Multi-ion Mach–Zehnder interferometer with artificial nonlinear interactions*

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We propose a method to implement a Mach–Zehnder interferometry based upon a string of trapped ions with artificial nonlinear interactions. By manipulating the coupling strength between two involved internal states of the ions, we could achieve the beam splitting/recombination with NOON states. Using current techniques for manipulating trapped ions, we discuss the experimental feasibility of our scheme and analyze some undesired uncertainty under realistic experimental environment.

Keywords: Mach–Zehnder interferometer, trapped ions, nonlinear interactions

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1. Introduction

Due to long coherence time of some specific hyperfine states, high controllability of operations, and high efficiency of state detection, the trapped atomic ion system has been considered as a promising candidate for quantum information processing. In particular, since they could be nearly perfectly prepared in some entangled states, the trapped ions have attracted considerable attention for high-precision quantum metrology.[1–5]

It is generally believed that the measurement precision can be enhanced from the standard quantum limit (SQL) or shot noise limit to the Heisenberg limit by utilizing multipartite entanglement.[6,7] For example, it has demonstrated that the measurement precision may reach the Heisenberg limit by using an NOON state $(|N\rangle_a|0\rangle_b + |0\rangle_a|N\rangle_b)/\sqrt{2}$, which is a superposition of N particles in mode a with zero particle in mode b and vice versa. However, most of the relevant proposals on high-precision interferometry of trapped ions are subject to limited numbers of ions or to the requirement for individual addressing of ions, which restricts the scalability of those interferometry schemes.

The Mach–Zehnder (MZ) interferometer has taken an increasingly important role in studying quantum physics recently.[8,9] The MZ interferometry consists of a beam splitter for splitting the incoming state and another beam splitter for state recombination. It has been shown that the adiabatic MZ interferometry is an optimal candidate for high-precision measurement.[10–12]

In this article, we consider how to accomplish MZ interferometry using large numbers of entangled trapped ions. To this end, we have to realize a nonlinear giant-spin Hamiltonian with anisotropic interaction. The key point is how to generate the nonlinear interactions and then achieve the beam splitting and recombination. By applying a pair of lasers to manipulate independently their center-of-mass (COM) modes (either transverse or longitudinal vibrational modes), the nonlinear giant-spin Hamiltonian with anisotropic interaction could be simulated bases upon an array of trapped ions in a linear trap. By adjusting the coupling strength (i.e., the Rabi frequency) between the two involved hyperfine states, the MZ interferometry could be carried out with NOON states. Based upon the current techniques for manipulating trapped ions, we discuss the experimental feasibility of our scheme and the difficulty due to decoherence in realistic experimental environment.

The paper is structured as follows. In the next section, we specifically describe the physical system and derive its effective Hamiltonian. Then we show how to realize the MZ interferometer by using NOON states. After that we discuss the experimental feasibility and analyze the undesired uncertainty due to decoherence. In the last section, we briefly summarize

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our results.

2. Model

We consider N identical ions confined in a linear trap, as shown in Fig. 1, where the three ionic states under our consideration are denoted by two hyperfine ground spin states ($|\downarrow\rangle$, $|\uparrow\rangle$) and an excited state $|\mathrm{e}\rangle$, respectively. In our scheme, all the ions are irradiated simultaneously by two traveling-wave laser beams with different frequencies, which coherently couple the hyperfine ground states through the optically stimulated Raman transitions by adiabatically eliminating the excited state $|\mathrm{e}\rangle$. Here, we assume the detuning of the optical fields from the electronic resonance to be much larger than the excited state linewidth and the corresponding coupling strength. As indicated in Fig. 1(a), under a usual rotating wave approximation to the laser frequencies, the interaction Hamiltonian between the fields and the ions is written as ($\hbar = 1$)[13]

$$H_1 = \nu a^\dagger a - \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \Omega_z \eta_z [(a^\dagger + a)\sigma_z^i \mathrm{e}^{-\mathrm{i}(\nu-\Delta)t} + \mathrm{h.c.}], \quad (1)$$

where $\sigma_z^i = |\downarrow\rangle_i\langle\downarrow| - |\uparrow\rangle_i\langle\uparrow|$ is the population inversion operator for the i-th ion, $a^\dagger$ (a) is the creation (annihilation) operator of the center-of-mass (COM) mode and Ω_z is the uniform effective Rabi frequency, ν is the vibrational frequency of the COM mode and Δ is the detuning from the vibrational frequency, η_z denotes the Lamb–Dicke parameter $k_z\sqrt{\hbar/2M\nu}$ with the ion mass M and the laser wavevector difference $k_z = k_{B,z} - k_{A,z}$.

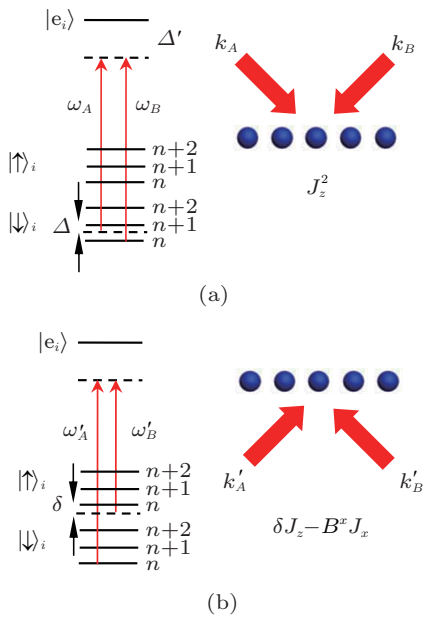


Fig. 1. (color online) Laser coupling scheme for simulating the Hamiltonian (6). (a) Physical realization of the nonlinear term of J_z^2: a Stark shift regarding the level $|\downarrow\rangle_i$ is created if two laser beams uniformly radiate the ions for the excitation from $|\downarrow\rangle_i$ to $|\mathrm{e}\rangle_i$ with a large detuning Δ'. (b) Physical realization of the terms of J_x and J_z: two additional laser beams are used to induce the carrier transitions corresponding to the term of J_x; and therefore the detuning to the resonant transition $\delta = \omega_0 - (\omega'_A - \omega'_B)$ will generate the longitudinal-field term δJ_z.

As indicated in Fig. 1(a), irradiating the ions to the longitudinal COM mode by applying two non-copropagating laser beams with the wavevector difference k_z along the z axis, we have the Hamiltonian below in the rotating frame regarding $H_0 = (\nu - \Delta)a^\dagger a$ under the rotating-wave approximation

$$H_2 = \mathrm{e}^{\mathrm{i}H_0 t} H_1 \mathrm{e}^{-\mathrm{i}H_0 t} = \Delta a^\dagger a - \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \Omega_z \eta_z (a^\dagger + a)\sigma_z^i.$$

Then, applying a canonical transformation $\mathrm{e}^{-S} H_2 \mathrm{e}^{S}$ with $S = \sum_{i=1}^{N} (\Omega_z \eta_z / \sqrt{N}\Delta)(a^\dagger - a)\sigma_z^j$,[14] we obtain

$$H_3 = \Delta a^\dagger a - \frac{2\Omega_z^2 \eta_z^2}{N\Delta} \sum_{i,j}^{N} \sigma_z^i \sigma_z^j. \quad (2)$$

Cooling the COM motion of the ions to its ground states, we obtain an effective nonlinear spin–spin interaction

$$\tilde{H}_3 = -\lambda J_z^2, \quad (3)$$

with $\lambda = 8\Omega_z^2 \eta_z^2/(N\Delta)$ and $J_z = \sum_{i=1}^{N} \sigma_z^i/2$.

The transverse-field term $-B^x J_x$ is also generated with additional lasers. As shown in Fig. 1(b), we set $\omega'_A - \omega'_B = \omega_0$, implying a resonant Raman process with respect to the two ground spin states of the ions. Such operations yield the transverse-field term[14]

$$\tilde{H}_2 = -B^x \sum_{i=1}^{N} \sigma_x^i/2 = -B^x J_x, \quad (4)$$

with B^x an effective Rabi frequency[14,15] and $J_x = \sum_{i=1}^{N} \sigma_x^i/2$.

If the Raman beams for generating the transverse-field term ($-B^x J_x$) are non-resonant, i.e., the frequency difference of the two lasers $\omega''_A - \omega''_B = \omega_0 - \delta$ with the detuning δ, we can obtain the longitudinal-field term

$$\tilde{H}_1 = \delta \sum_{i=1}^{N} \sigma_z^i/2 = \delta J_z, \quad (5)$$

see the schematic diagram Fig. 1(b).

Consequently, by tuning frequency differences and radiation directions of the pair of lasers along with independent manipulation of transverse or longitudinal COM modes, we may obtain the following effective Hamiltonian under some approximations and canonical transformations

$$H_{\mathrm{eff}} = \tilde{H}_1 + \tilde{H}_2 + \tilde{H}_3 = \delta J_z - B^x J_x - \lambda J_z^2, \quad (6)$$

where the collective angular momentum for all the ions are defined as $J_k = \sum_{i=1}^{N} \sigma_k^i/2$ with $k = (x, y, z)$.

3. MZ interferometry with NOON states

The MZ interferometer consists of a beam splitter for splitting input states and another beam splitter for recombining output states. Based upon the trapped-ion system (6), the two beam splitters for an MZ interferometer could be achieved by adiabatic quantum evolution through dynamical bifurcations.

Due to all ions in the ground state $|\downarrow\rangle$ initially and the external perturbation interacting with all ions in the same way,

our systematic joint internal state remains symmetric when the ions interchange. As a result, the trapped-ion system is regarded as an ensemble of N spin-1/2 particles and could be described in the collective spin representation.[3] Let $|jm\rangle$ represents the joint eigenstate of the SU(2) Casimir operators J^2 and J_z, which satisfy the relations

$$J^2|jm\rangle = j(j+1)|jm\rangle,$$
$$J_z|jm\rangle = m|jm\rangle,$$

with $j = N/2$ and $m = N/2, N/2-1, \ldots, -N/2$,[16,17] $|N/2,-N/2\rangle$ ($|N/2,N/2\rangle$) means all ions in the hyperfine state $|\downarrow\rangle$ ($|\uparrow\rangle$).

We first consider the case $\delta = 0$. In the strong coupling limit $B^x \gg |\lambda|$, the ground state $|0\rangle$ and the first excited state $|1\rangle$ for the Hamiltonian $H_{\rm eff}$ are non-degenerated, where $|0\rangle = \exp(i\pi J_y/2)|j = N/2, m = N/2\rangle$ is a spin coherent state, and $|1\rangle$ is a superposition of different states $|jm\rangle$. When we adiabatically tune B^x from the strong coupling limit ($B^x \gg |\lambda|$) to the weak coupling limit ($B^x \ll |\lambda|$), the ground state $|0\rangle$ and the first excited state $|1\rangle$ evolve into the states $|N/2,-N/2\rangle$ and $|N/2,N/2\rangle$, respectively. If $B^x = 0$, the two lowest states turn to be degenerated. Therefore, in such an adiabatic passage, there exists a transition from non-degeneracy to degeneracy between $|0\rangle$ and $|1\rangle$, which can be regarded as a bifurcation,[18–20] as shown in Figs. 2(a) and 2(b). Similar to a single-particle MZ interferometry, where the ground state $|0\rangle$ and the first exited state $|1\rangle$ are utilized as two paths, the two states of maximum spin $|N/2,-N/2\rangle$ and $|N/2,N/2\rangle$ can also be used as two paths for an N-particle MZ interferometry. Therefore, the achievement of the maximally path-entangled state

$$|\Psi_{\rm p}\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{N}{2},\frac{N}{2}\right\rangle + \left|\frac{N}{2},-\frac{N}{2}\right\rangle\right)$$

implies accomplishment of the beam splitting. Then, we turn off the lasers for a duration T so that the maximally path-entangled state $|\Psi_{\rm p}\rangle$ evolves into

$$|\Psi_{\rm p}'\rangle = \frac{1}{\sqrt{2}}\left(e^{-iN\omega_0 T/2}\left|\frac{N}{2},\frac{N}{2}\right\rangle + e^{iN\omega_0 T/2}\left|\frac{N}{2},-\frac{N}{2}\right\rangle\right), \quad (7)$$

where ω_0 is the frequency difference between the two hyperfine spin states $|\downarrow\rangle$ and $|\uparrow\rangle$. Experimentally, the fidelity of the obtained NOON state $|\Psi_{\rm p}\rangle$ is relevant to the ferromagnetic order parameter, which could be achieved by measuring the populations of the spin-down and -up states via spin-dependent fluorescence.[21]

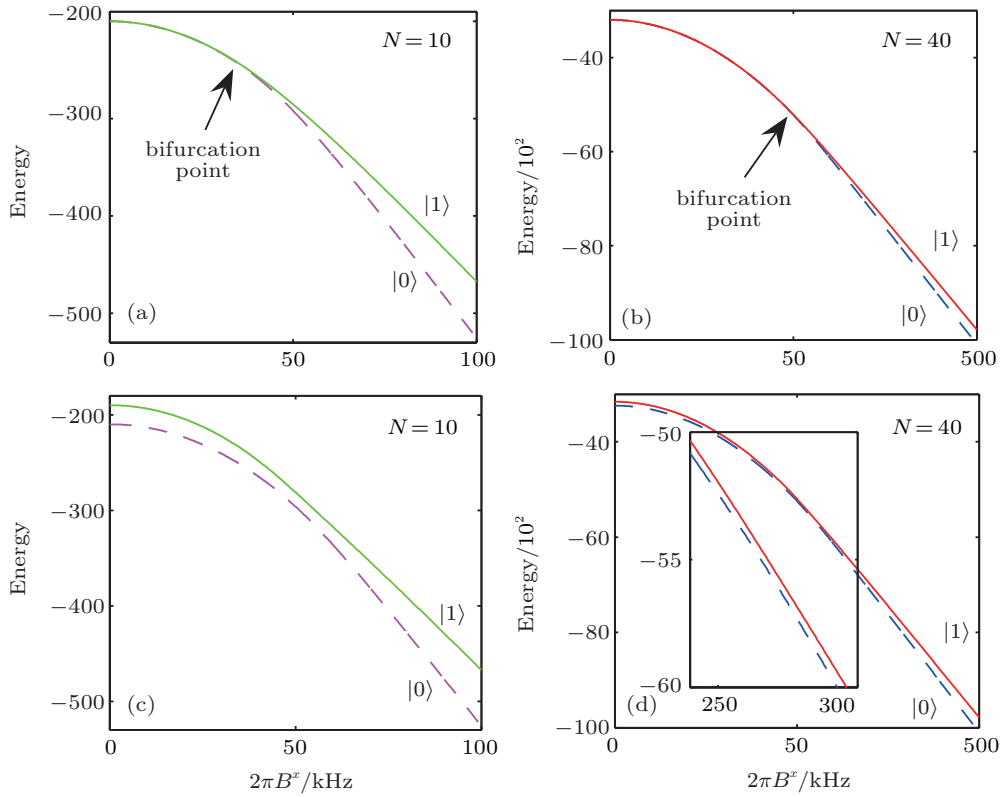


Fig. 2. (color online) The energy spectra of the ground state $|0\rangle$ (dashed line) and the first excited state $|1\rangle$ (solid line) with respect to B^x for different values of δ and N. If $\delta = 0$, there is a bifurcation from degeneracy to non-degeneracy of the two lowest states $|0\rangle$ and $|1\rangle$ when B^x increases, where (a) $N = 10$ and (b) $N = 40$. The beam splitting can be achieved by adiabatic passage through such a bifurcation. If $\delta \neq 0$, the degeneracy breaks down even in the weak limit of B^x, where $\delta = \lambda/4$, with $N = 10$ (c) and $N = 40$ (d). The other parameters are chosen as $\eta_z = 0.1$, $\Delta = 0.01$ MHz, $\Omega_z = 2\pi \times 100$ kHz ($N = 10$), and $2\pi \times 200$ kHz ($N = 40$).

There are two different methods for extracting the relative phase information between the two paths of the N-particle MZ interferometry. In the case of few trapped ions, this could be carried out using controlled-NOT gates $\mathrm{CNOT}^1_{\mathrm{rest}}$ with the first ion being the control qubit and the remaining ions being the target ones, followed by a Hadamard operation (H_1) on the first ion.[3] The relative phase information can be obtained by measuring the population of the first ion in the state $|\downarrow\rangle$ or $|\uparrow\rangle$. Taking $N=5$ as an example, this individually addressing method can be summarized briefly as follows:

$$\begin{aligned}|\Psi'_{\mathrm{p}}\rangle &\xrightarrow{N=5} \frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}5\omega_0 T/2}|\uparrow\uparrow\uparrow\uparrow\uparrow\rangle+\mathrm{e}^{\mathrm{i}5\omega_0 T/2}|\downarrow\downarrow\downarrow\downarrow\downarrow\rangle\right)\\ &\xrightarrow{\mathrm{CNOT}^1_{\mathrm{rest}}} \frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}5\omega_0 T/2}|\uparrow\downarrow\downarrow\downarrow\downarrow\rangle+\mathrm{e}^{\mathrm{i}5\omega_0 T/2}|\downarrow\downarrow\downarrow\downarrow\downarrow\rangle\right)\\ &\xrightarrow{H_1} \cos\left(\frac{5}{2}\omega_0 T\right)|\downarrow\downarrow\downarrow\downarrow\downarrow\rangle+\mathrm{i}\sin\left(\frac{5}{2}\omega_0 T\right)|\uparrow\downarrow\downarrow\downarrow\downarrow\rangle. \quad (8)\end{aligned}$$

However, for systems of large numbers of ions, individual addressing becomes challenging and it becomes very difficult to extract the relative phase with the controlled-NOT gate $\mathrm{CNOT}^1_{\mathrm{rest}}$. If we apply the inverse process of the first beam splitter for recombining the two paths, the relative phase information indeed could be transferred into the population information of the two lowest states for the limit of $B^x \gg \lambda$. However, it is not easy to directly distinguish these two lowest states. Fortunately, the two lowest states for symmetric ($\delta=0$) and asymmetric ($\delta\neq 0$) systems of $B^x \ll \lambda$ are almost identical and the two lowest states ($|N/2,-N/2\rangle$, $|N/2,N/2\rangle$) become non-degenerate for an asymmetric system ($0<\delta<\lambda/4$), see Figs. 2(c) and 2(d). Therefore, to extract the relative phase, after applying the inverse process of the beam splitting process, we suddenly apply a proper nonzero δ ($0<\delta<\lambda/4$) and then adiabatically decrease B^x from $B^x \gg \lambda$ to $B^x=0$. This recombination procedure for extracting the relative phase with two adiabatic processes could be summarized as

$$\begin{aligned}|\Psi'_{\mathrm{p}}\rangle &\xrightarrow{\text{adiabatic process of } \delta=0,\ B^x=0\ \to B^x\gg\lambda} \frac{\cos(N\omega_0 T/2)\,|0\rangle-\mathrm{i}\sin(N\omega_0 T/2)\,|1\rangle}{\sqrt{2}}\\ &\xrightarrow{\text{adiabatic process of } \delta\neq 0,\ B^x\gg\lambda\to B^x=0} \frac{\cos(N\omega_0 T/2)|N/2,N/2\rangle-\mathrm{i}\sin(N\omega_0 T/2)|N/2,-N/2\rangle}{\sqrt{2}}. \quad (9)\end{aligned}$$

Obviously, by measuring the population with all the ions in the state $|\downarrow\rangle$ or $|\uparrow\rangle$, the relative phase between the two paths could be obtained. Specifically, the population of the first excited state $P_1=\sin^2(N\phi/2)$ and the corresponding phase sensitivity is $\Delta\phi=\Delta\widehat{P}_1/(\partial\langle\widehat{P}_1\rangle/\partial\phi)=1/N$ could be obtained, where $\widehat{P}_1=|1\rangle\langle 1|$ is the population operator and $\Delta\widehat{P}_1$ is the variance of $\widehat{P}_1$. Obviously, this is the Heisenberg limit, where the relative phase ϕ is measured more precisely than the disentanglement case by $\sqrt{N}$ times.[6,7] Since $\phi=\omega_0 T$ and ω_0 depends on the atomic configuration and the local magnetic field, our MZ interferometry is of practical applications in understanding the atomic configuration and measuring the local magnetic field.

4. Experimental feasibility

Our scheme is feasible with current ion trap technology. If we employ a line of trapped $^{40}\mathrm{Ca}^+$, the states $|\downarrow\rangle$, $|\uparrow\rangle$, and $|\mathrm{e}\rangle$ are denoted by $\mathrm{S}_{1/2}(m_j=-1/2)$, $\mathrm{S}_{1/2}(m_j=1/2)$, and $\mathrm{P}_{1/2}(m_j=-1/2)$.[22] To adiabatically eliminate the ionic excited state $|\mathrm{e}\rangle$, the detuning of the optical fields from electronic resonance should be much larger than the excited state linewidth and the corresponding coupling strength, e.g., $\Delta'=2\pi\times 20$ GHz. On the other hand, generation of the nonlinear term λJ_z^2 in the Hamiltonian (6) requires that the detuning Δ should be much smaller than the frequency ν of the longitudinal COM mode. As a result, in our case, we may take $\nu=1$ MHz and $\Delta=0.01\nu$. Besides, the coherence time of the motional states reaches 100 ms.[23] To change the effective coupling B^x adiabatically, we may set $B^x(t)=\alpha-\beta t$, where α and β are nonnegative parameters, for example, $\alpha=500$ kHz and $\beta=50$ kHz/ms with $0\le t\le 10$ ms in the case of $N=40$. Consider the system in a free evolution by $T=5$ ms. After a duration for about 35 ms including three adiabatic processes plus the free evolution, we could obtain the relative phase by measuring the population of all the ions in the state $|N/2,-N/2\rangle$ or $|N/2,+N/2\rangle$.

Higher measurement precision for the trapped-ion MZ interferometer requires more ions involved. With more ions involved, however, besides more attention should be paid to larger coupling strength, we need also to consider the detrimental influence from decoherence. Fortunately, currently available technologies enable the effective coupling strength to be of the order of MHz, which meets the requirement of our proposal. Therefore, we focus on the influence from decoherence below.

5. Estimation of frequency uncertainty under decoherence

There are several sources of decoherence. Here, as an example, we only consider the dephasing decoherence influence on the estimation of the ionic Zeeman splitting ω_0. The maximally entangled state of N trapped ions undertaking MZ interferometry decreases the measurement uncertainty of ω_0 by $\sqrt{N}$ times compared with the trapped ions in product states.[3,5] If decoherence due to environment is involved, the measurement precision would be seriously affected[24–26]

because entangled states are more sensitive to decoherence than product states. A recent experiment[27] reported that the NOON state with 8 to 14 trapped ions is subject to correlated dephasing noise mostly because of magnetic field fluctuation and shows super-decoherence, where the coherence decays quadratically with the number of qubits. This correlated dephasing could be modeled by the Lindblad equation in the free evolution

$$\dot{\rho} = -\mathrm{i}\omega_0[J_z,\rho] + \gamma(2J_z\rho J_z - J_z^2\rho - \rho J_z^2), \tag{10}$$

where γ is the correlated dephasing rate, which is $N^2\gamma_0$ with γ_0 the dephasing rate of a single qubit. Here, we consider $\gamma_0 = \gamma_0'\Omega_z^2/\Delta'^2$ to be the effective two-level dephasing rate[28] due to the two-photon Raman process, where γ_0' is the spontaneous decay rate of the excited state $|\mathrm{e}\rangle$ for a single qubit. The uncertainty $\Delta\omega/\omega_0$ shown in Fig. 3 is calculated in the equivalent spin representation, where the details are given in the appendix A.

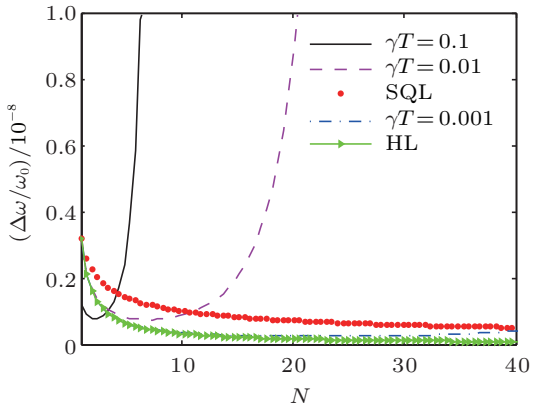


Fig. 3. (color online) The frequency uncertainty $\Delta\omega/\omega_0$ versus the total number of ions N. The dotted curve represents the estimated uncertainty in SQL corresponding to the product state, and the dash-dotted curve is the Heisenberg limit (HL) with the NOON state. We take $T = 5$ ms and $\omega_0 \approx 2\pi \times 3$ GHz at the magnetic field $B = 0.1$ T.

Repeating the implementation of our proposal by one hundred times, we find that the correlated dephasing seriously spoils the measurement precision of the entangled state, yielding the precision even lower than the SQL. By setting $\gamma_0' \approx 2\pi \times 20$ MHz, $\Omega_z = 2\pi \times 200$ kHz, and $\Delta' = 2\pi \times 20$ GHz, and considering free evolution by $T = 5$ ms in each implementation, we have $\gamma T = 0.1$ and the uncertainty of ω_0 in the case of three-ion entanglement is larger than the SQL, see Fig. 3. In order to enhance the precision, we have to reduce γ. A possible way is to enlarge Δ' by ten times, which leads to γT to be about 0.001, and thereby the measurement precision would approach the Heisenberg limit. Further improvement could be made by using refocusing pulses during the adiabatic operations to suppress the decay from $|\mathrm{e}\rangle$.

6. Summary and conclusion

In summary, we have shown a method to adiabatically carry out an MZ interferometer based upon multiple trapped ions with artificial nonlinear interactions. Influence from decoherence has also been discussed in our treatment. Since our interferometry proposal involves large numbers of trapped ions and works by global operations, we argue that the proposal would be useful for high-precision quantum metrology with trapped ions toward the Heisenberg limit.[29]

Quantum squeezing is significant for both high-precision metrology and explaining correlations and entanglement.[30,31] We should also emphasize that, although our discussion above has only focused on the case of $\Delta > 0$ and $\lambda > 0$ for a trapped-ion MZ interferometer, our model enables the study of spin squeezing in an ion trap by setting $\Delta < 0$ for $\lambda < 0$, which could be done by changing the directions of the laser beams k_A' and k_B' to make $B^x < 0$.[32] Our model also enables the investigation of the interaction blockade of spin flip with trapped ions for weak B^x.[33]

Appendix A: Calculation of $\Delta\omega$

For simplicity, we only consider the decoherence of the system during a free evolution. Suppose the system to be prepared in the maximally path-entangled state

$$\rho_{\mathrm{p}}(0) = \left(\left|\frac{N}{2},\frac{N}{2}\right\rangle + \left|\frac{N}{2},-\frac{N}{2}\right\rangle\right) \times \left(\left\langle\frac{N}{2},\frac{N}{2}\right| + \left\langle\frac{N}{2},-\frac{N}{2}\right|\right)\Big/2,$$

where $|N/2,N/2\rangle$ and $|N/2,-N/2\rangle$ stand for all ions in $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively. After a time duration of T, the system evolves into

$$\begin{aligned}\rho_{\mathrm{p}}(\omega_0,T) = \frac{1}{2}\Bigg(&\left|\frac{N}{2},\frac{N}{2}\right\rangle\left\langle\frac{N}{2},\frac{N}{2}\right| + \left|\frac{N}{2},-\frac{N}{2}\right\rangle\left\langle\frac{N}{2},-\frac{N}{2}\right| \\ &+ \mathrm{e}^{-\gamma TN^2+\mathrm{i}\omega_0 TN}\left|\frac{N}{2},-\frac{N}{2}\right\rangle\left\langle\frac{N}{2},\frac{N}{2}\right| \\ &+ \mathrm{e}^{-\gamma TN^2-\mathrm{i}\omega_0 TN}\left|\frac{N}{2},\frac{N}{2}\right\rangle\left\langle\frac{N}{2},-\frac{N}{2}\right|\Bigg).\end{aligned} \tag{A1}$$

According to Refs. [25] and [26], the quantum Fisher information is given as

$$F_Q = \mathrm{Tr}\left[\rho_{\mathrm{p}}(\omega_0,T)A^2\right] = N^2T^2\,\mathrm{e}^{-2\gamma TN^2}, \tag{A2}$$

where the Hermitian operator A is the "symmetric logarithmic derivative" with its matrix elements defined as $A_{ij} = 2[\rho_{\mathrm{p}}'(\omega_0,T)]_{ij}/(p_i+p_j)$, $\rho_{\mathrm{p}}'(\omega_0,T) = \partial\rho_{\mathrm{p}}(\omega_0,T)/\partial T$, with p_i and p_j the eigenvalues of $\rho_{\mathrm{p}}(\omega_0,T)$, conditional on $p_i+p_j=0$ and $A_{ij}=0$.

The frequency uncertainty $\Delta\omega$ satisfies the quantum Cramér–Rao bound[34,35]

$$\Delta\omega \geq \frac{1}{\sqrt{kF_{\mathrm{Q}}}} = \frac{1}{\sqrt{k}NT\,\mathrm{e}^{-\gamma TN^2}}, \tag{A3}$$

in the measurements on a set of k probes.

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