give us null results in Eq. (43). The non-vanishing brane generating dynamical equations are the ones

for $\underline{G_{\mu\nu}}$

$$-3A'' - 6(A')^2 = +2V + \frac{1}{4}\mathcal{A}_{ij}\phi^{i\prime}\phi^{j\prime} + \frac{1}{2}g_{i\beta}\phi^{i\prime}\chi^{\beta\prime} + g_{\tau\rho}\chi^{\tau\prime}\chi^{\rho\prime}, \tag{44a}$$

for $\underline{G_{44}}$

$$6(A')^2 = -2V + \frac{1}{4}\mathcal{A}_{ij}\phi^{i\prime}\phi^{j\prime} + \frac{1}{2}g_{i\beta}\phi^{i\prime}\chi^{\beta\prime} + g_{\tau\rho}\chi^{\tau\prime}\chi^{\rho\prime}. \tag{44b}$$

These two equations can be combined to yield

$$A'' = -\frac{1}{6}\mathcal{A}_{ij}\phi^{i\prime}\phi^{j\prime} - \frac{1}{3}g_{i\beta}\phi^{i\prime}\chi^{\beta\prime} - \frac{2}{3}g_{\tau\rho}\chi^{\tau\prime}\chi^{\rho\prime}, \tag{45}$$

which together with Eq. (44b) is more appropriate for the first-order formalism in search for brane solutions. On the other hand, as discussed in Ref. [28], one can show that solutions of the first-order equations

$$\varphi^{a\prime} = \frac{1}{2}g^{ab}\partial_{\varphi^b}W(\varphi^c), \quad A' = -\frac{1}{3}W(\varphi^c), \tag{46}$$

are also solutions of Eq. (44a), (44b), and (45) and the corresponding scalar field equations. Here supergravity originated superpotential W is related to the ordinary potential V via

$$V(\varphi^c) = \frac{1}{8}g^{ab}\partial_{\varphi^a}W(\varphi^c)\partial_{\varphi^b}W(\varphi^c) - \frac{1}{3}\big(W(\varphi^c)\big)^2. \tag{47}$$

In this expression, without sub-labelling, we have denoted the set of scalars ϕ^i and χ^β generally as φ^c. The scalar manifold metric g_{ab}, which is introduced in Eq. (1), can be directly read from Eq. (6), which is also explicitly studied in Ref. [56].

3.3. Non-minimal coupling

In this subsection, we will consider the generalization of the action Eq. (38) in the gravity–scalar coupling sense and we will derive the dynamical field equations of the non-minimal coupling of the scalars to the gravity. The action which contains the modified gravity-bulk scalar coupling takes the form

$$S=\int\Big(-\frac{1}{4}f\,R_{AB}\wedge *e^{AB}-*V-\frac{1}{4}\mathrm{tr}(*\mathrm{d}\mathcal{M}^{-1}\wedge\mathrm{d}\mathcal{M})\Big), \quad (48)$$

where $f=f(\phi^i,\chi^\beta)$ is a generic function of the bulk scalars. In Appendix A, we present the details of the variation of the gravity term in Eq. (48). By using Eq. (A17)[4] via the total variation of Eq. (48) we can write down the modified Einstein equations as

$$-\frac{1}{2}fG_{AB}+\frac{1}{2}\big(\nabla_A\nabla_B(f)-\nabla_C\nabla^C(f)h_{AB}\big)=-Vh_{AB}+\mathcal{T}_{AB}. \quad (49)$$

Now if we consider the 5D manifold with the metric (28) since $G_{4\nu}$; $\nabla_4\nabla_\nu(f)$, as well as $h_{4\nu}$, $\mathcal{T}_{4\nu}$ vanish Eq. (49) gives us null equations for the $G_{4\nu}$ and $G_{\nu 4}$ components. Similarly, by substituting the other components via Eqs. (28), (34), (36), (41) in Eq. (49) we get the non-vanishing finite-brane generating dynamical equations of the non-minimal gravity coupling of the bulk scalars as

for $G_{\mu\nu}$

$$\begin{aligned}&3f\left(A''+2(A')^2\right)+f''+3f'A'\\&=-2V-\frac{1}{4}\mathcal{A}_{ij}\phi^{i\prime}\phi^{j\prime}-\frac{1}{2}g_{i\beta}\phi^{i\prime}\chi^{\beta\prime}-g_{\tau\rho}\chi^{\tau\prime}\chi^{\rho\prime},\end{aligned} \quad (50a)$$

for G_{44}

$$\begin{aligned}-6f(A')^2-4f'A' &= 2V-\frac{1}{4}\mathcal{A}_{ij}\phi^{i\prime}\phi^{j\prime}-\frac{1}{2}g_{i\beta}\phi^{i\prime}\chi^{\beta\prime}\\&\quad-g_{\tau\rho}\chi^{\tau\prime}\chi^{\rho\prime}.\end{aligned} \quad (50b)$$

By again using the second one in the first, we obtain

$$\begin{aligned}3fA''+f''-f'A' &= -\frac{1}{2}\mathcal{A}_{ij}\phi^{i\prime}\phi^{j\prime}-g_{i\beta}\phi^{i\prime}\chi^{\beta\prime}\\&\quad-2g_{\tau\rho}\chi^{\tau\prime}\chi^{\rho\prime},\end{aligned} \quad (51)$$

which again together with G_{44}-component equation in Eq. (50) is more appropriate for a first-order formulation to search for finite brane solutions. We should state that like the minimal coupling case, one can also work on the construction of an equivalent first-order formalism in terms of a superpotential W.

3.4. Scalar field equations

We now consider the scalar sector and we will present the component form of the scalar field equations for both of the couplings discussed above. Beforehand, to be able to switch our formulation to the component form, we should remind the reader of the identity

$$e^A\wedge *e^B=h^{AB}*1, \quad (52)$$

which is valid for a generic moving co-frame. Furthermore, for a one-form $A=A_Ce^C$ and a scalar field φ, we have

$$\begin{aligned}\mathrm{d}*A&=\big(\nabla_CA^C\big)*1=\big(\partial_CA^C+A^C\Gamma^B_{CB}\big)*1,\\\mathrm{d}*\mathrm{d}\varphi&=\big(\nabla^2\varphi\big)*1,\end{aligned} \quad (53)$$

where ∇^2 is defined in Eq. (37). Now for the definitions in Eq. (39) let us introduce the coefficients

$$\begin{aligned}\mathrm{d}g_{k\beta}&=X_{ik\beta}\,\mathrm{d}\phi^i+X_{\theta k\beta}\,\mathrm{d}\chi^\theta,\\\mathrm{d}g_{\gamma\theta}&=X_{i\gamma\theta}\,\mathrm{d}\phi^i+X_{\kappa\gamma\theta}\,\mathrm{d}\chi^\kappa,\end{aligned} \quad (54)$$

where we define

$$\begin{aligned}X_{ik\beta}&=\frac{\partial g_{k\beta}}{\partial\phi^i},\quad X_{\theta k\beta}=\frac{\partial g_{k\beta}}{\partial\chi^\theta},\\X_{i\gamma\theta}&=\frac{\partial g_{\gamma\theta}}{\partial\phi^i},\quad X_{\kappa\gamma\theta}=\frac{\partial g_{\gamma\theta}}{\partial\chi^\kappa}.\end{aligned} \quad (55)$$

Following the definitions introduced in Eq. (39) and furthermore by making use of the above definitions, from Eq. (18) and Eq. (19) we can express the scalar field equations in a more compact form as

$$\begin{aligned}&\mathcal{A}_{ik}\,\mathrm{d}*\mathrm{d}\phi^i+g_{k\beta}d*\mathrm{d}\chi^\beta\\&\qquad=\Big(\frac{1}{2}g_{ki\beta}-X_{ik\beta}\Big)\,\mathrm{d}\phi^i\wedge*\mathrm{d}\chi^\beta\\&\qquad\quad+\big(\tilde{g}_{k\tau\gamma}-X_{\gamma k\tau}\big)\,\mathrm{d}\chi^\gamma\wedge*\mathrm{d}\chi^\tau+F_k(V),\\&\frac{1}{2}g_{i\theta}\,\mathrm{d}*\mathrm{d}\phi^i+2g_{\theta\gamma}d*\mathrm{d}\chi^\gamma\\&\qquad=\Big(\frac{1}{2}h_{i\theta\beta}-\frac{1}{2}X_{\beta i\theta}-2X_{i\theta\beta}\Big)\,\mathrm{d}\phi^i\wedge*d\chi^\beta\\&\qquad\quad+\big(h_{\theta\tau\gamma}+h_{\theta\gamma\tau}-2X_{\gamma\theta\tau}\big)\,\mathrm{d}\chi^\gamma\wedge*\mathrm{d}\chi^\tau\\&\qquad\quad-\frac{1}{2}X_{ji\theta}\,\mathrm{d}\phi^j\wedge*d\phi^i+F_\theta(V),\end{aligned} \quad (56)$$

where we have introduced

(i) Minimal coupling

$$\begin{aligned}F_k(V)&=-4\partial_{\phi^k}V*1\equiv\mathcal{F}_k(V)*1,\\F_\theta(V)&=-2\partial_{\chi^\theta}V*1\equiv\mathcal{F}_\theta(V)*1,\end{aligned}$$

(ii) Non-minimal coupling

$$\begin{aligned}F_k(V)&=\Big(-4\partial_{\phi^k}V-\frac{1}{4}\big(\partial_{\phi^k}f\big)R\Big)*1\equiv\mathcal{F}_k(V)*1,\\F_\theta(V)&=\Big(-2\partial_{\chi^\theta}V-\frac{1}{4}\big(\partial_{\chi^\theta}f\big)R\Big)*1\equiv\mathcal{F}_\theta(V)*1.\end{aligned} \quad (57)$$

[4] By also bearing in mind the appropriate normalization for the energy–momentum tensor Eq. (26) which relates it to the component-form variation of Eq. (A17).

The F-terms include the contributions to the scalar sector coming from the variation of the potential term for both of the cases, whereas there is an extra contribution for the non-minimal coupling case, which arises from the variation of the modified Hilbert–Einstein term in the action. Now by using Eq. (52) and Eq. (53), we can first express the component form of the scalar field Eqs. (56) then since we assume that $\phi^i = \phi^i(y)$ and $\chi^\beta = \chi^\beta(y)$ via the relations in Eqs. (36) and (37), we can obtain the scalar field equations for the 5D-bulk geometry arising from the metric (28). The result reads

(I) Dilaton equations

$$\mathcal{A}_{ik}\big(-\phi^{i\prime\prime}-4A'\phi^{i\prime}\big)+g_{k\beta}\big(-\chi^{\beta\prime\prime}-4A'\chi^{\beta\prime}\big)$$
$$=-\big(\frac{1}{2}g_{ki\beta}-X_{ik\beta}\big)\phi^{i\prime}\chi^{\beta\prime}-\big(\tilde{g}_{k\tau\gamma}-X_{\gamma k\tau}\big)\chi^{\gamma\prime}\chi^{\tau\prime}+\mathcal{F}_k(V),$$

(II) Axion equations

$$\frac{1}{2}g_{i\theta}\big(-\phi^{i\prime\prime}-4A'\phi^{i\prime}\big)+2g_{\theta\gamma}\big(-\chi^{\gamma\prime\prime}-4A'\chi^{\gamma\prime}\big)$$
$$=-\big(\frac{1}{2}h_{i\theta\beta}-\frac{1}{2}X_{\beta i\theta}-2X_{i\theta\beta}\big)\phi^{i\prime}\chi^{\beta\prime}$$
$$-\big(h_{\theta\tau\gamma}+h_{\theta\gamma\tau}-2X_{\gamma\theta\tau}\big)\chi^{\gamma\prime}\chi^{\tau\prime}$$
$$+\frac{1}{2}X_{ji\theta}\phi^{j\prime}\phi^{i\prime}+\mathcal{F}_\theta(V). \tag{58}$$

These equations are the coupled scalar equations for the brane forming scalar and the bulk scalars, which must be solved simultaneously with Eqs. (44a), (44b), and (45) for the minimal coupling and with Eqs. (50a), (50b), and (51) for the non-minimal coupling cases.

4. Conclusion

In Section 2, we have discussed the elements of the general action which couples gravity to the symmetric space sigma model in the solvable Lie algebra parametrization of the symmetric target space. We have performed the variation of the action to obtain the corresponding Einstein equations. The variation also leads to the scalar field equations, which coincide with the ones derived in Ref. [47], which are obtained by direct application of the Euler–Lagrange equations. After identifying the energy–momentum one-forms from the Einstein equations, we have calculated the energy–momentum tensor associated with a generic symmetric space sigma model action. Then in Section 3, we have turned our attention to the thick brane scenarios of fundamental interactions. Following the discussion about the constituents of the warped 5D bulk geometry, which accommodates finite thick brane solutions, we have focused on the dynamics of the bulk scalars in this framework. In this respect, we have used the results of the previous section to derive the dynamical equations for the models of thick branes, in which the bulk scalar interactions are assumed to be localized on the finitely sized brane solution that is formed by one of the scalars. As we have discussed before, such a model is described by a σ-model-gravity coupling in the context of warped bulk geometry. As we have mentioned in the Introduction, our focus has been on the interactions generated by symmetric space sigma models, which appear as the most common scalar sectors in the dimensional reduction of supergravities. Our analysis has included two distinct cases of minimal and non-minimal scalar-gravity couplings and for both of these cases, we have derived the component form of the dynamical equations appropriate for a first-order formalism. Finally, we also obtained the scalar field equations for both of the cases in component form for the warped bulk geometry.

Although our analysis is performed for a general σ-model, coupling it is done for the flat-brane case. One may consider extensions of the interaction formalism presented here to the anti-de Sitter (AdS) or de Sitter (dS) brane geometries to study localized gravity in these models, as well as the corresponding dual CFT renormalization group flow equations.[55] Similarly, one may also consider cosmological solutions in the 4D-sector including scalar interactions. Other localized interactions of the bulk (specifically for various supergravity multiplets) can also be considered in connection with the scalar sector, which is studied in detail here as it needs special care owing to its nonlinear sigma model structure. We should state that the formulation presented here is purely algebraic and general; it is applicable to the sigma models whose target spaces are generic symmetric spaces of the form G/K. Therefore, the results provide valuable and case-free tools for any other coupling extensions. From this point of view, our formulation presents a formal framework for the study of specifically chosen localized supergravity theories in the context of thick brane scenarios. In this work, although we have derived the necessary dynamical field equations for a class of models, we have not attempted to construct the first-order formalisms and studied their solutions. Starting from the currently occurring Grasmannian scalar manifold models and supersymmetry predicted superpotentials of supergravity theories, one may work out for either generic multiplets or for specific supergravity theories interaction carrying extended thick brane solutions in parallel with the rich literature on non-interacting single scalar models. One may also study bulk scalar interactions on the brane for higher dimensional bulk geometries in connection with higher dimensional supergravities. We should also remark that similar analysis can also be extended for generalized scalar and gravity dynamics.[54,57] Finally, apart from the specific directions we have mentioned above, the results of this work can find extensions in various aspects of thick brane dynamics, including the search for particular solutions and probing phenomenology of localized gravity, gauge, and matter fields in various geometries.[20,58–65]

In relevance to the mathematical content of the present work, one may also slightly change the point of view from the warped-geometric braneworld cosmological scenario to the consideration of the coupling of Section 2 in its own right. That is to say, the sigma model can purely be considered as a nonlinear source to the gravity sector. It can separately be used as an ingredient in the construction of UV completion of general relativity (GR) in a nonlinear context. On the gauge theory side in the UV completion by spontaneous symmetry breaking of the low energy effective sigma model theory, which is also closely related to the string theories, sigma models play the central role. One may expect a similar contribution to the quantum gravity model building, which also emerges from the unifying and underlying string theories. Recently, a ghost-free nonlinear extension of the Fierz–Pauli[66] massive gravity has been constructed.[67–72] In its physically truncated form[71] of this nonlinear massive gravity, the basic ingredient of the sigma model kinetic term, namely the induced metric (3), appears as an argument of the graviton potential, which gives the physical metric its mass as a result of a gravitational Higgs mechanism. In that formalism (3) becomes the so-called fiducial metric $f_{\mu\nu}$; the basic field, which is a sigma model type kinetic term, enables the coupling of the Skückelberg scalars to the physical metric. This emergence suggests that sigma models may play a more generalized role with their entire geometrical content in massive as well as massless nonlinear extensions of GR. We should share an observation here that introducing sigma models in massive graviton constructions, one would need to make changes in the link structure between the sites of the theory space construction of Ref. [67], which inspired the de Rham, Gabadadze, Tolley (dRGT) programme of massive gravity. In Ref. [67], the links were taken as general coordinate transformations. To introduce sigma models, one should also impose the embedding structure of them onto the links, possibly one may introduce immersion maps between various sites in the theory space and consider objects that not only transform covariently under both of the general coordinate transformations via links, but also admit respect to the immersion structure. This way of introducing sigma models at link level may also resemble the situation valid at low energies for certain gauge theory constructions.

Apart from this above-mentioned fundamental role, nonlinear sigma models can also be used as a direct nonlinear source in nonlinear massive gravity. This can possibly be done in three ways. Firstly in Ref. [67], the dual site on which the fiducial metric lives is also taken to be 4D, so that the links, namely the Stückelberg fields, which can be considered as coordinate transformations, have the correct number of dofs (for which one is identified as the Boulware–Deser (BD) ghost to be erased). One may introduce more than four scalars in the context of a sigma model immersion framework, for which only four of them will get a vev via $\phi^\mu = x^\mu + \delta\phi^\mu$ where $\delta\phi^\mu \equiv \pi^\mu$ are the Goldstone bosons, which represent small fluctuations around a background spacetime. Then the relevance of the rest of the scalars can be inspected in the modification of GR to explain DM, DE, or inflationary structures arising via nonlinear level interactions of these extra geometrical dofs. Secondly, in Ref. [73], it has been shown that in the nonlinear massive gravity dRGT theory, when one chooses the fiducial metric flat, the homogeneous and isotropic FRW-type cosmological solutions do not exist due to the same mechanism which cancels the BD ghost. The de Sitter (which is a symmetric space) choice of the fiducial metric is currently being studied (see for example Refs. [74] and [75]). These constructions admit FRW-type solutions, showing once more the special place of de Sitter space in massive gravity theories, but still they possess some physical problems to be resolved. At this point, in connection with the formalism presented in this work and in Ref. [56], one may introduce a parametric generalization for the de Sitter case, namely taking the fiducial metric as the most general form of the induced metric of the symmetric space sigma model, which may assist in the search for the true fiducial background to obtain a physical homogeneous and isotropic universe model from the massive gravity theory. Finally, as a third interface between the nonlinear massive gravity and the nonlinear sigma models, like in the attempts to introduce dynamics to the fiducial metric[76–81], one may take the fiducial metric as the induced metric on the immersed p-brane and consider p-brane massive gravity interactions as a new nonlinear source to the dRGT massive gravity.

Appendix A: Variational details

Here we present the details of the variation of the gravity-σ-model coupling of Section 2 and the non-minimal gravity–scalar coupling of Section 3.

A1 SSSM-gravity coupling

In spite of the fact that the results coincide with the standard variational methods, here we will present a detailed account of the global variation of the total action in Eq. (14). When one applies a variation operator on Eq. (14), one can consider the variation of the orthogonal moving co-frame $\{e^A\}$; δe^C, which contains the variation of the metric h_{AB} in it and the variation of the scalar fields, namely $\delta\phi^j, \delta\chi^\tau$ separately. Therefore, one may collect the relative terms in two disjoint groups as in Eq. (15). The first term, which contains the gravitational variation, becomes

$$\begin{aligned}\delta S_1 = \int \Big(& (-1)^{D-2}\mathrm{d}(*e^{AB}\wedge\delta\omega_{AB}) \\ & + \delta e^C \wedge \Big(*e_{ABC}\wedge R^{AB} - V * e_C \\ & - (-1)^{D-1}\frac{1}{8}\mathcal{A}_{ij}(\mathrm{d}\phi^i \wedge i_C * \mathrm{d}\phi^j + i_C \mathrm{d}\phi^j \wedge *\mathrm{d}\phi^i)\end{aligned}$$

$$
\begin{aligned}
&-(-1)^{D-1}\frac{1}{4}\mathcal{B}_{i\alpha}\exp\Big(\frac{1}{2}\alpha_j\phi^j\Big)\\
&\times\boldsymbol{\Omega}^{\alpha}_{\beta}(\mathrm{d}\phi^i\wedge i_C*\mathrm{d}\chi^{\beta}+i_C\mathrm{d}\chi^{\beta}\wedge*\mathrm{d}\phi^i)\\
&-(-1)^{D-1}\frac{1}{2}\mathcal{C}_{\alpha\beta}\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\exp\Big(\frac{1}{2}\beta_j\phi^j\Big)\\
&\times\boldsymbol{\Omega}^{\alpha}_{\tau}\boldsymbol{\Omega}^{\beta}_{\rho}(\mathrm{d}\chi^{\tau}\wedge i_C*\mathrm{d}\chi^{\rho}+i_C\mathrm{d}\chi^{\rho}\wedge*\mathrm{d}\chi^{\tau})\Big)\Big), \qquad \text{(A1)}
\end{aligned}
$$

and the second term, which is composed of the scalar field variations, reads

$$
\begin{aligned}
\delta S_2=\int\Big((-1)^{D-1}&\frac{1}{8}\mathcal{A}_{ij}(\delta\,\mathrm{d}\phi^j\wedge*\mathrm{d}\phi^i+\delta\,\mathrm{d}\phi^i\wedge*\mathrm{d}\phi^j)\\
&+(-1)^{D-1}\frac{1}{4}\mathcal{B}_{i\alpha}\Big(\delta\Big(\exp\Big(\frac{1}{2}\alpha_j\phi^j\Big)\boldsymbol{\Omega}^{\alpha}_{\beta}\Big)\mathrm{d}\chi^{\beta}\wedge*\mathrm{d}\phi^i\\
&+\exp\Big(\frac{1}{2}\alpha_j\phi^j\Big)\boldsymbol{\Omega}^{\alpha}_{\beta}(\delta\,\mathrm{d}\chi^{\beta}\wedge*\mathrm{d}\phi^i+\delta\,\mathrm{d}\phi^i\wedge*\mathrm{d}\chi^{\beta})\Big)\\
&+(-1)^{D-1}\frac{1}{2}\mathcal{C}_{\alpha\beta}\Big(\delta\Big(\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\exp\Big(\frac{1}{2}\beta_j\phi^j\Big)\boldsymbol{\Omega}^{\alpha}_{\tau}\boldsymbol{\Omega}^{\beta}_{\gamma}\Big)\\
&\times\mathrm{d}\chi^{\gamma}\wedge*\mathrm{d}\chi^{\tau}+\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\exp\Big(\frac{1}{2}\beta_j\phi^j\Big)\\
&\times\boldsymbol{\Omega}^{\alpha}_{\tau}\boldsymbol{\Omega}^{\beta}_{\gamma}(\delta\,\mathrm{d}\chi^{\gamma}\wedge*\mathrm{d}\chi^{\tau}+\delta\,\mathrm{d}\chi^{\tau}\wedge*\mathrm{d}\chi^{\gamma})\Big)\\
&-\Big(\partial_iV\delta\phi^i+\partial_{\alpha}V\delta\chi^{\alpha}\Big)*1\Big), \qquad \text{(A2)}
\end{aligned}
$$

where we define $\partial_iV\equiv\partial V/\partial\phi^i$ and $\partial_{\alpha}V\equiv\partial V/\partial\chi^{\alpha}$. One obtains the Einstein Eqs. (16) by equating the coefficients of the variations of the co-frame to zero in Eq. (A1). As usual, the first term in Eq. (A1), which is an exact differential form, gives a null result in the variation due to the standard assumptions of the variational method[51], thus it does not contribute to the Einstein equations. In general, when N is an r-chain, one can apply the Stoke theorem and then one can assume that the variation of the independent fields vanishes on the boundary ∂N of N. If $\partial N=\emptyset$, then directly the integral of the first term vanishes. We can furthermore simplify Eq. (A2) to write it in its final form in which we gather the coefficients of the variations $\delta\phi^i$ and $\delta\chi^{\theta}$. The result becomes

$$
\begin{aligned}
\delta S_2=\int\Big((-1)^{D-1}&\frac{1}{8}\mathcal{A}_{ij}\mathrm{d}(\delta\phi^j\wedge*\mathrm{d}\phi^i)+(-1)^{D-1}\frac{1}{8}\mathcal{A}_{ij}\mathrm{d}(\delta\phi^i\wedge*\mathrm{d}\phi^j)+(-1)^{D-1}\frac{1}{4}\mathcal{B}_{i\alpha}\mathrm{d}\Big(\delta\chi^{\beta}\wedge\exp\Big(\frac{1}{2}\alpha_j\phi^j\Big)\boldsymbol{\Omega}^{\alpha}_{\beta}*\mathrm{d}\phi^i\Big)\\
&+(-1)^{D-1}\frac{1}{4}\mathcal{B}_{i\alpha}\mathrm{d}\Big(\delta\phi^i\wedge\exp\Big(\frac{1}{2}\alpha_j\phi^j\Big)\boldsymbol{\Omega}^{\alpha}_{\beta}*\mathrm{d}\chi^{\beta}\Big)+(-1)^{D-1}\frac{1}{2}\mathcal{C}_{\alpha\beta}\mathrm{d}\Big(\delta\chi^{\gamma}\wedge\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\exp\Big(\frac{1}{2}\beta_j\phi^j\Big)\boldsymbol{\Omega}^{\alpha}_{\tau}\boldsymbol{\Omega}^{\beta}_{\gamma}*\mathrm{d}\chi^{\tau}\Big)\\
&+(-1)^{D-1}\frac{1}{2}\mathcal{C}_{\alpha\beta}\mathrm{d}\Big(\delta\chi^{\tau}\wedge\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\exp\Big(\frac{1}{2}\beta_j\phi^j\Big)\boldsymbol{\Omega}^{\alpha}_{\tau}\boldsymbol{\Omega}^{\beta}_{\gamma}*\mathrm{d}\chi^{\gamma}\Big)+\delta\phi^n\Big(-\partial_nV*1-(-1)^{D-1}\frac{1}{8}\mathcal{A}_{in}\mathrm{d}*\mathrm{d}\phi^i\\
&-(-1)^{D-1}\frac{1}{8}\mathcal{A}_{nj}\mathrm{d}*\mathrm{d}\phi^j+(-1)^{D-1}\frac{1}{8}\mathcal{B}_{i\alpha}\alpha_n\exp\Big(\frac{1}{2}\alpha_j\phi^j\Big)\boldsymbol{\Omega}^{\alpha}_{\beta}\mathrm{d}\chi^{\beta}\wedge*\mathrm{d}\phi^i-(-1)^{D-1}\frac{1}{4}\mathcal{B}_{n\alpha}\mathrm{d}\Big(\exp\Big(\frac{1}{2}\alpha_j\phi^j\Big)\boldsymbol{\Omega}^{\alpha}_{\beta}*\mathrm{d}\chi^{\beta}\Big)\\
&+(-1)^{D-1}\frac{1}{2}\mathcal{C}_{\alpha\beta}\Big(\frac{1}{2}\alpha_n\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\exp\Big(\frac{1}{2}\beta_j\phi^j\Big)+\frac{1}{2}\beta_n\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\exp\Big(\frac{1}{2}\beta_j\phi^j\Big)\Big)\boldsymbol{\Omega}^{\alpha}_{\tau}\boldsymbol{\Omega}^{\beta}_{\gamma}\mathrm{d}\chi^{\gamma}\wedge*\mathrm{d}\chi^{\tau}\Big)\\
&+\delta\chi^{\theta}\Big(-\partial_{\theta}V*1+(-1)^{D-1}\frac{1}{4}\mathcal{B}_{i\alpha}\exp\Big(\frac{1}{2}\alpha_j\phi^j\Big)\mathcal{D}^{\alpha}_{\theta\beta}\mathrm{d}\chi^{\beta}\wedge*\mathrm{d}\phi^i-(-1)^{D-1}\frac{1}{4}\mathcal{B}_{i\alpha}\mathrm{d}(\exp\Big(\frac{1}{2}\alpha_j\phi^j\Big)\boldsymbol{\Omega}^{\alpha}_{\theta}*\mathrm{d}\phi^i)\\
&+(-1)^{D-1}\frac{1}{2}\mathcal{C}_{\alpha\beta}(\mathcal{D}^{\alpha}_{\theta\tau}\boldsymbol{\Omega}^{\beta}_{\gamma}+\boldsymbol{\Omega}^{\alpha}_{\tau}\mathcal{D}^{\beta}_{\theta\gamma})\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\exp\Big(\frac{1}{2}\beta_j\phi^j\Big)\mathrm{d}\chi^{\gamma}\wedge*\mathrm{d}\chi^{\tau}\\
&-(-1)^{D-1}\frac{1}{2}\mathcal{C}_{\alpha\beta}\mathrm{d}\Big(\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\exp\Big(\frac{1}{2}\beta_j\phi^j\Big)\boldsymbol{\Omega}^{\alpha}_{\tau}\boldsymbol{\Omega}^{\beta}_{\theta}*\mathrm{d}\chi^{\tau}\Big)\\
&-(-1)^{D-1}\frac{1}{2}\mathcal{C}_{\alpha\beta}\mathrm{d}\Big(\exp\Big(\frac{1}{2}\alpha_i\phi^i\Big)\exp\Big(\frac{1}{2}\beta_j\phi^j\Big)\boldsymbol{\Omega}^{\alpha}_{\theta}\boldsymbol{\Omega}^{\beta}_{\gamma}*\mathrm{d}\chi^{\gamma}\Big)\Big)\Big), \qquad \text{(A3)}
\end{aligned}
$$

where we have introduced the matrix functions

$$
\mathcal{D}_{\theta}\equiv\frac{\partial\boldsymbol{\Omega}}{\partial\chi^{\theta}}. \qquad \text{(A4)}
$$

As we have pointed out earlier, we do not need their explicit form in our formulation in this work. The reader may refer to their formal derivation and therein their exact definitions in Ref. [47]. If we equate the coefficients of $\delta\phi^n$ and $\delta\chi^{\theta}$ to zero, also by again disregarding the exact-forms in Eq. (A3), which give surface terms, we may obtain the field equations for the scalars (18) and (19). These field equations are the ones which were already derived in Ref. [47].

A2 Non-minimal gravity–scalar coupling

Here we will take a look at the variation of the gravitational term in Eq. (48), which is the differing term from Eq. (38). We will derive the variation for a generic D-dimensional manifold N. By applying a standard analysis, we can write

$$
\begin{aligned}
\delta\Big(-\frac{1}{4}\int f*R\Big)&=-\frac{1}{4}\int\delta f*R-\frac{1}{4}\int f\,\delta(*R)\\
&=-\frac{1}{4}\int\big(\partial_if\,\delta\phi^i+\partial_{\alpha}f\,\delta\chi^{\alpha}\big)*R\\
&\quad-\frac{1}{4}\int f\,G_{AB}\delta h^{AB}*1\\
&\quad-\frac{1}{4}\int f\,\delta R_{AB}h^{AB}*1. \qquad \text{(A5)}
\end{aligned}
$$

For a coordinate moving co-frame starting from the definition of the Ricci tensor via Eq. (32) and by using the metric compatibility

$$
\nabla_Ch^{AB}=0, \qquad \text{(A6)}
$$

after some algebra one can show that

$$-\frac{1}{4}\int f\,\delta R_{AB}h^{AB}*1=-\frac{1}{4}\int f\,\nabla_C K^C*1, \tag{A7}$$

where

$$K^C=h^{AB}\delta\Gamma^C_{BA}-h^{AC}\delta\Gamma^B_{BA}. \tag{A8}$$

Furthermore, we have

$$\begin{aligned}-\frac{1}{4}\int f\,\nabla_C K^C*1=&-\frac{1}{4}\int\nabla_C(f\,K^C)*1\\&+\frac{1}{4}\int\nabla_C(f)K^C*1.\end{aligned} \tag{A9}$$

By using the metric compatibility Eq. (A6) once more, we can show that the second term in Eq. (A9) can be written as

$$\begin{aligned}\frac{1}{4}\int\nabla_C(f)K^C*1=&-\frac{1}{4}\int\nabla_C(f)\nabla_B\delta h^{CB}*1\\&-\frac{1}{2}\int\nabla_C(f)\delta\Gamma^B_{AB}h^{CA}*1.\end{aligned} \tag{A10}$$

Now the first term on the right-hand side of this equation can further be expressed as

$$\begin{aligned}-\frac{1}{4}\int\nabla_C(f)\nabla_B\delta h^{CB}*1=&-\frac{1}{4}\int\nabla_B\big(\nabla_C(f)\delta h^{CB}\big)*1\\&+\frac{1}{4}\int\nabla_B\big(\nabla_C(f)\big)\delta h^{CB}*1.\end{aligned} \tag{A11}$$

By using the volume form via

$$*1=\mathrm{d}x^D\sqrt{|\det h|}, \tag{A12}$$

and the identity

$$N^C\delta\Gamma^B_{CB}=\delta(\nabla_C N^C)-\nabla_C(\delta N^C), \tag{A13}$$

where

$$N^C=\nabla_A(f)h^{AC}, \tag{A14}$$

after some algebra by also using the Stoke theorem, we can express the second term on the right-hand side of Eq. (A10) as

$$\begin{aligned}&-\frac{1}{2}\int\nabla_C(f)\delta\Gamma^B_{AB}h^{CA}*1\\&=-\frac{1}{2}\int_{\partial N}\mathrm{d}x^{(D-1)}\delta\Big(\sqrt{|\det\tilde{h}|}\Big)N^C n_C\\&\quad+\frac{1}{2}\int\mathrm{d}x^D\delta\Big(\sqrt{|\det h|}\Big)\nabla_C N^C,\end{aligned} \tag{A15}$$

where $\{n_C\}$ is the unit normal vector to the boundary ∂N of N and in the first term on the right-hand side $\tilde{h}$ must be taken as the image of the D-dimensional metric h under the inclusion map of the boundary ∂N in the manifold N. Since

$$\delta(\sqrt{|\det h|})=-\frac{1}{2}\sqrt{|\det h|}h_{AB}\delta h^{AB}, \tag{A16}$$

we can finally write

$$\begin{aligned}\delta\Big(-\frac{1}{4}\int f*R\Big)=&-\frac{1}{4}\int\big(\partial_i f\,\delta\phi^i+\partial_\alpha f\,\delta\chi^\alpha\big)*R\\&-\frac{1}{4}\int f\,G_{AB}\delta h^{AB}*1\\&+\frac{1}{4}\int\nabla_B\big(\nabla_C(f)\big)\delta h^{CB}*1\\&-\frac{1}{4}\int\nabla_C\big(\nabla_A(f)\,h^{AC}\big)h_{BD}\delta h^{BD}*1\\&-\frac{1}{4}\int\nabla_C(f\,K^C)*1\\&-\frac{1}{4}\int\nabla_B\big(\nabla_C(f)\delta h^{CB}\big)*1\\&-\frac{1}{2}\int_{\partial N}\mathrm{d}x^{(D-1)}\delta\big(\sqrt{|\det\tilde{h}|}\big)N^C n_C.\end{aligned} \tag{A17}$$

It is obvious that due to the vanishing of the variation of the fields on the boundary ∂N, the last three terms above, which are the surface terms, identically vanish. We remind the reader of the action of the covariant derivative on the functional f

$$\begin{aligned}\nabla_B(f)&=\frac{\partial f}{\partial\phi^i}\frac{\partial\phi^i}{\partial x^B}+\frac{\partial f}{\partial\chi^\alpha}\frac{\partial\chi^\alpha}{\partial x^B},\\\nabla_A\nabla_B(f)&=\partial_A\partial_B f-\partial_C f\,\Gamma^C_{BA}.\end{aligned} \tag{A18}$$

Thus in Eq. (A17), we may identify

$$\nabla_C\nabla^C(f)\equiv\nabla_C\big(\nabla_B(f)h^{BC}\big)=\big(\partial_A\partial_B f-\partial_C f\,\Gamma^C_{BA}\big)h^{BA}. \tag{A19}$$

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