Analysis of dynamical properties for the two-site Bose–Hubbard model with an algebraic method*

Meng Xiang-Jia(孟祥佳)[a], Feng Hai-Ran(冯海冉)[b], and Zheng Yu-Jun(郑雨军)[a]†

[a] School of Physics, Shandong University, Jinan 250100, China
[b] Department of Physics and Information Engineering, Jining University, Qufu 273155, China

(Received 15 June 2013; revised manuscript received 17 September 2013; published online 10 February 2014)

In this work, we propose an algebraic recursion method to study the dynamical evolution of the two-site Bose–Hubbard model. We analyze its properties from the viewpoints of single partite purity, energy, and trace distance, in which the model is considered as a typical bipartite system. The analytical expressions for the quantities are derived. We show that the purity can well reflect the transition between different regimes for the system. In addition, we demonstrate that the transition from the delocalization regime to the self-trapping regime with the ratio η increasing not only happens for an initially local state but also for any initial states. Furthermore, we confirm that the dynamics of the system presents a periodicity for $\eta = 0$ and the period is $t_c = \pi/2J$ when the initial state is symmetric.

Keywords: two-site Bose–Hubbard system, dynamical evolution, purity, energy, trace distance

PACS: 03.75.Kk, 03.75.Lm, 32.80.Pj **DOI:** 10.1088/1674-1056/23/4/040305

1. Introduction

As an important and useful example to describe many-body quantum systems, the Bose–Hubbard (BH) model has received a great deal of attention in recent years. Many interesting properties and applications of the BH model, such as quantum phase transition,[1–3] entanglement generation,[4] tunneling dynamics,[5–7] etc., are investigated.[8–15] Due to the fact that the model can be realized in experiment through confining ultracold bosonic atoms in an optical lattice, thus opens up a possibility to directly observe and control the evolution of a quantum system in some special conditions.[16–20]

One of the most important properties for the BH model is the quantum phase transition. In Ref. [1], the authors demonstrated that at a low temperature limit (zero temperature), a phase transition arose from a superfluid phase to a Mott insulator via increasing the ratio $\eta = NU/J$ (U is the amplitude of the on-site interaction and J is the tunneling matrix element) for the ground state of the BH model. Then, Greiner *et al.* observed the quantum phase transition behavior[21] by borrowing the idea of confining ultracold dilute gas of bosonic atoms in an optical lattice. During this period, other properties for the BH model were also explored and studied.[22–27] Recently, Underwood *et al.* presented a scheme for universal quantum computation based on spinless interacting bosonic quantum walkers using a two-dimensional BH model,[28] therefore, a new practical application of the BH model arose in quantum information and quantum computation.

In previous studies, two lines of research can be pursued: one is devoted to studying the stationary properties for the BH model; the other one is focused on the dynamical evolution of the model, in which many theoretical approaches have been proposed and adopted, such as the mean-field theory,[29] the density-matrix renormalization group method,[30,31] the exact diagonalization Hamiltonian method,[32,33] and the process-chain approach.[34] In this paper, we will propose an alternative method to investigate the dynamical evolution of the standard two-site BH model. Our method is much more clear and explicit to understand the evolutional mechanism of the system. Four physical quantities, the probability distribution, purity, energy, and trace distance are used to describe the dynamical properties of the model. The analytical expressions of the quantities are derived based on the new algebraic method. From the analytical expressions, we note that the influence of the intrasystem parameters on the previous quantities is obvious. So we investigate the dynamics of the quantities with the case of $\eta = NU/J$ ranges from zero to a large value. Our investigations show that the dynamical quantities exhibit rich and complicated behaviors. From the behaviors we reprove that as the ratio η increases, the distribution of the particles in the two modes demonstrates a transition from a Josephson-like oscillation to a self-trapping one. In previous works that studied the dynamics of the two-site BH model, they all considered an initially local state. But here we demonstrate that the transition is not dependent on the initial state. This is one of emphasises in this paper. In addition, for the limit case of $\eta = 0$ (i.e., we ignore the interaction between the bosons), we find that the dynamics of the system also show a periodicity but the period is $t_c = \pi/2J$ based on a symmetric initial state and it is not dependent on the total number of particles.

*Project supported by the National Natural Science Foundation of China (Grant Nos. 91021009 and 21073110).
†Corresponding author. E-mail: yzheng@sdu.edu.cn
© 2014 Chinese Physical Society and IOP Publishing Ltd

Moreover, we discuss the energy transfer and information flow between two subsystems by investigating the energy and trace distance.

The paper is constructed as follows: in Section 2, we review the two-site BH model and present the algebraic theoretical framework employed in this paper. The definitions of the purity, energy, and trace distance are also given in this section. In Section 3, we present some interesting results for the dynamical quantities; all works are finished based on two symmetirc initial states. We end with a conclusion in Section 4.

2. The theoretical framework

The Hamiltonian of the standard two-site Bose–Hubbard model is written as follows:[35]

$$\hat{H} = -\hbar J(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + \frac{\hbar U}{2}[\hat{n}_1(\hat{n}_1 - 1) + \hat{n}_2(\hat{n}_2 - 1)], \quad (1)$$

where $\hat{a}_k$ and $\hat{a}_k^\dagger$ $(k = 1,2)$ are bosonic annihilation and creation operators for the two modes and they satisfy $[\hat{a}_k, \hat{a}_l^\dagger] = \delta_{kl}$. $\hat{n}_k = \hat{a}_k^\dagger \hat{a}_k$ is the particle number operator for the k-th mode. U is the boson–boson interaction amplitude within each mode, $U > 0$ and $U < 0$ correspond to repulsive and attractive interactions, respectively. J is the coupling strength between the two modes (or the tunneling strength between the two wells), its value equals half of the energy gap between the first excited state and the ground state of a single boson. For the sake of simplicity, we assume $J > 0$ and $\hbar = 1$ throughout this work.

For the bipartite system (every mode is considered as a subsystem), we assume that the total number of particles N is conserved, then the quantum state for the system can be expressed in a Hilbert space $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$, which is expanded with a set of Fock basis: $|0,N\rangle, |1,N-1\rangle, |2,N-2\rangle, \ldots, |N,0\rangle$. Obviously, the dimension of the Hilbert space is $N+1$. For an arbitrary initial state, the wave function for the whole system at time t can be written in this space as

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\hat{H}t}|\psi(0)\rangle = \sum_{m=0}^{N} C_m(t)|m,N-m\rangle, \quad (2)$$

where $C_m(t)$ is a complex coefficient. Through calculation and after some algebras, $C_m(t)$ can be written in a summation form as follows:

$$C_m(t) = \sum_{k=0}^{\infty} M_k^{m,N-m}, \quad (3)$$

where the recursion formula $M_k^{m,N-m}$ is given by

$$\begin{aligned} M_k^{m,N-m} &= -\frac{\mathrm{i}t}{k}\Big(V_1 M_{k-1}^{m,N-m} + V_2 M_{k-1}^{m+1,N-m-1} \\ &\quad + V_3 M_{k-1}^{m-1,N-m+1}\Big), \end{aligned} \quad (4)$$

and

$$\begin{aligned} V_1 &= \frac{U}{2}[m(m-1) + (N-m)(N-m-1)], \\ V_2 &= -J\sqrt{(m+1)(N-m)}, \\ V_3 &= -J\sqrt{m(N-m+1)}. \end{aligned} \quad (5)$$

The "starting condition" $M_0^{m,N-m}$ of the recursion formula $M_k^{m,N-m}$ is determined by the choice of the initial state.

Then the density matrix for the system can be obtained by

$$\begin{aligned} \rho(t) &= |\psi(t)\rangle\langle\psi(t)| \\ &= \sum_{m=0}^{N}\sum_{m'=0}^{N} C_m(t)C_{m'}^*(t)|m,N-m\rangle\langle m',N-m'|. \end{aligned} \quad (6)$$

Correspondingly, the reduced density matrix for subsystem 1, tracing out subsystem 2, can be expressed as

$$\begin{aligned} \rho_1(t) &= \mathrm{Tr}_2\rho(t) = \sum_{\kappa=0}^{N}\langle\kappa|\rho(t)|\kappa\rangle \\ &= \sum_{m=0}^{N}\sum_{m'=0}^{N} C_m(t)C_{m'}^*(t)|m\rangle\langle m'|\delta_{N-m,N-m'} \\ &= \sum_{m=0}^{N}|C_m(t)|^2|m\rangle\langle m|, \end{aligned} \quad (7)$$

where m and κ are possible numbers of particles in modes 1 and 2, respectively. It is distinct that the reduced density matrix $\rho_1(t)$ is diagonal in the subspace $\{|m\rangle, m = 0,1,\ldots,N\}$, and $|C_m(t)|^2$ is the existing probability of the state $|m,N-m\rangle$ at time t.

In order to better understand the recursive process for the system, in Fig. 1, we take into account the initial state $|n_1,n_2\rangle$ as an example. The evolutional mechanism of the initial state is illustrated. The effect of the intrasystem parameters U and J on the evolution of the system is apparent. The U is trying to keep the existence of the source state, while J plays a divergence role for the source state. Based on the previous analytical derivation and physical interpretation, in the next section, we will successively investigate the dynamical properties of the probability distribution, purity, energy, and trace distance of the system. The probability distribution $|C_m(t)|^2$ is very easy to be understood and obtained. The definitions of the other three quantities are as follows:

(i) In order to present the mixedness of the subsystem for the model, the purity is employed. It is defined as

$$P = \mathrm{Tr}\{\rho_1^2(t)\}, \quad (8)$$

where $\rho_1(t)$ is the reduced density matrix for subsystem 1. For a d-dimensional system, purity ranges from $1/d$ for a completely mixed state to 1 for a pure state.

(ii) For a composite system, the energy of a single subsystem, as the expectation value of Hamiltonian H_1, is a good indicator of the energy transfer between subsystems. It can be obtained by

$$E_1 = \mathrm{Tr}\{\rho(t)H_1\}, \quad (9)$$

where H_1 is the Hamiltonian of subsystem 1. $\rho(t)$ is the density matrix for the composite system.

(iii) The trace distance, one of the most employed quantifiers for distinguishability of quantum states, is defined as:[36,37]

$$D\{\rho_1^{(1)}(t),\rho_1^{(2)}(t)\}=\frac{1}{2}\mathrm{Tr}|\rho_1^{(1)}(t)-\rho_1^{(2)}(t)|, \tag{10}$$

where the modulus of an operator A is defined by $|A|=\sqrt{A^{\dagger}A}$. $\rho_1^{(1)}(t)$ and $\rho_1^{(2)}(t)$ are two quantum states for subsystem 1.

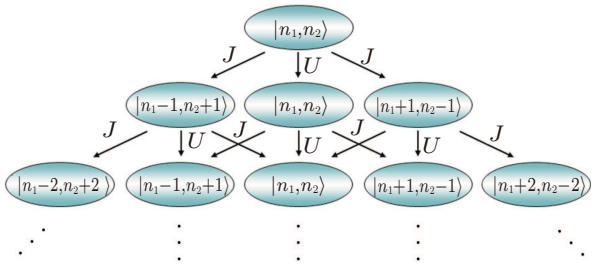


Fig. 1. (color online) Schematic illustration of the evolutional mechanism of the initial state $|n_1,n_2\rangle$.

The three quantities are all good measures to represent the correlation between subsystems in a composite system and they are employed from different physical backgrounds. So, in the following, we investigate the dynamical properties for the two-site BH system via studying the above physical quantities. Some interesting results will be presented.

3. Results and discussion

In this work, we consider two initial states, that is, Fock state $|\psi_1(0)\rangle$ and *NOON* state $|\psi_2(0)\rangle$

$$|\psi_1(0)\rangle = \left|\frac{N}{2},\frac{N}{2}\right\rangle, \tag{11}$$

$$|\psi_2(0)\rangle = \frac{1}{\sqrt{2}}(|0,N\rangle+|N,0\rangle). \tag{12}$$

The choice of the two initial states is motivated by the following facts: first, both of them are symmetric, and contrast with the asymmetric local state $|0,N\rangle$ (or $|N,0\rangle$), which is often used as an initial state in studying the dynamics of the BH model; second, they have the same total number of particles, but the former one is a product state and the later is a typical entangled state. Last, they all can be prepared in experiment through adjusting the ratio η and this has significant applications in quantum information and quantum computation, such as in quantum metrology and quantum sensing.

The "starting condition" for the initial Fock state is

$$M_0^{N/2,N/2}=1,\ \ M_0^{m,N-m}=0,\ \ (m\neq N/2), \tag{13}$$

and for the initial *NOON* state it is

$$M_0^{0,N}=M_0^{N,0}=\frac{1}{\sqrt{2}},\ \ M_0^{m,N-m}=0,\ \ (m\neq 0,N). \tag{14}$$

In the following we focus on the effect of ratio $\eta=NU/J$ on the dynamical properties of the system.

3.1. Dynamics of the probability distribution

We first investigate the dynamics of the probability distribution for the system. From the previous analytical expressions, we can see that the time-dependent quantum states of the system for two initial states have the same forms of Eqs. (2) and (6). But the values of $|C_m(t)|^2$ are different for the two cases.

In Fig. 2, we plot the probability distribution as a function of time t for several values of η with the total number of particles $N=40$. The left column corresponds to the initial Fock state of Eq. (11) and the right column is for the initial *NOON* state of Eq. (12). In panels (a1) and (b1), we first consider the limit case of $\eta=0$. This means that we ignore the interactions between bosons. Interestingly, no matter for the case of initial state (11) or (12), the dynamics of the probability distribution all demonstrate periodicity (although here we do not post a much longer time evolution). The periodic behavior is consistent with the results of previous works that the initial state is based on an asymmetric state $|0,N\rangle$. Actually, this periodicity can be considered as an optical analogy of the bosonic Josephson oscillation in the non-interacting regime. In addition, focusing on panel (a1) only, we can note that during the evolution, a crossover from the Fock state to a coherent-like state occurs, and various cases of the probability distribution emerge. The coherence and the superposition among different bases are important in quantum information processing.

From other panels for the cases of $\eta\neq 0$, i.e. the interaction between bosons is considered, we can see that the dynamical evolution of the probability distribution becomes complicated. But an important conclusion can also be obtained. It is shown that, from panel (a1) to panel (a3) on the left column, as η increases, the probability distribution concentrates to the intermediate value of m and the width of the peak of $|C_m(t)|^2$ becomes narrow. This indicates that the probabilities for the adjacent states of $|N/2,N/2\rangle$ become large while the local states, such as $|0,N\rangle$, $|N,0\rangle$, $|1,N-1\rangle$, and $|N-1,1\rangle$, disappear gradually. The hardening of the localization corresponds to a reduction of the coherence for the system. And if the value of η increases continually, the state of the system approaches to the initial Fock state. The same situation can also be obtained from panel (b1) to panel (b3), where when the value of η becomes very large, the quantum state of the system closes to the initial state infinitely. Then the system enters into the self-trapping regime. This phenomenon can be explained from the nature of η. Because the large η implies that the interaction strength U between bosons is relatively stronger than the coupling strength J between two modes. So the coupling strength is not strong enough to break the interaction between bosons. It is difficult for the boson to transfer from one

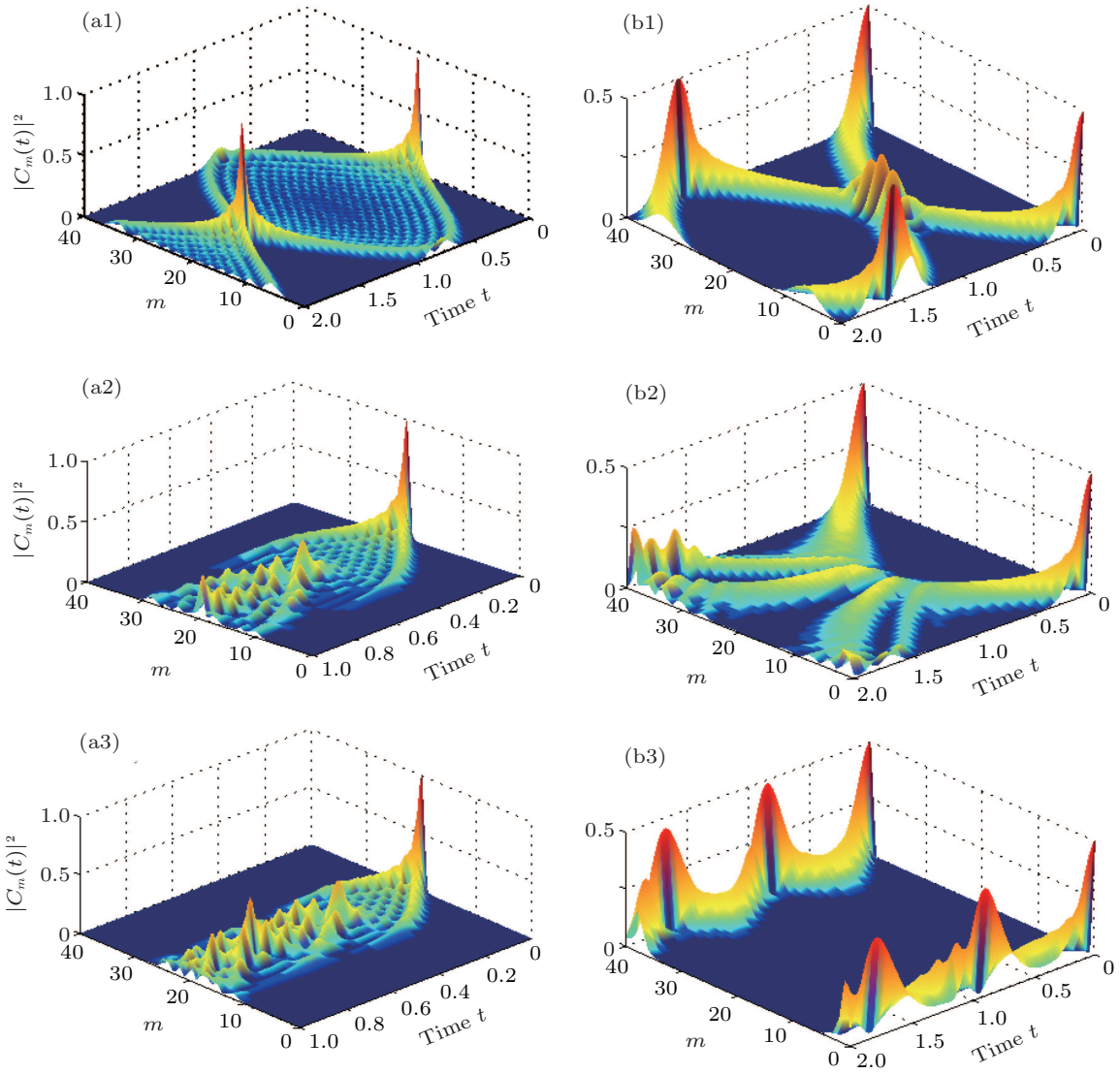


Fig. 2. (color online) The probability distribution $|C_m(t)|^2$ as a function of time t ($m = 0 \to N$). The parameter of $\eta = NU/J = 0$ (a1), 20 (a2), 40 (a3), 0 (b1), 4 (b2), 8 (b3). Panels (a1)–(a3) and (b1)–(b3) correspond to initial states $|\psi_1(0)\rangle$ and $|\psi_2(0)\rangle$, respectively. Other parameters are $J = 1, N = 40$.

mode to the other. Thus, the system presents the self-trapping regime. Actually, although we focus on the evolution mechanism for a large number of bosons (i.e., the collective behavior of all bosons), the real reason causing this collective behavior is the single boson tunnelling mechanism. For the case that there is only one or two bosons in the two modes, the tunnelling mechanism is very clear and the corresponding results have been proved in experiment. For a single boson, the two possible states $|1,0\rangle$ and $|0,1\rangle$ are coupled by the tunnel matrix element J. For the case of two bosons, the states $|2,0\rangle$ and $|0,2\rangle$ both directly couple to the state $|1,1\rangle$ via the tunnelling term. For weak interactions (η is very small), the two bosons can transfer between two modes freely. With η increasing, the two bosons cannot transfer between two modes independently, the correlated tunnelling process emerges. With a further increase of η, i.e., for strong interactions $U \gg J$, the energy difference between the states $|2,0\rangle$ and $|1,1\rangle$ is much larger than the coupling, resulting in a strong detuning and therefore in suppressed transitions between these states. The weak self-trapping phenomenon occurs. This rule is consistent with the case of many bosons.

Through calculations we also find that the probability distribution shows symmetry for $+\eta$ and $-\eta$, this reflects that the repulsive and attractive interactions between bosons have the same effect on the evolution of the BH system. It should be mentioned that although Fig. 2 refers to a particular value of the total number of particles $N = 40$, our results are checked for different values of N, and the same characteristics for the dynamics of the probability distribution are observed.

3.2. Dynamics of purity and energy

In the previous subsection, we have studied the dynamics of the probability distribution. This characterises the properties of the whole system. In order to investigate the properties of the subsystem and the correlation between the subsystems, the purity and the energy are employed. According to the def-

initions, they are readily calculated as follows:

$$P(t) = \mathrm{Tr}\{\rho_1^2(t)\} = \sum_{m=0}^{N} |C_m(t)|^2 \cdot |C_m(t)|^2, \qquad (15)$$

and

$$E(t) = \mathrm{Tr}\{\rho(t)H_1\} = \sum_{m=0}^{N} |C_m(t)|^2 \frac{U}{2} m(m-1). \qquad (16)$$

As shown in Eqs. (15) and (16), the dynamical quantities $P(t)$ and $E(t)$ depend strongly on the probability distribution $|C_m(t)|^2$. We first focus on the properties of purity. The behaviors of purity as a function of time t for different values of η for two initial states are shown in Fig. 3, where panels (a) and (b) are also for the cases of the initial states (11) and (12), respectively. From the first row of the figure, we can see that, for the limit case of $\eta = 0$, the evolutions of purity for two initial states all demonstrate periodic behaviors and the period agrees with the result of the probability distribution. Clearly, through calculation we find that the period is involved with the coupling strength J, it is $t_c = \pi/2J$. Compared with the result $t_{\mathrm{Rabi}} = \pi/J$ of Ref. [38], this can be approved because here we choose a symmetric initial state while in Ref. [38] they considered an asymmetric initial state. Moreover, the case for the value of the purity which equals 1 appears in the top-left panel for the Fock state. According to the definition of the linear entropy $Sl(t) = 1 - \mathrm{Tr}[\rho_1^2(t)]$, which is used to describe the entanglement between two subsystems, this illustrates that entanglement sudden death (ESD) occurs within this special condition. In addition, for the two kinds of initial states, when the value of η is relatively small, the amplitude of the purity is oscillational decay with time evolution. But as η increases, especially when the value of $\eta \geq 1$, the oscillational behavior of the purity almost disappears. The purity is always kept in a low level and approaches to a steady value with time evolution. This not only means that the mixedness of the subsystem is large but also means that the entanglement between two subsystems is strong. For this case, the near-stability of the purity also reflects the phenomenon that when the value of η is very large, the system will enter into the self-trapping regime. In addition, from the results of purity, the Josephson oscillation regime for the system is apparent when the value of η is very small not only for the Fock state but also for the *NOON* state. According to the dynamics of the purity, we also obtain the dynamical properties of entanglement between the two subsystems. Furthermore, it is worth noting that all the results of the purity are also symmetric about $\eta = 0$ (the case of $+\eta$ is equal to the one of $-\eta$). Studying the dynamical behavior of the purity for composite quantum systems is meaningful, because any quantum system cannot be completely isolated, they always interact with their environments more or less. The changing of the purity for one quantum (sub)system is a good reflection of the information flow between the (sub)system and its environment.

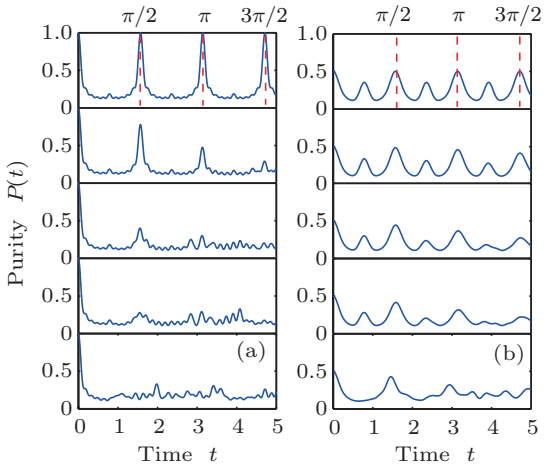


Fig. 3. (color online) The purity $P(t)$ as a function of time t for the initial states $|\psi_1(0)\rangle$ (a) and $|\psi_2(0)\rangle$ (b). The values of the ratio $\eta = NU/J$ from the top to the bottom are $0, 0.4, 0.8, 1, 5$. Other parameters $J = 1$ and $N = 10$.

Another notable physical quantity is the energy. The relationship between the energy transfer and the entanglement decoherence in a composite system has been studied in Ref. [39]. In Fig. 4, we present the behaviors of the energy $E(t)/U$ for subsystem 1 according to Eq. (16) with various sets of parameters (here, in order to study the properties of $E(t)$ under a unified standard, we calculate $E(t)/U$). All the values of parameters are given in the caption. Panels (a) and (b) correspond to the cases of the initial Fock state and the *NOON* state, respectively. In the first row, we also consider the limit case of $\eta = 0$. It is apparent that the evolution of the energy is periodic, but the period is not consistent with the ones of the purity and the probability distribution. We have mentioned that the employed quantities all have a strong dependence on $|C_m(t)|^2$, so why does the situation of the different period emerge? From the analytical expression of $E(t)$, the answer can be found that the energy of the ground state and the first excited state of subsystem 1 is not included indeed. In addition, when the value of η is small, such as $\eta < 1$, the dynamics of the energy is oscillationally damping with time evolution, which adopts a similar behavior to the purity, except for having a different frequency. However, with η increasing continually, especially for the case of $\eta \geq 5$, the behavior of the energy $E(t)$ becomes complicated. This reflects that the energy transfer between two subsystems is becoming irregular. Then, a furthermore physical interpretation is that the interaction U between bosons plays a more important role than the coupling strength J between two modes in the evolution of the system. This leads to a chaotic evolution behavior for the system. Although we investigate the dynamics of the energy for subsystem 1 with various cases, but contrasted with the dynamics of the probability distribution and the purity, we recognize that the ability of the energy

to describe the dynamical properties for the system is poorer than the other two quantities. One reason is we have pointed out from the expression of Eq. (16) that we actually ignore the energy of the ground state and the first excited state for the subsystem. The other reason is we can see that although the energy has the same order as the probability distribution, the number of particles of subsystem 1 will have a correction to the energy. Finally, it should be emphasized that the complete energy dissipation does not occur in this model. This is an obvious and essential difference with the open quantum systems.

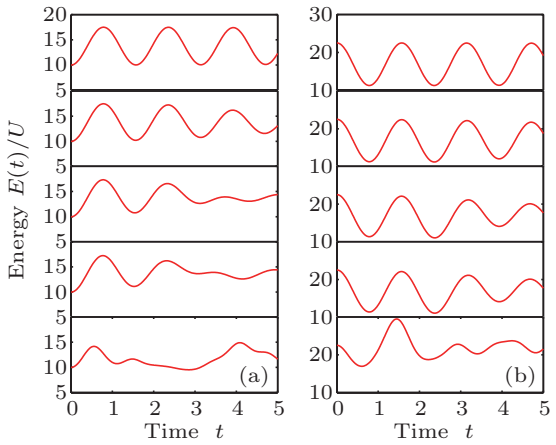


Fig. 4. (color online) The energy $E(t)/U$ as a function of time t for the initial states $|\psi_1(0)\rangle$ (a) and $|\psi_2(0)\rangle$ (b). The values of the ratio $\eta = NU/J$ from the top to the bottom are $0, 0.4, 0.8, 1, 5$. Other parameters $J = 1$ and $N = 10$.

3.3. Dynamics of the trace distance

Trace distance is one of the most employed quantifiers to describe the distinguishability of quantum states. A change of the trace distance between two quantum states for a system indicates an information flow between the system and its environment. In this work, we regard subsystem 1 as the target system and subsystem 2 as the environment. We study the dynamics of the trace distance between two quantum states corresponding to the initial Fock state and the *NOON* state for subsystem 1. The trace distance as a function of time t is plotted in Fig. 5 with various cases of η. As shown in the figure, the trace distance demonstrates a periodic evolution for $\eta = 0$ (blue line). But, as η increases, the behavior of the trace distance becomes intricate and it starts oscillating with time evolution. When η takes a larger value, the trace distance is almost unchanged, as exhibited for the case of $\eta = 20$. In addition, we note that for all cases of η, the trace distance always decreased at the early time of evolution and the rate of the decrease becomes slower with η increasing. This means that the distinguishability for the two time-dependent quantum states becomes smaller than both the initial states. A more intuitional description is that the distance between the two quantum states in a Bloch sphere becomes closer with evolution. Actually, it is not difficult to understand why when η takes a larger value, the trace distance is almost a constant. In the previous discussion, we demonstrated that when the value of η is large, such as $\eta \geq 5$, no matter for the cases of initial Fock state or *NOON* state, the time-dependent quantum states are almost "frozen" in their initial states. So, in this case, the distinguishability between two time-dependent quantum states is almost unchanged.

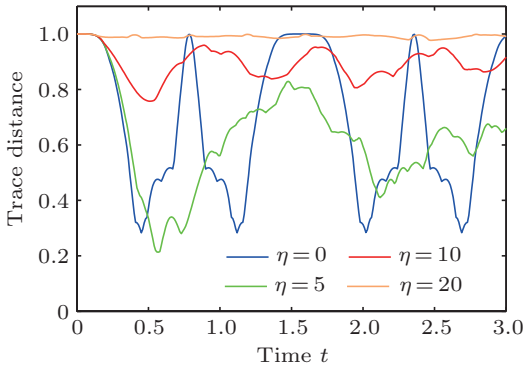


Fig. 5. (color online) The trace distance as a function of time t. The other parameters are $J = 1$, $N = 10$.

4. Conclusion

In this paper, we analytically investigated the dynamical properties for the standard two-site Bose–Hubbard (BH) model based on an algebraic recursion method. This method describes the dynamical evolution of the two-site Bose–Hubbard system clearly and explicitly. The probability distribution, purity, energy, and trace distance are employed to describe the dynamical properties for the system. All these studies are finished based on two symmetric initial states: the Fock state and the *NOON* state.

From the analytical expressions of all quantities, we show that the purity, energy, and trace distance all depend strongly on the probability distribution of the quantum state. The ratio $\eta = NU/J$ has a significant impact on the dynamics of the quantities. According to the behavior of the probability distribution, we find the regime transition of the BH system from the Josephson oscillation to the self-trapping as η increases from zero to a large value. We demonstrated that the emergency of the regime transition is not only based on an initially local state $|0, N\rangle$ but is also based on an arbitrary initial state, such as a product state and a entangled state. Through investigating the behaviors of purity and energy, we discussed the correlation between the two subsystems. When the value of η is relatively small, the behaviors of the purity and energy are oscillationally damping with time evolution. This reflects that the information flow between the two subsystems is regular and often. But with the value of η increases, the quantities show rich and complex behaviors. And, when η

increases up to a certain value, the time-dependent quantum state of the system seems to be suppressed. In this situation, the tunneling effect between two modes is restricted and the self-trapping phenomenon for the system occurs. In addition, we discussed the distinguishability of the two quantum states corresponding to both the initial states by studying the dynamics of the trace distance. Moreover, the limit case of $\eta = 0$ is considered as an emphasis, the probability distribution and the purity all demonstrate periodic behaviors for both symmetric initial states and the period is $t_c = \pi/2J$. The entanglement sudden death is also found under the special case. In the end, we should emphasize again, that first, the evolution mechanism for the two-site BH system, which is described by our proposed algebraic recursion method, is based on an arbitrary initial state. Secondly, the rule of the regime transition for the BH system with the parameter η increasing is valid for all initial states, although we only consider two symmetric initial states in this work, this has been proved by our calculation. And lastly, for the special case of $\eta = 0$, the dynamics of the system demonstrates periodicity for any initial states, and for an asymmetric initial state, the period is $t_c = \pi/J$, while for a symmetric state, the period is $t_c = \pi/2J$.

Actually, the increasing of η means that the on-site interaction between bosons becomes strong and the coupling between two modes becomes weak. The regime transition from the Josephson oscillation to the self-trapping with η increases is a gradual process. From all results for the investigated quantities in previous ones, we recognize that for the non-interacting regime $\eta = 0$, the dynamical properties for the system demonstrate meaningful characteristics. However, this requires a very low experimental temperature and an extremely diluted atomic gas sample to be achieved in experiment.

References

[1] Fisher M P A, Weichman P B, Grinstein G and Fisher D S 1989 *Phys. Rev. B* **40** 546
[2] Kühner T D and Monien H 1998 *Phys. Rev. B* **58** R14741
[3] Li B and Chen J B 2010 *Chin. Phys. Lett.* **27** 123701
[4] Romero-Isart O, Eckert K, Rodó C and Sanpera A 2007 *J. Phys. A: Math. Theor.* **40** 8019
[5] Tomadin A, Mannella R and Wimberger S 2007 *Phys. Rev. Lett.* **98** 130402
[6] Bissbort U, Deuretzbacher F and Hofstetter W 2012 *Phys. Rev. A* **86** 023617
[7] Cao H and Zhao Q 2010 *Acta Phys. Sin.* **59** 2187 (in Chinese)
[8] Jaksch D, Bruder C, Cirac J I, Gardiner C W and Zoller P 1998 *Phys. Rev. Lett.* **81** 3108
[9] Anders P, Gull E, Pollet L, Troyer M and Werner P 2010 *Phys. Rev. Lett.* **105** 096402
[10] Trefzger C and Sengupta K 2011 *Phys. Rev. Lett.* **106** 095702
[11] Chen B L, Kou S P, Zhang Y and Chen S 2010 *Phys. Rev. A* **81** 053608
[12] Sakmann K, Streltsov A I, Alon O E and Cederbaum L S 2010 *Phys. Rev. A* **82** 013620
[13] Collura M, Aufderheide H, Roux G and Karevski D 2012 *Phys. Rev. A* **86** 013615
[14] Pino M, Prior J, Somoza A M, Jaksch D and Clark S R 2012 *Phys. Rev. A* **86** 023631
[15] Xie Y D 2012 *Acta Phys. Sin.* **61** 023201 (in Chinese)
[16] Ziegler K 2010 *Phys. Rev. A* **81** 034701
[17] Longhi S 2011 *J. Phys. B: At. Mol. Opt. Phys.* **44** 051001
[18] Melé-Messeguer M, Juliá-Díaz B and Polls A 2011 *J. Low. Temp. Phys.* **165** 180
[19] Mazzarella G, Salasnich L, Parola A and Toigo F 2011 *Phys. Rev. A* **83** 053607
[20] Dell'Anna L 2012 *Phys. Rev. A* **85** 053608
[21] Greiner M, Mandel O, Esslinger T, Hansch T W and Bloch I 2002 *Nature* **415** 39
[22] Buonsante P and Vezzani A 2007 *Phys. Rev. Lett.* **98** 110601
[23] Deng X and Santos L 2011 *Phys. Rev. B* **84** 085138
[24] Ng K K, Chen Y C and Tzeng Y C 2010 *J. Phys.: Condens. Matter* **22** 185601
[25] Xu Y and Li J B 2012 *Acta Phys. Sin.* **61** 110207 (in Chinese)
[26] Wang Y M and Liang J Q 2012 *Chin. Phys. B* **21** 060305
[27] Zhou L, Liu Z J and Yan W B 2011 *Chin. Phys. B* **20** 074205
[28] Underwood M S and Feder D L 2012 *Phys. Rev. A* **85** 052314
[29] Oktel M Ö, Niţă M and Tanatar B 2007 *Phys. Rev. B* **75** 045133
[30] Ejima S, Fehske H and Gebhard F 2011 *Europhys. Lett.* **93** 30002
[31] Rossini D and Fazio R 2012 *New J. Phys.* **14** 065012
[32] Zhang J M and Dong R X 2010 *Eur. J. Phys.* **31** 591
[33] Wang L, Hao Y and Chen S 2008 *Eur. Phys. J. D* **48** 229
[34] Teichmann N, Hinrichs D and Holthaus M 2009 *Phys. Rev. B* **79** 224515
[35] Mazzarella G, Moratti M, Salasnich L and Toigo F 2010 *J. Phys. B: At. Mol. Opt. Phys.* **43** 065303
[36] Breuer H P, Laine E M and Piilo J 2009 *Phys. Rev. Lett.* **103** 210401
[37] Laine E M, Piilo J and Breuer H P 2010 *Phys. Rev. A* **81** 062115
[38] Juliá-Díaz B, Dagnino D, Lewenstein M, Martorell J and Polls A 2010 *Phys. Rev. A* **81** 023615
[39] Yu T 2007 *Phys. Lett. A* **361** 287