An approximation for the boundary optimal control problem of a heat equation defined in a variable domain*

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In this paper, we consider a numerical approximation for the boundary optimal control problem with the control constraint governed by a heat equation defined in a variable domain. For this variable domain problem, the boundary of the domain is moving and the shape of the boundary is defined by a known time-dependent function. By making use of the Galerkin finite element method, we first project the original optimal control problem into a semi-discrete optimal control problem governed by a system of ordinary differential equations. Then, based on the aforementioned semi-discrete problem, we apply the control parameterization method to obtain an optimal parameter selection problem governed by a lumped parameter system, which can be solved as a nonlinear optimization problem by a Sequential Quadratic Programming (SQP) algorithm. The numerical simulation is given to illustrate the effectiveness of our numerical approximation for the variable domain problem with the finite element method and the control parameterization method.

Keywords: boundary optimal control, heat equation, variable domain, finite element method, control parameterization method

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1. Introduction

Interest in the problems governed by various partial differential equations (PDEs) defined in variable domains has been growing in the last several decades because of their applications in physical science, chemical engineering, and many industrial processes. This kind of problem is classical and has been studied by many authors, including solidification with heat transfer and fluid flow,[1] chemical reaction within spherical particles,[2] drug release from matrix systems,[3] the formation of sedimentary ocean deltas,[4] and directional solidification of binary alloys with a curvature-dependent melting temperature.[5] For more applications and studies of this kind of problem, we would like to mention some other related works.[6–9]

In the existing published literature on variable domain problems, one of the main topics is the numerical method for approximations of PDEs' solutions. In Ref. [10], the author considered general linear parabolic equations in a given time-dependent domain and presented a general class of Galerkin-type approximations with applications to space-time finite elements, which are continuous with respect to the space variables, but admit discontinuities with respect to time at each step. Unconditional stability was proved and a general error estimate was established. In Ref. [11], a discretization of the one-dimensional heat equation in a time-dependent domain was considered. A generalization of the Crank–Nicholson scheme for the discretization with respect to time was constructed. Moreover, an application of this discretization method to a one-dimensional Stefan problem was given. Based on a finite-difference solution of equations of motion on an orthogonal curvilinear coordinate system, the authors in Ref. [12] presented a numerical technique for solving axisymmetric (or two-dimensional) free-boundary problems of fluid mechanics. There are many other numerical methods for approximations of PDEs defined in the variable domains.[2,13,14]

By far, many of the papers studying variable domain problems are concerned with numerical methods for various PDEs. But few works are devoted to numerical approximations of optimal control problems defined in variable domains. In this paper, we consider a numerical approach for solving an optimal boundary control problem with control constraint governed by a heat equation in a variable domain. In this case, the boundary of the variable domain is moving and the shape of the boundary is defined by a known time-dependent function.

Assume that $s : [0,T] \longrightarrow \mathbb{R}$ is a continuously differentiable function satisfying

$$s(0) = s_0,\ s(T) = L \ \text{ and } \ s(t) > 0,\ t \in [0,T]. \tag{1}$$

Write

$$Q = \{(x,t);\ 0 < x < s(t),\ 0 < t < T\}$$

and define the following set of functions by

$$\mathcal{U} = \{v \in L^2(0,T);\ a \le v(t) \le b \text{ a.e. } t \in [0,T]\}.$$

where $a, b \in \mathbb{R}$ are two given constants.

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The problem we shall study in this paper is the following boundary optimal control problem governed by a parabolic equation:

$$(P)\quad \min_{u\in\mathcal{U}}\left\{\frac{1}{2}\int_0^T\int_0^{s(t)}(y(x,t)-y^d(x,t))^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^T|u(t)-u^d(t)|^2\,\mathrm{d}t\right\}$$

subject to

$$\begin{cases} y_t - y_{xx} = f, \quad 0<x<s(t),\ 0<t\le T,\\ y_x(0,t)+\alpha y(0,t)=u(t),\ y(s(t),t)=0,\\ 0<t\le T,\\ y(x,0)=y^0(x), \quad 0<x<s_0, \end{cases} \tag{2}$$

where $y^0\in L^2(0,s_0)$, $f\in L^2(Q)$, $\alpha\in\mathbb{R}$ is the given constant and $y^d\in L^2(Q), u^d\in L^2(0,T)$ are two given target functions. Here and in what follows, we shall omit the notation t in the functions of t and the notations (x,t) in the functions of (x,t) to make sure there is no confusion.

As the development of the theory of optimal controls for PDEs,[15,16] the related theoretic results are expected to be used in fields of applied sciences. Thus, researchers are getting more and more interested in problems on the numerical approximation of optimal controls for PDEs.[17–19] In this paper, we will study the numerical approximation for the problem (P), which is a boundary optimal control problem governed by a heat equation defined in a variable domain. By introducing a new variable, we can get an optimal control problem $(\bar{P})$ with a fixed domain. Then, we shall set up a semi-discrete optimal control problem projected by the problem $(\bar{P})$ with the discretization of spacial variables. There are many spacial discretization methods, such as the difference method,[20] meshless method,[21,22] finite element method,[23,24] and so on. In our paper, by the Galerkin finite element method, we can obtain the semi-discrete optimal control problem (P_h), which is an optimal control governed by a system of ordinary differential equations. Based on the aforementioned semi-discrete problem, the control parameterization technique, which has been successfully applied to standard optimal control problems governed by finite dimensional systems,[25–27] will be used to approximate the admissible controls by suitable piecewise constant functions. In this case, the optimal control problem can be reduced to an optimal parameter selection problem (P_h^{p}) and can be solved as a nonlinear optimization problem by numerical optimization methods such as SQP. Finally, a numerical simulation is given to show that the numerical results are in good agreement with the exact solutions.

2. Reformulation of the original problem (P)

We see from Eq. (1) that $s(t)>0$ on $[0,T]$. By introducing the following new variable

$$z=\frac{x}{s(t)} \tag{3}$$

and the new unknown function

$$\bar{y}(z,t)=y(zs(t),t), \tag{4}$$

we can rewrite the controlled equation (2) as

$$\begin{cases} \bar{y}_t-\dfrac{s'(t)}{s(t)}z\bar{y}_z-\dfrac{1}{s^2(t)}\bar{y}_{zz}=\bar{f},\\ 0<z<1,\ 0<t\le T,\\ \dfrac{1}{s(t)}\bar{y}_x(0,t)+\alpha\bar{y}(0,t)=u(t),\ \bar{y}(1,t)=0,\\ 0<t\le T,\\ \bar{y}(z,0)=\bar{y}^0(z), \quad 0<z<1, \end{cases} \tag{5}$$

where $\bar{y}^0(z)=y^0(zs_0)$ and $\bar{f}(z,t)=f(zs(t),t)$. We denote by $\bar{Q}$ the product set $(0,1)\times(0,T)$, then we have $\bar{f}\in L^2(\bar{Q})$.

Thus, it follows from the transformations (3) and (4) that the problem (P) can be restated as

$$(\bar{P})\quad \min_{u\in\mathcal{U}}\left\{\frac{1}{2}\int_0^T\int_0^1(\bar{y}(z,t)-\bar{y}^d(z,t))^2 s(t)\,\mathrm{d}z\,\mathrm{d}t + \frac{1}{2}\int_0^T|u(t)-u^d(t)|^2\,\mathrm{d}t\right\}$$

subject to Eq. (5), where $\bar{y}^d(z,t)=y^d(zs(t),t)\in L^2(\bar{Q})$.

3. Semi-discrete approximation of $(\bar{P})$

In this section, we shall set up a semi-discrete approximation problem (P_h) for the problem $(\bar{P})$ with the finite element method. Associated with a parameter h with $h>0$, let $\mathcal{T}^h$ be a standard triangulation of $[0,1]$ into a finite number of subintervals using the grid points

$$0=z_1<z_2<\cdots<z_N<z_{N+1}=1,$$

where the mesh size of the triangulation $\mathcal{T}^h$ is h. Thus, $\mathcal{T}^h$ can be written as

$$\mathcal{T}^h=\{I_j=[z_j,z_{j+1}]\ ;\ j=1,2,\ldots,N\}.$$

Associated with every triangulation $\mathcal{T}^h$, we define a finite dimensional space as follows:

$$V^h=\{v_h\in C[0,1]\ ;\ v_h|_{I_j}\in P_1(I_j)\ \text{for}\ j=1,2,\ldots,N,\ v_h(1)=0\},$$

where $P_1(I)$ denotes the space of all polynomials of a degree less than or equal to 1, which are defined on the subinterval $I_j\in\mathcal{T}^h$. It is clear that $V^h\subset H_R^1(0,1)=\{v\in H^1(0,1);\ v(1)=0\}$. Write P_h for the L^2-projection from $L^2(0,1)$ onto V^h, defined by

$$(P_hv,v_h)=(v,v_h),\quad \forall v\in L^2(0,1), v_h\in V^h, \tag{6}$$

where $(\cdot,\cdot)$ is the usual inner product of $L^2(0,1)$. Then, the finite element approximation of Eq. (5) consists in finding $y_h(t) = y_h(\cdot,t)$, which belongs to V^h for $t \in [0,T]$, and satisfies

$$\begin{cases} (\partial_t y_h(t), v_h) + \dfrac{1}{s^2(t)}(\partial_z y_h(t), \partial_z v_h) \\ \quad - \dfrac{s'(t)}{s(t)}(z\partial_z y_h, v_h) - \dfrac{\alpha}{s(t)} y_h(0,t) v_h(0) \\ = (\bar{f}, v_h) - \dfrac{1}{s(t)} u(t) v_h(0), \ \forall v_h \in V^h, \ 0 < t \le T, \\ y_h(0) = P_h \bar{y}^0. \end{cases} \tag{7}$$

Let $\{\phi_i;\ i = 1,2,\ldots,N\}$ be the basis for the finite dimensional space V^h. Without loss of generality, we can take the basis functions to be continuous and piecewise linear polynomials that satisfy

$$\phi_i(z_j) = \begin{cases} 1, & \text{for } i = j; \\ 0, & \text{for } i \ne j, \end{cases}$$

for $i = 1,2,\ldots,N, j = 1,2,\ldots,N+1$. Thus, we can write

$$y_h(z,t) = \sum_{j=1}^{N} Y_j(t)\phi_j(z), \quad y_h(z,0) = \sum_{j=1}^{N} Y_j^0 \phi_j(z). \tag{8}$$

Substituting the first equation of Eq. (8) into Eq. (7) and taking $v_h = \phi_i$, we have

$$\begin{aligned} &\sum_{j=1}^{N}(\phi_i,\phi_j)\dot{Y}_j(t) + \frac{1}{s^2(t)}\sum_{j=1}^{N}(\partial_z\phi_i,\partial_z\phi_j)Y_j(t) \\ &- \frac{s'(t)}{s(t)}\sum_{j=1}^{N}(z\phi_i,\partial_z\phi_j)Y_j(t) \\ &- \frac{\alpha}{s(t)}\sum_{j=1}^{N}\phi_i(0)\phi_j(0)Y_j(t) \\ &= (\bar{f},\phi_i) - \frac{1}{s(t)}u(t)\phi_i(0), \quad \forall v_h \in V^h, \ 0 < t \le T. \end{aligned} \tag{9}$$

Moreover, by the definition of the L^2-projection (P_h), substituting the second equation of Eq. (8) into Eq. (6) yields

$$\sum_{j=1}^{N}(\phi_i,\phi_j)Y_j^0 = (y^0,\phi_i). \tag{10}$$

Thus, combining Eqs. (9) and (10), we see that equation (5) is equivalent to the following system of controlled ordinary differential equations

$$\begin{cases} \dot{Y}(t) = \tilde{F}(Y(t), \\ u(t)) = M^{-1}[-A(t)Y(t) + F(t) - B(t)u(t)], \\ 0 < t \le T, \\ Y(0) = Y^0, \end{cases} \tag{11}$$

where

$$\begin{aligned} &Y(t) = [Y_i(t)]_{N\times 1}, \quad Y^0 = [Y_i^0]_{N\times 1}, \\ &A(t) = \frac{1}{s^2(t)}K - \frac{s'(t)}{s(t)}W - \frac{\alpha}{s(t)}P, \\ &M = [(\phi_i,\phi_j)]_{N\times N}, \quad K = [(\partial_z\phi_i,\partial_z\phi_j)]_{N\times N}, \\ &W = [(z\phi_i,\partial_z\phi_j)]_{N\times N}, \quad P = [\phi_i(0)\phi_j(0)]_{N\times N}, \end{aligned}$$

and

$$F(t) = [(\bar{f},\phi_i)]_{N\times 1}, \quad B(t) = \frac{1}{s(t)}[\phi_i(0)]_{N\times 1}.$$

Moreover, if we write

$$\xi = [(\bar{y}^0,\phi_i)]_{N\times 1},$$

we can compute Y^0 through

$$Y^0 = M^{-1}\xi.$$

To get the semi-discrete form of the cost functional, we replace the target function $\bar{y}_d$ in the cost functional $(\bar{P})$ by

$$P_h\bar{y}^d(x,t) = \sum_{j=1}^{N} Y_j^d(t)\phi_j(x) \quad \text{for } t \in [0,T].$$

Again by Eq. (6), we have

$$\sum_{j=1}^{N}(\phi_i,\phi_j)Y_j^d(t) = (\bar{y}^d(t),\phi_i).$$

Thus,

$$Y^d(t) = M^{-1}\eta(t),$$

where

$$Y^d(t) = [Y_i^d(t)]_{N\times 1}, \quad \eta(t) = [(\bar{y}^d(t),\phi_i)]_{N\times 1}.$$

Then, the semi-discrete form of the boundary optimal control problem $(\bar{P})$ reads

$$\begin{aligned} (P_h) \quad \min_{u\in\mathcal{U}} \Big\{ &\frac{1}{2}\int_0^T s(t)(Y(t) - Y^d(t), M(Y(t) - Y^d(t)))_{\mathbb{R}^N}\,\mathrm{d}t \\ &+ \frac{1}{2}\int_0^T |u(t) - u^d(t)|^2\,\mathrm{d}t \Big\} \end{aligned} \tag{12}$$

subject to Eq. (11), where $(\cdot,\cdot)_{\mathbb{R}^N}$ denotes the inner product of $\mathbb{R}^N$.

4. The approximation for (P_h) with the control parameterization method

4.1. Control parameterization

To solve problem (P_h), we will apply the control parameterization technique, which is a versatile approach that has been successfully applied to a wide variety of practical problems. Control parametrization involves approximating the control function by a piecewise-constant function with possible discontinuities at a set of pre-assigned switching points, so that the original optimal control problem can be reduced to an optimal parameter selection problem. The heights of this piecewise-constant function are regarded as decision variables to be selected optimally.

To be more specific, we subdivide the time horizon $[0,T]$ into p subintervals $[t_{k-1},t_k), k=1,2,\ldots,p$, where $p\geq 1$ is a fixed integer and $t_k, k=0,1,\ldots,p$, are fixed knot points such that

$$0=t_0<t_1<t_2<\cdots<t_{p-1}<t_p=T. \tag{13}$$

Then, the control function $u(t)$ is approximated by a constant vector on each subinterval

$$u(t)\approx\sigma^k,\quad t\in[t_{k-1},t_k),\quad k=1,2,\ldots,p, \tag{14}$$

where σ^k denotes the approximate value of $u(t)$ on subinterval $[t_{k-1},t_k)$. The piecewise-constant approximation (14) can be expressed as follows:

$$u(t)\approx u^{\mathrm{p}}(t)=\sum_{k=1}^{\mathrm{p}}\sigma^k\chi_{[t_{k-1},t_k)}(t),\quad t\in[0,T], \tag{15}$$

where $\chi_{[t_{k-1},t_k)}$ is the indicator function of the interval $[t_{k-1},t_k)$ defined by

$$\chi_{[t_{k-1},t_k)}(t)=\begin{cases}1, & \text{if } t\in[t_{k-1},t_k),\\ 0, & \text{otherwise.}\end{cases} \tag{16}$$

Define

$$\begin{aligned}U=\{&\boldsymbol{\sigma}=[\sigma^1,\sigma^2,\ldots,\sigma^{\mathrm{p}}]^*\in\mathbb{R}^{\mathrm{p}};\\ &a\leq\sigma^i\leq b, i=1,2,\ldots,p\}.\end{aligned}$$

After the control parameterization, the dynamic system (11) becomes

$$\begin{cases}\dot{Y}(t)=\sum_{k=1}^{\mathrm{p}}\tilde{F}(Y(t),\sigma^k)\chi_{[t_{k-1},t_k)}(t),\ \ t\in[0,T],\\ Y(0)=Y^0.\end{cases} \tag{17}$$

Let $Y^{\mathrm{p}}(\cdot|\boldsymbol{\sigma})$ denote the solution of system (17) corresponding to $\boldsymbol{\sigma}\in U$. To determine $Y^{\mathrm{p}}(\cdot|\boldsymbol{\sigma})$, we can solve Eq. (17) sequentially over the subintervals $[t_{k-1},t_k), k=1,2,\ldots,p$. Now, the cost functional of problem (P_h) becomes

$$\begin{aligned}\mathcal{J}^{\mathrm{p}}(\boldsymbol{\sigma})=&\frac{1}{2}\int_0^T s(t)(Y^{\mathrm{p}}(t|\boldsymbol{\sigma})-Y^d,M(Y^{\mathrm{p}}(t|\boldsymbol{\sigma})-Y^d))_{\mathbb{R}^N}\,\mathrm{d}t\\ &+\frac{1}{2}\sum_{k=1}^{\mathrm{p}}\int_{t_{k-1}}^{t_k}\left|\sigma^k-u^d(t)\right|^2\mathrm{d}t.\end{aligned} \tag{18}$$

We may state the approximate optimal parameter selection problem as follows:

$$(P_h^{\mathrm{p}})\quad\min_{\boldsymbol{\sigma}\in U}\mathcal{J}^{\mathrm{p}}(\boldsymbol{\sigma}) \tag{19}$$

subject to the lumped parameter system (17).

4.2. Solving the problem (P_h^{p})

Problem (P_h^{p}) is an optimal parameter selection problem in the canonical form,[27] which can be solved as a nonlinear optimization problem using the SQP method. However, to do this, we need the gradient of the cost functional (18) with respect to the decision parameters. We now show that this gradient can be computed using the sensitivity method described in Ref. [28].

We first define the state variation with respect to σ^k as follows:

$$S^k(t|\boldsymbol{\sigma})=\frac{\partial Y^{\mathrm{p}}(t|\boldsymbol{\sigma})}{\partial\sigma^k},\quad k=1,2,\ldots,p. \tag{20}$$

Furthermore, let

$$\delta_{kl}=\begin{cases}1, & \text{if } k=l,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\hat{\delta}_{kl}=\begin{cases}1, & \text{if } k\leq l,\\ 0, & \text{otherwise.}\end{cases}$$

For each $l=1,2,\ldots,p$, it follows from Eq. (17) that

$$\begin{aligned}Y^{\mathrm{p}}(t|\boldsymbol{\sigma})=&\ Y^{\mathrm{p}}(t_{l-1}|\boldsymbol{\sigma})+\int_{t_{l-1}}^t\tilde{F}(Y^{\mathrm{p}}(s|\boldsymbol{\sigma}),\sigma^l)\,\mathrm{d}s,\\ &t\in[t_{l-1},t_l).\end{aligned} \tag{21}$$

If $k<l$, then differentiating Eq. (21) with respect to σ^k yields

$$\begin{aligned}\frac{\partial Y^{\mathrm{p}}(t|\boldsymbol{\sigma})}{\partial\sigma^k}=&\ \frac{\partial Y^{\mathrm{p}}(t_{l-1}|\boldsymbol{\sigma})}{\partial\sigma^k}\\ &+\int_{t_{l-1}}^t\frac{\partial\tilde{F}(Y^{\mathrm{p}}(s|\boldsymbol{\sigma}),\sigma^l)}{\partial Y}\frac{\partial Y^{\mathrm{p}}(s|\boldsymbol{\sigma})}{\partial\sigma^k}\,\mathrm{d}s,\\ &t\in[t_{l-1},t_l),\end{aligned} \tag{22}$$

where

$$\frac{\partial\tilde{F}(Y^{\mathrm{p}}(s),u^{\mathrm{p}}(s))}{\partial Y}=-M^{-1}A(s). \tag{23}$$

If $k=l$, differentiating Eq. (21) with respect to σ^k yields

$$\begin{aligned}\frac{\partial Y^{\mathrm{p}}(t|\boldsymbol{\sigma})}{\partial\sigma^k}=&\ \frac{\partial Y^{\mathrm{p}}(t_{l-1}|\boldsymbol{\sigma})}{\partial\sigma^k}\\ &+\int_{t_{l-1}}^t\frac{\partial\tilde{F}(Y^{\mathrm{p}}(s|\boldsymbol{\sigma}),\sigma^l)}{\partial Y}\frac{\partial Y^{\mathrm{p}}(s|\boldsymbol{\sigma})}{\partial\sigma^k}\,\mathrm{d}s\\ &+\int_{t_{l-1}}^t\frac{\partial\tilde{F}(Y^{\mathrm{p}}(s|\boldsymbol{\sigma}),\sigma^l)}{\partial u}\,\mathrm{d}s,\quad t\in[t_{l-1},t_l).\end{aligned} \tag{24}$$

If $k>l$, then it is obvious that

$$\frac{\partial Y^{\mathrm{p}}(t|\boldsymbol{\sigma})}{\partial\sigma^k}=\mathbf{0},\quad t\in[t_{l-1},t_l). \tag{25}$$

Combining Eqs. (22), (24), and (25), we obtain

$$\begin{aligned}\frac{\partial Y^{\mathrm{p}}(t|\boldsymbol{\sigma})}{\partial\sigma^k}=&\ \hat{\delta}_{kl}\frac{\partial Y^{\mathrm{p}}(t_{l-1}|\boldsymbol{\sigma})}{\partial\sigma^k}\\ &+\int_{t_{l-1}}^t\hat{\delta}_{kl}\frac{\partial\tilde{F}(X^{\mathrm{p}}(s|\boldsymbol{\sigma}),\sigma^l)}{\partial Y}\frac{\partial Y^{\mathrm{p}}(s|\boldsymbol{\sigma})}{\partial\sigma^k}\,\mathrm{d}s\\ &+\int_{t_{l-1}}^t\delta_{kl}\frac{\partial\tilde{F}(Y^{\mathrm{p}}(s|\boldsymbol{\sigma}),\sigma^l)}{\partial u}\,\mathrm{d}s,\\ &t\in[t_{l-1},t_l).\end{aligned} \tag{26}$$

By differentiating Eq. (26) with respect to time t, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial Y^{\mathrm{p}}(t|\boldsymbol{\sigma})}{\partial \sigma^k}\right\}\\ &= \hat{\delta}_{kl}\frac{\partial \tilde{F}(Y^{\mathrm{p}}(t|\boldsymbol{\sigma}),\sigma^l)}{\partial Y}\frac{\partial Y^{\mathrm{p}}(t|\boldsymbol{\sigma})}{\partial \sigma^k}\\ &\quad+\delta_{kl}\frac{\partial \tilde{F}(Y^{\mathrm{p}}(t|\boldsymbol{\sigma}),\sigma^l)}{\partial u}, \quad t\in[t_{l-1},t_l).\end{aligned} \tag{27}$$

Moreover,

$$\frac{\partial Y^{\mathrm{p}}(0|\boldsymbol{\sigma})}{\partial \sigma^k}=\mathbf{0}. \tag{28}$$

It follows from Eqs. (27) and (28) that, for each $k=1,2,\ldots,p$, the state variation defined in Eq. (20) satisfies the following dynamic system

$$\begin{aligned}\dot{S}^k(t|\boldsymbol{\sigma}) &= \hat{\delta}_{kl}\frac{\partial \tilde{F}(Y^{\mathrm{p}}(t|\boldsymbol{\sigma}),\sigma^l)}{\partial Y}S^k(t|\boldsymbol{\sigma})\\ &\quad+\delta_{kl}\frac{\partial \tilde{F}(Y^{\mathrm{p}}(t|\boldsymbol{\sigma}),\sigma^l)}{\partial u},\\ &\quad t\in[t_{l-1},t_l),\ \ l=1,2,\ldots,p,\end{aligned} \tag{29}$$

with the initial condition

$$S^k(0|\boldsymbol{\sigma})=\mathbf{0}. \tag{30}$$

As a result, by using the chain rule, we can derive the gradient of $\mathcal{J}^{\mathrm{p}}(\boldsymbol{\sigma})$ as follows:

$$\begin{aligned}\frac{\partial \mathcal{J}^{\mathrm{p}}(\boldsymbol{\sigma})}{\partial \sigma^k} &= \int_0^T s(t)[Y^{\mathrm{p}}(t|\boldsymbol{\sigma})-Y^d]^T M S^k(t|\boldsymbol{\sigma})\,\mathrm{d}t\\ &\quad+\int_{t_{k-1}}^{t_k}\left|\sigma^k-u^d(t)\right|\mathrm{d}t, \quad k=1,2,\ldots,p.\end{aligned} \tag{31}$$

By incorporating these formulae into a nonlinear programming algorithm such as SQP, we can solve the problem (P_h^{p}) numerically.

5. Numerical simulation

In this section, we present a numerical simulation for the boundary optimal control problem (P). Take $\alpha=1$, $T=1$, $b=0$, $a=-1$,

$$s(t)=\mathrm{e}^{0.1t}, \quad t\in[0,1],$$

$$y^0(x)=0, \quad x\in[0,1],$$

and

$$f(x,t)=(2t-0.1t^2)\mathrm{e}^{-0.1t}x-2t, \quad (x,t)\in Q$$

in Eq. (2). Moreover, take

$$y^d(x,t)=(\mathrm{e}^{-0.1t}x-1)t^2, \quad (x,t)\in Q$$

and

$$u^d(t)=(\mathrm{e}^{-0.1t}-1)t^2, \quad t\in[0,1]$$

in the cost functional of (P). Thus, for problem $(\bar{P})$, we have

$$\bar{y}^0(z)=0, \quad z\in[0,1],$$

$$\bar{f}(z,t)=(2t-0.1t^2)z-2t, \quad (z,t)\in\bar{Q},$$

and

$$\bar{y}^d(z,t)=(z-1)t^2, \quad (z,t)\in\bar{Q}$$

by Eq. (3). Under this setting, we see that the exact solution (i.e., the optimal control function) for the problem $(\bar{P})$ is known as

$$u(t)=(\mathrm{e}^{-0.1t}-1)t^2, \quad t\in[0,1],$$

which is also the optimal control for the original problem (P). Moreover, we can get the corresponding state function for the problem $(\bar{P})$ as

$$\bar{y}(z,t)=(z-1)t^2, \quad (z,t)\in\bar{Q}.$$

In this case, we have $u=u^d$ for $t\in[0,1]$, $\bar{y}=\bar{y}^d$ for $(z,t)\in\bar{Q}$, and then, the optimal value of the cost functional of problem $(\bar{P})$ is 0.

To get the numerical result for the problem $(\bar{P})$, we should first obtain the form of the semi-discrete problem (P_h) with the finite element method. For any $1\le i\le N$, the basis function ϕ_i is given by

$$\phi_i(x)=\begin{cases}N(x-x_{i-1}), & x_{i-1}\le x<x_i,\\ -N(x-x_{i+1}), & x_i\le x<x_{i+1},\\ 0, & \text{otherwise.}\end{cases}$$

By direct computations, we can obtain

$$M=\begin{pmatrix}1/3N & 1/6N & 0 & 0 & \cdots & 0\\ 1/6N & 2/3N & 1/6N & 0 & \cdots & 0\\ 0 & 1/6N & 2/3N & 1/6N & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 2/3N\end{pmatrix},$$

$$K=\begin{pmatrix}N & -N & 0 & 0 & \cdots & 0\\ -N & 2N & -N & 0 & \cdots & 0\\ 0 & -N & 2N & -N & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 2N\end{pmatrix},$$

$$W=\begin{pmatrix}-1/6N & 1/6N & 0 & 0 & \cdots & 0\\ -1/3N & -1/3N & 4/6N & 0 & \cdots & 0\\ 0 & -5/6N & -1/3N & 7/6N & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & -1/3N\end{pmatrix},$$

$$P=\begin{pmatrix}1 & 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 0\end{pmatrix},$$

and

$$B(t) = \mathrm{e}^{-0.1t}(1,0,0,\cdots,0)^*.$$

Moreover, we have

$$F(t) = (2t - 0.1t^2)\begin{pmatrix} 1/6N^2 \\ 1/N^2 \\ 2/N^2 \\ \vdots \\ N-1/N^2 \end{pmatrix} - 2t\begin{pmatrix} 1/2N \\ 1/N \\ 1/N \\ \vdots \\ 1/N \end{pmatrix}$$

and

$$Y^d(t) = t^2M^{-1}\begin{pmatrix} 1/6N^2 \\ 1/N^2 \\ 2/N^2 \\ \vdots \\ N-1/N^2 \end{pmatrix} - t^2M^{-1}\begin{pmatrix} 1/2N \\ 1/N \\ 1/N \\ \vdots \\ 1/N \end{pmatrix}.$$

In this simulation study, we take $N = 10$, which means the finite element triangulation $\mathcal{T}^h$ has divided the interval $[0,1]$ into 10 subintervals.

Then, by making use of the control parameterization method, the admissible controls can be approximated by piecewise constant functions and the optimal control problem (P_h) can be reduced to an optimal parameter selection problem (P_h^{p}). We will study three cases for $p = 10$, $p = 20$, and $p = 30$, respectively.

The following simulations are carried out by using the SQP procedure with an active strategy within the MATLAB environment and run in a personal computer. The numerical results of the optimal controls for the problem (P_h^{p}) are compared with the exact solution for the problem (P) in Figs. 1, 2, and 3, which show that the numerical results are in good agreement with the exact solution. Moreover, when p is increased from 5 to 10 and 20, the errors between the numerical values and the analytic values are decreased obviously.

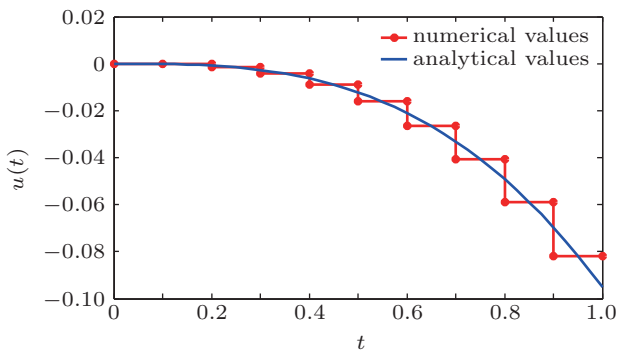


Fig. 1. (color online) Numerical values and analytical values for $p = 10$.

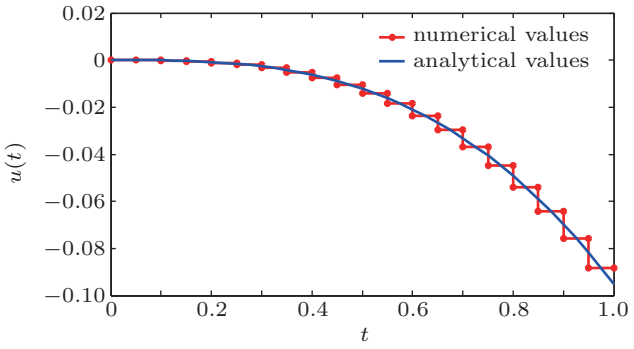


Fig. 2. (color online) Numerical values and analytical values for $p = 20$.

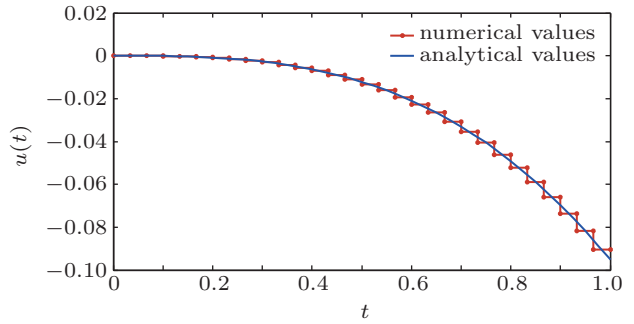


Fig. 3. (color online) Numerical values and analytical values for $p = 30$.

6. Conclusion

This paper is concerned with the numerical approximation for the optimal boundary control problem with control constraint governed by a heat equation in a variable domain, where the boundary of the variable domain boundary is defined by a known time-dependent function. Following the schedules of the Galerkin finite element discretization and control parameterization technique, the original optimal control problem can be reduced to an optimal parameter selection problem and can be solved by numerical optimization methods. A numerical simulation is given to illustrate the effectiveness of our numerical approximation for the variable domain problem.

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