Riccati-type Bäcklund transformations of nonisospectral and generalized variable-coefficient KdV equations*

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We extend the method of constructing Bäcklund transformations for integrable equations through Riccati equations to the nonisospectral and the variable-coefficient equations. By taking nonisospectral and generalized variable-coefficient Korteweg–de Vries (KdV) equations as examples, their Bäcklund transformations are obtained under a more generalized constrain condition. In addition, the Lax pairs and infinite numbers of conservation laws of these equations are given. Especially, some classical equations such as the cylindrical KdV equation are just the special cases of the constrain condition.

Keywords: Bäcklund transformation, Lax pair, conservation law, Cole–Hopf transformation

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1. Introduction

The Bäcklund transformations have emerged over the past decades as a pivotal tool in the study of soliton. The significance of the Bäcklund transformations is basically twofold.[1] On one hand, the Bäcklund transformations may be used to construct an infinity sequence of exact solutions of integrable nonlinear evolution equations by purely algebraic superposition principles.[2–5] On the other hand, the Bäcklund transformations may be closely related to many important integrable properties such as the inverse scattering method, the Painlevé property, Lax pairs, and an infinite number of conservation laws.[6–8] Therefore, a nonlinear evolution equation is believed to be integrable if it possesses an auto-Bäcklund transformation or possesses a hetero-Bäcklund transformation to another integrable nonlinear evolution equation, which is one definition of the integrability.

So far, the systematic approaches for obtaining a Bäcklund transformation include Clairin's method, Chen's method, Hirota's bilinear method, and Wahlquist–Estabrook procedure.[2,3,9] The study of the scalar Riccati differential equations dates back to the early days of modern mathematical analysis, since such equations represent one of the simplest types of nonlinear ordinary differential equations.[10] Recently, Grundland and Levi constructed the Bäcklund transformations of several integrable systems from the higher-order Riccati equations.[11]

In this paper, we will investigate the nonisospectral[12] and the generalized variable-coefficient Korteweg–de Vries (vc-KdV) equations[13,14] given respectively by

$$u_t + h_1(u_{3x} + 6uu_x) + 4h_2u_x - h_3(2u + xu_x) = 0, \quad (1)$$

$$u_t + h_1u_{xxx} + h_2uu_x + h_3u_x + h_4u = 0, \quad (2)$$

where h_1, h_2, h_3, and h_4 are arbitrary functions with respect to variable t. The Riccati type Bäcklund transformations of these equations are obtained firstly, and some important integrable properties such as Lax pairs and infinite numbers of conservation laws are derived simultaneously. It is worth noting that the Bäcklund transformation of the generalized vc-KdV equation is obtained under a constrain condition, which is more generalized than ever, and some important classical equations such as the cylindrical KdV equation and so on are just special cases of this condition.

The paper is organized as follows. The Bäcklund transformation of the nonisospectral vc-KdV equation is given in Section 2, through which the Lax pair and infinity numbers of conservation laws are given. The Bäcklund transformation, the Lax pair, and infinity numbers of conservation laws of the generalized vc-KdV equation under a constrain condition are given in Section 3. Based on the results in Section 3, some special interesting cases are discussed in Section 4. Discussion and conclusion are given in Section 5.

2. Riccati-type Bäcklund transformation, Lax pair, and infinite numbers of conservation laws of the nonisospectral vc-KdV equation

Consider the nonisospectral vc-KdV equation (1), which includes several important physical equations such as the classical constant-coefficient KdV equation

$$u_t + u_{3x} + 6uu_x = 0, \quad (3)$$

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and the vc-KdV equation[15]

$$u_t + u_{xxx} + 6uu_x + \gamma u + [(c_0 + \gamma x)u]_x = 0. \tag{4}$$

Some significant properties, such as infinite numbers of conservation laws, the Bäcklund transformations, and the Lax pairs of Eq. (1) have been obtained.[4,12,16–18] We will construct the Riccati-type Bäcklund tranformation in the same form as that of the KdV equation,[19] which plays an important role in finding the N-soliton solution and seeking the nonlocal symmetry related to the Bäcklund transformation.

Now, by introducing the potential variable

$$v(x,t) = \int_x^{\infty} u(y,t)\,\mathrm{d}y, \tag{5}$$

the potential nonisospectral vc-KdV equation

$$v_t = -h_1(3v_x^2 + v_{xxx}) - 4h_2 v_x + h_3(v + xv_x) \tag{6}$$

can be obtained. In order to obtain the Riccati-type Bäcklund transformation, we assume

$$v_x = \alpha + \beta v + \gamma v^2. \tag{7}$$

It is easy to verify that equation (7) is compatible with Eq. (6) if

$$\gamma = -\frac{1}{2},\quad \alpha = -\frac{\beta}{2} - \beta_x + \lambda\,\mathrm{e}^{\int 2h_3\,\mathrm{d}t}, \tag{8}$$

where β is any solution of the potential nonisospectral vc-KdV equation (6). Then we can derive the spatial part of the auto-Bäcklund transformation for Eq. (6)

$$(v+\beta)_x = \lambda\,\mathrm{e}^{\int 2h_3 dt} - \frac{(\beta - v)^2}{2}. \tag{9a}$$

The time part can be derived by directly computing, that is

$$\begin{aligned}(v+\beta)_t \;=\; & h_1(v-\beta)(v_{xx} - \beta_{xx}) + (-4h_2 + h_3 x)(v_x + \beta_x)\\ & -2h_1(v_x^2 + \beta_x^2 + v_x\beta_x) + h_3(\beta + v).\end{aligned} \tag{9b}$$

According to the linearization of the Riccati equation, by using the Cole–Hopf transformation

$$(v - \beta) = -2\,(\ln\phi)_x, \tag{10}$$

the Lax pair of Eq. (6) can be obtained as

$$\phi_{xx} = \left(-v_x + \frac{\lambda}{2}\,\mathrm{e}^{\int 2h_3\,\mathrm{d}t}\right)\phi, \tag{11a}$$

$$\phi_t = h_1 v_{xx}\phi - \left(2h_1\left(v_x + \lambda\,\mathrm{e}^{\int 2h_3\,\mathrm{d}t}\right) + 4h_2 - h_3 x\right)\phi_x. \tag{11b}$$

Based on the Lax pair, infinite numbers of conservation laws of Eq. (6) can be obtained. To this end, we introduce a transformation

$$\phi_x = \eta\phi, \tag{12}$$

on account of which the Lax pair (11) can be translated into a Riccati type equation

$$\eta_x + \eta^2 + v_x - \varepsilon^2 = 0, \tag{13}$$

and a divergence type equation

$$\eta_t + [h_1(2v_x\eta + 4\eta\varepsilon^2 - v_{2x}) + 4h_2\eta - xh_3\eta]_x = 0, \tag{14}$$

where $\varepsilon^2 = \frac{1}{2}\lambda\,\mathrm{e}^{2\int h_3\,\mathrm{d}t}$.

To proceed, inserting the expansion

$$\eta = \varepsilon + \sum_{n=1}^{\infty} \mathcal{I}_n(q, q_x, \ldots)\varepsilon^{-n} \tag{15}$$

into Eq. (13) and equating the coefficients for powers of ε, we explicitly obtain the recursion relation for the conserved densities $\mathcal{I}_n'$s as follows:

$$\begin{aligned}
&\mathcal{I}_1 = -\frac{1}{2}v_x = -\frac{1}{2}u,\\
&\mathcal{I}_2 = -\frac{1}{2}\mathcal{I}_{1,x} = \frac{1}{4}v_{2x} = \frac{1}{4}u_x,\\
&\mathcal{I}_3 = -\frac{1}{2}(\mathcal{I}_{2,x} + \mathcal{I}_1^2) = -\frac{1}{8}(v_{3x} + v_x^2) = -\frac{1}{8}(u_{2x} + u^2),\\
&\mathcal{I}_4 = -\frac{1}{2}(I_{3,x} + 2I_1 I_2) = \frac{1}{16}(u_{3x} + 4uu_x),\\
&\ldots,\\
&\mathcal{I}_{n+1} = -\frac{1}{2}\left(\mathcal{I}_{n,x} + \sum_{k=1}^{n}\mathcal{I}_k\mathcal{I}_{n-k}\right),\quad n = 2,3,\ldots
\end{aligned} \tag{16}$$

By applying the nonisospectral condition

$$\varepsilon_t = h_3\varepsilon, \tag{17}$$

substituting expansion (15) into the divergence type equation (14) leads to

$$\begin{aligned}
&\sum_{n=1}^{\infty}\mathcal{I}_{n,t}\varepsilon^{-n} + \Big[h_1\Big(2u\sum_{n=1}^{\infty}\mathcal{I}_n\varepsilon^{-n} + 4\sum_{n=1}^{\infty}\mathcal{I}_{n+2}\varepsilon^{-n}\Big) + 4h_2\sum_{n=1}^{\infty}\mathcal{I}_n\varepsilon^{-n}\\
&- xh_3\sum_{n=1}^{\infty}\mathcal{I}_n\varepsilon^{-n} - h_3\sum_{n=1}^{\infty}n\partial_x^{-1}\mathcal{I}_n\varepsilon^{-n}\Big]_x = 0,
\end{aligned} \tag{18}$$

which provides us the infinite numbers of conservation laws of Eq. (6),

$$\mathcal{I}_{n,t} + \mathcal{F}_{n,x} = 0,\quad n = 1,2,\ldots, \tag{19}$$

where the fluxes $\mathcal{F}_n'$s are given by

$$\begin{aligned}
&\mathcal{F}_1 = -\frac{1}{2}h_1(u_{2x} + 3u^2) - 2h_2 u + \frac{1}{2}h_3(xu + \partial_x^{-1}u),\\
&\mathcal{F}_2 = \frac{1}{4}h_1(u_{3x} + 6uu_x) + h_2 u_x - \frac{1}{4}h_3(xu_x + 2u),\\
&\ldots,\\
&\mathcal{F}_n = h_1(2u\mathcal{I}_n + 4\mathcal{I}_{n+2}) + 4h_2\mathcal{I}_n - xh_3\mathcal{I}_n - nh_3\partial_x^{-1}\mathcal{I}_n,\\
&\qquad n = 3,4,\ldots
\end{aligned} \tag{20}$$

The first equation of conservation law (19) is exactly the nonisospectral vc-KdV equation (6).

Therefore, in the sense of admitting the auto-Bäcklund transformation, the Lax pair, and the infinite numbers of conservation laws, the nonisospectral KdV equation (6) is completely integrable.

3. Riccati-type Bäcklund transformation, Lax pair, and infinite numbers of conservation laws of the generalized vc-KdV equation

Consider a more generalized vc-KdV equation (2), the corresponding potential equation is

$$v_t = -h_1 v_{xxx} - \frac{1}{2} h_2 v_x^2 - h_3 v_x - h_4 v, \tag{21}$$

where $u = v_x$. The nonlinear excitations of a Bose gas of impenetrable bosons with a longitudinal confinement can be described by a vc-KdV equation,[20] and equation (21) is a special model without the dissipative term.

Similar to the nonisospectral vc-KdV equation, the Bäcklund transformation of the generalized vc-KdV equation is given by the Riccati equation (7). By directly computing, we can obtain the solution

$$\begin{aligned}&\gamma = -\frac{h_2}{12h_1},\\ &\alpha = -\frac{3h_1}{h_2}(2\beta_x + \beta^2) + \frac{2(h_1h_2h_4 - h_{2t}h_1 + h_{1t}h_2)x}{h_1h_2^2} + \lambda(t),\end{aligned} \tag{22}$$

under the constrain condition

$$\begin{aligned}&2(-3h_2h_{2t}h_{1t} - 4h_1h_2h_4h_{2t} + 3h_2^2h_4h_{1t} + 2h_1h_2^2h_4^2\\ &-h_1h_2h_{2tt} + h_2^2h_{1tt} + h_1h_2^2h_{4t} + 3h_1h_{2t}^2)x/(h_1h_2^3)\\ &+2h_3(h_1h_2h_4 - h_{2t}h_1 + h_{1t}h_2)/(h_1h_2^2)\\ &+\lambda(t)(2h_1h_2h_4 - h_{2t}h_1 + h_{1t}h_2)/(h_1h_2) + \frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) = 0,\end{aligned} \tag{23}$$

where β satisfies the following equation:

$$\begin{aligned}\beta_t &= -h_1(3\beta_x^2 + \beta_{xxx}) - h_3\beta_x\\ &\quad - \frac{2(h_1h_2h_4 - h_{2t}h_1 + h_{1t}h_2)\beta}{h_1h_2}.\end{aligned} \tag{24}$$

The coefficient of x in Eq. (23) is a Riccati equation about h_4

$$\begin{aligned}h_{4t} &= -2h_4^2 + \frac{(4h_{2t}h_1 - 3h_2h_{1t})h_4}{h_1h_2}\\ &\quad + \frac{3h_{2t}h_{1t}h_2 - h_2^2h_{1tt} - 3h_{2t}^2h_1 + h_2h_1h_{2tt}}{h_1h_2^2},\end{aligned} \tag{25}$$

and the solution of $\lambda(t)$ can be solved by the 0-term of x in Eq. (23) as

$$\lambda(t) = \mathrm{e}^{-\rho(t)}\left(2\int \frac{h_3\,\mathrm{e}^{\rho(t)}(h_1h_2h_4 + h_2h_{1t} - h_1h_{2t})}{h_1h_2^2}\,\mathrm{d}x + c_1\right), \tag{26}$$

where

$$\rho(t) = \int \frac{(2h_1h_2h_4 + h_2h_{1t} - h_1h_{2t})}{h_1h_2}\,\mathrm{d}t. \tag{27}$$

It is easy to verify that

$$h_4 = \frac{h_1h_{2t} - h_2h_{1t}}{h_1h_2} \tag{28}$$

is a particular solution of Riccati equation (25), then the general solution[21] can be given as

$$h_4 = \frac{h_1h_{2t} - h_2h_{1t}}{h_1h_2} + \frac{h_1}{\int 2h_1\,\mathrm{d}t + c_1}, \tag{29}$$

in which the particular solution corresponds to $c_1 = \infty$. The substitution of Eq. (29) reduces Eqs. (24) and (26) to the following form:

$$\beta_t = -h_1\beta_{xxx} - 3h_1\beta_x^2 - h_3\beta_x - \frac{\beta}{\psi}, \tag{30}$$

$$\lambda(t) = \frac{1}{h_2\psi}\left(-2\int h_3 t\,\mathrm{d}t + \lambda\right), \tag{31}$$

where

$$\psi = \frac{2\int h_1\,\mathrm{d}t + c_1}{h_1}. \tag{32}$$

Therefore, the spatial part of the Bäcklund transformation for generalized vc-KdV equation (21) and vc-KdV equation (30) under constrain condition (29) can be given as

$$\begin{aligned}\left(v + \frac{6h_1}{h_2}\beta\right)_x &= -\frac{h_2}{12h_1}\left(v - \frac{6h_1}{h_2}\beta\right)^2\\ &\quad + \frac{1}{h_2\psi}\left(2x - 2\int h_3\,\mathrm{d}t + \lambda\right).\end{aligned} \tag{33a}$$

The time part of the Bäcklund transformation under constrain condition (29) can be obtained from Eqs. (21) and (30) by directly computing, and its expression is

$$\begin{aligned}\left(v + \frac{6h_1}{h_2}\beta\right)_t &= \frac{h_2}{6}\left(v - \frac{6h_1}{h_2}\beta\right)\left(v_{xx} - \frac{6h_1}{h_2}\beta_{xx}\right)\\ &\quad - \left(\frac{12h_1^2}{h_2}\beta_x^2 + 2h_1\beta_x v_x + \frac{h_2}{3}v_x^2 + \frac{6h_1h_3}{h_2}\beta_x\right.\\ &\quad \left. + h_3v_x + \frac{6h_1}{h_2\psi}\beta + h_4v\right).\end{aligned} \tag{33b}$$

By using the Cole–Hopf transformation

$$v - \frac{6h_1\beta}{h_2} = -\frac{12h_1}{h_2}\frac{\partial}{\partial x}\ln\phi, \tag{34}$$

the Lax pair for Eq. (21) can be derived as

$$\phi_{xx} = -\frac{1}{12}\left(\frac{2h_2}{h_1}v_x - \frac{1}{h_1\psi}\left(2x - 2\int h_3\,\mathrm{d}t + \lambda\right)\right)\phi, \tag{35a}$$

$$\phi_t = \frac{1}{6}h_2v_{xx}\phi - \frac{1}{3}\left(3h_3 + h_2v_x + \frac{\lambda + 2x - 2\int h_3\,\mathrm{d}t}{\psi}\right)\phi_x, \tag{35b}$$

where ψ is given in Eq. (32).

By utilizing the same transformation (12), on account of which, the Lax pair (35) can be translated into a Riccati type equation

$$\eta_x + \eta^2 + \frac{h_2}{6h_1}u - \varepsilon^2 = 0, \tag{36}$$

and a divergence type equation

$$\eta_t + \left(\frac{h_2}{3}u\eta - \frac{h_2}{6}u_x + h_3\eta + 4h_1\eta\varepsilon^2\right)_x = 0, \tag{37}$$

where

$$\varepsilon^2 = \frac{2x - 2\int h_3\,\mathrm{d}t + \lambda}{12\left(2\int h_1\,\mathrm{d}t + c_1\right)}.$$

Inserting expansion (15) into Eq. (36) and equating the coefficients for powers of ε, we explicitly obtain the recursion relation for the conserved densities $\mathcal{I}_n'$s as follows:

$$\begin{aligned}
&\mathcal{I}_1 = -\frac{h_2}{12h_1}u,\\
&\mathcal{I}_2 = -\frac{1}{2}\mathcal{I}_{1,x} = \frac{h_2}{24h_1}u_x,\\
&\mathcal{I}_3 = -\frac{1}{2}(\mathcal{I}_{2,x} + \mathcal{I}_1^2) = -\frac{h_2}{288h_1^2}(6h_1u_{2x} + h_2u^2),\\
&\mathcal{I}_4 = -\frac{1}{2}(I_{3,x} + 2I_1I_2) = \frac{h_2}{288h_1^2}(3h_1u_{3x} + 2h_2uu_x),\\
&\cdots,\\
&\mathcal{I}_{n+1} = -\frac{1}{2}\left(\mathcal{I}_{n,x} + \sum_{k=1}^{n}\mathcal{I}_k\mathcal{I}_{n-k}\right), \quad n = 2,3,\ldots
\end{aligned} \tag{38}$$

By applying the nonisospectral condition

$$\varepsilon_t = -\frac{h_1\varepsilon}{2\int h_1\,\mathrm{d}t + c_1} - \frac{h_3}{12\varepsilon\left(2\int h_1\,\mathrm{d}t + c_1\right)}, \tag{39}$$

substituting expansion (15) into the divergence-type equation (37) leads to

$$\begin{aligned}
&\sum_{n=1}^{\infty}\mathcal{I}_{n,t}\varepsilon^{-n} + \left[\frac{h_2}{3}u\sum_{n=1}^{\infty}\mathcal{I}_n\varepsilon^{-n} + h_3\sum_{n=1}^{\infty}\mathcal{I}_n\varepsilon^{-n}\right.\\
&+4h_1\sum_{n=1}^{\infty}\mathcal{I}_{n+2}\varepsilon^{-n} + h_1\sum_{n=1}^{\infty}n\partial_x^{-1}\frac{\mathcal{I}_n}{2\int h_1\,\mathrm{d}t + c_1}\varepsilon^{-n}\\
&\left.+h_3\sum_{n=1}^{\infty}n\partial_x^{-1}\frac{\mathcal{I}_{n-2}}{12\left(2\int h_1\,\mathrm{d}t + c_1\right)}\varepsilon^{-n}\right]_x = 0,
\end{aligned} \tag{40}$$

which provides us the infinite numbers of conservation laws of Eq. (21)

$$\mathcal{I}_{n,t} + \mathcal{F}_{n,x} = 0, \quad n = 1,2,\ldots, \tag{41}$$

where the fluxes $\mathcal{F}_n'$s are given by

$$\begin{aligned}
&\mathcal{F}_1 = -\frac{h_2^2}{24h_1}u^2 - \frac{h_2}{12}u_{2x} - \frac{h_2h_3}{12h_1}u - \frac{h_2}{12\left(2\int h_1\,\mathrm{d}t + c_1\right)}\int u\,\mathrm{d}x,\\
&\mathcal{F}_2 = \frac{h_2^2}{24h_1}uu_x + \frac{h_2}{24}u_{3x} + \frac{h_2h_3}{24h_1}u_x + \frac{h_2}{12\left(\int h_1\,\mathrm{d}t + c_1\right)}u,\\
&\cdots,\\
&\mathcal{F}_n = \frac{h_2}{3}u\mathcal{I}_n + h_3\mathcal{I}_n + 4h_1\mathcal{I}_{n+2} + h_1n\partial_x^{-1}\frac{\mathcal{I}_n}{2\int h_1\,\mathrm{d}t + c_1}\\
&\quad + h_3n\partial_x^{-1}\frac{\mathcal{I}_{n-2}}{12\left(2\int h_1\,\mathrm{d}t + c_1\right)}, \quad n = 3,4,\ldots,
\end{aligned} \tag{42}$$

with the constrain condition (29). The first equation of conservation law (41) is exactly the generalized vc-KdV equation (21).

Remark Zhang *et al.*[13] derived the Bäcklund transformation, the Lax pairs, and the exact solutions of Eq. (2) by using the Hirota bilinear method[22] under constrain condition

$$h_1 = c_0h_2\,\mathrm{e}^{-\int h_4\,\mathrm{d}t}, \tag{43}$$

Fan[23] constructed the bilinear representation, the Bäcklund transformation, the Lax pairs, and the conservation laws of this equation by using the Bell polynomial under the same constrain condition (43). In fact, the constrain condition (43) is equivalent to our particular solution (28). That is to say, we obtain the Bäcklund transformation and the Lax pair of Eq. (21) under a more generalized constrain condition (29).

4. Some special cases

We now discuss some special interesting cases.

4.1. Case 1 of $h_4 = 0$

It is obvious that equation (29) is an integral equation, if we take $h_4 = 0$ in Eq. (29) and solve it directly, some particular solutions will be lost. Therefore, taking $h_4 = 0$ into Eq. (25), the solution

$$h_1 = h_2\left(c_2 + c_3\int h_2\,\mathrm{d}t\right) \tag{44}$$

can be obtained. Especially, if we also take $h_3 = 0$, constrain condition (44) is just the condition that the common vc-KdV equation

$$u_t + h_1u_{xxx} + h_2uu_x = 0 \tag{45}$$

pass the painlevé test.[24] In fact, we have noticed that constrain condition (29) or (44) has nothing to do with coefficient h_3, and Zhang *et al.*[13] gave the same conclusion.

In this case, equations (21) and (24) become

$$v_t = -h_1v_{xxx} - \frac{1}{2}h_2v_x^2 - h_3v_x, \tag{46}$$

$$\beta_t = -h_1\beta_{xxx} - 3h_1\beta_x^2 - h_3\beta_x - \frac{c_3h_2}{c_2 + c_3\int h_2\,\mathrm{d}t}\beta. \tag{47}$$

And the Bäcklund transformation of these two equations under constrain condition (44) is

$$\begin{aligned}
&\left(v + \frac{6h_1}{h_2}\beta\right)_x\\
&= -\frac{h_2}{12h_1}\left(v - \frac{6h_1}{h_2}\beta\right)^2 + \frac{2c_3x - 2c_3\int h_3\,\mathrm{d}t + \lambda}{c_2 + c_3\int h_2\,\mathrm{d}t},
\end{aligned} \tag{48a}$$

$$\left(v+\frac{6h_1}{h_2}\beta\right)_t$$
$$= \frac{h_2}{6}\left(v-\frac{6h_1}{h2}\beta\right)\left(v_{xx}-\frac{6h_1}{h_2}\beta_{xx}\right)-\frac{1}{3}h_2v_x^2-2h_1\beta_xv_x$$
$$-\frac{12h_1^2}{h_2}\beta_x^2-h_3v_x-\frac{6h_1h_3}{h_2}\beta_x-6c_2h_2\beta. \quad (48b)$$

By using the same Cole–Hopf transformation (34), the Lax pair of Eq. (46) can be derived as

$$\phi_{xx}=\left(-\frac{h_2}{6h_1}v_x-\frac{h_2\left(-2c_3x+2c_3\int h_3\,\mathrm{d}t-\lambda\right)}{12h_1\left(c_2+c_3\int h_2\,\mathrm{d}t\right)}\right)\phi, \quad (49a)$$

$$\phi_t=\left(-\frac{1}{3}h_2v_x+\frac{\left(-2c_3x+2c_3\int h_3\,\mathrm{d}t-\lambda\right)h_2}{3\left(c_2+c_3\int h_2\,\mathrm{d}t\right)}-h_3\right)\phi_x$$
$$+\frac{1}{6}h_2v_{xx}\phi. \quad (49b)$$

Especially, taking $h_3=0$, $h_1=at^m$, and $h_2=bt^n$, equations (46) and (47) become

$$v_t=-at^mv_{xxx}-\frac{1}{2}bt^nv_x^2, \quad (50)$$

$$\beta_t=-at^m\left(3\beta_x^2+\beta_{xxx}\right)-\frac{m-n}{t}\beta. \quad (51)$$

We obtain the following results for which equation (44) holds:

$$\text{(i)}\ \ m=n,\ \ c_2=\frac{a}{b},\ \ c_3=0, \quad (52)$$

$$\text{(ii)}\ \ m=2n+1,\ \ c_2=0,\ \ c_3=\frac{a}{b}(n+1), \quad (53)$$

which coincide with the conclusions of Nirmala *et al.*[14] Through a nonlinear transformation

$$v\to\frac{6a}{b}vt^{(m-n)}, \quad (54)$$

we can transform Eq. (50) to the equation

$$v_t=-at^m\left(3v_x^2+v_{xxx}\right)-\frac{m-n}{t}v. \quad (55)$$

It is obvious that equation (55) has the same form as Eq. (51). Therefore, by substituting Eqs. (52)–(54) into Eqs. (48) and (49), the auto-Bäcklund transformation and the Lax pair of Eq. (55) can be obtained as follows:

$$(v+\beta)_x=-\frac{(v-\beta)^2}{2}+\frac{bt^n(n+1)(2c_3x+\lambda)}{at^m\left(c_2(n+1)+c_3bt^{(n+1)}\right)}, \quad (56a)$$

$$(v+\beta)_t=at^m(v-\beta)(v_{xx}-\beta_{xx})-2at^m(v_x^2+v_x\beta_x+\beta_x^2)$$
$$-\frac{(m-n)}{t}(\beta+v), \quad (56b)$$

$$\phi_{xx}=\left(-v_x+\frac{bt^n(n+1)(2c_3x+\lambda)}{12at^m(c_2(n+1)+c_3bt^{(n+1)})}\right)\phi, \quad (57a)$$

$$\phi_t=at^mv_{xx}\phi+\left(-2at^mv_x-\frac{bt^nc_3(n+1)(2x+\lambda)}{3(c_2(n+1)+c_3bt^{(n+1)})}\right)\phi_x. \quad (57b)$$

When $\alpha=1$, $m=0$, $n=1/2$, equation (55) transforms to the well-known cylindrical KdV equation.[25,26] At the same time, the corresponding auto-Bäcklund transformation and Lax pair can be obtained from Eqs. (56) and (57). The cylindrical KdV equation has been studied by several authors, and some integrable properties are well known for this equation.[27–29]

4.2. Case 2 of $h_2=6h_1$

In this case, the generalized vc-KdV equation (21) and the vc-KdV equation (24) satisfy the same equation

$$v_t=-h_1v_{xxx}-3h_1v_x^2-h_3v_x-\frac{h_1}{2\int h_1\,\mathrm{d}t+c_1}v, \quad (58)$$

$$\beta_t=-h_1\beta_{xxx}-3h_1\beta_x^2-h_3\beta_x-\frac{h_1}{2\int h_1\,\mathrm{d}t+c_1}\beta. \quad (59)$$

Therefore, the auto-Bäcklund transformation of this equation can be obtained from Eq. (33) as

$$(v+\beta)_x=-\frac{1}{2}(v-\beta)^2+\frac{\left(2x-2\int h_3\,\mathrm{d}t+\lambda\right)}{6\left(\int 2h_1\,\mathrm{d}t+c_1\right)}, \quad (60a)$$

$$(v+\beta)_t=-(v_x+\beta_x)h_3+\left((v-\beta)(v_{xx}-\beta_{xx})-2v_x^2\right.$$
$$\left.-2v_x\beta_x-2\beta_x^2-\frac{v+\beta}{2\int h_1\,\mathrm{d}t+c_1}\right)h_1. \quad (60b)$$

The Lax pair of this equation can be obtained from Eq. (35) as

$$\phi_{xx}=\left(-v_x-\frac{-2x+3\int h_3\,\mathrm{d}t-\lambda}{12\left(\int h_1\,\mathrm{d}t+c_1\right)}\right)\phi, \quad (61a)$$

$$\phi_t=v_{xx}h_1\phi-\left(2h_1v_x+h_3+\frac{\left(2x-2\int h_3\,\mathrm{d}t+\lambda\right)h_1}{3(\int 2h_1\,\mathrm{d}t+c_1)}\right)\phi_x. \quad (61b)$$

Especially, if we take $h_1=1$ and $h_3=0$, this equation becomes the generalized cylindrical KdV equation

$$v_t=-v_{xxx}-3v_x^2-\frac{v}{2t+c_1}. \quad (62)$$

By taking $h_1=1$ and $h_3=0$ in Eqs. (60) and (61), the auto-Bäcklund transformation and the Lax pair of the generalized cylindrical KdV equation can be obtained as follows:

$$(v+\beta)_x=-\frac{(v-\beta)^2}{2}+\frac{2x+\lambda}{3(2t+c_1)}, \quad (63a)$$

$$(v+\beta)_t=(v-\beta)(v_{xx}-\beta_{xx})-2v_x^2-2v_x\beta_x$$
$$-2\beta_x^2-\frac{v+\beta}{2t+c_1}, \quad (63b)$$

$$\phi_{xx}=\left(-v_x+\frac{2x+\lambda}{12(2t+c_1)}\right)\phi, \quad (64a)$$

$$\phi_t = \left(-2v_{xx} - \frac{2x+\lambda}{3(2t+c_1)}\right)\phi_x + v_{xx}\phi_x. \tag{64b}$$

The case of $h_2 = ch_1$, where c is any constant, is equivalent to this case, a scale transformation $v \to 6v/c$ can transform it to the case of $h_2 = 6h_1$. In addition, if $\alpha = 1$, $m = 0$, $n = 1/2$ in Eqs. (56) and (57) and $c_1 = 1$ in Eqs. (63) and (64), they are exactly the same formulas.

4.3. Case 3 of $c_1 = \infty$

In this case, the general solution (29) becomes the particular solution (28), that is

$$h_4 = \frac{h_1 h_{2t} - h_2 h_{1t}}{h_1 h_2}, \tag{65}$$

which is equivalent to the constrain condition (43) obtained by Zhang *et al.*[13]

In this case, β satisfies the following special vc-KdV equation:

$$\beta_t = -h_1(\beta_{xxx} + 3\beta_x^2) - h_3\beta_x. \tag{66}$$

By the use of condition (22) and Eq. (7), the Bäcklund transformation for generalized vc-KdV equation (21) and special vc-KdV equation (66) is derived as follows:

$$\left(v + \frac{6h_1}{h_2}\beta\right)_x = -\frac{h_2}{12h_1}\left(v - \frac{6h_1}{h_2}\beta\right)^2 + \frac{\lambda h_1}{h_2}, \tag{67a}$$

$$\begin{aligned}\left(v + \frac{6h_1}{h_2}\beta\right)_t =& \frac{h_2}{6}\left(v - \frac{6h_1}{h_2}\beta\right)\left(v_{xx} - \frac{6h_1}{h_2}\beta_{xx}\right) - \frac{h_2}{3}v_x^2 \\ & - 2h_1 v_x\beta_x - h_3 v_x - \frac{12h_1^2}{h_2}\beta_x^2 - \frac{6h_1h_3}{h_2}\beta_x \\ & - h_4 v.\end{aligned} \tag{67b}$$

The Lax pair of Eq. (21) under constrain condition (65) can be obtained by Cole–Hopf transformation transformation (34) as

$$\phi_{xx} = -\frac{1}{12}\left(\frac{2h_2}{h_1}v_x - \lambda\right), \tag{68a}$$

$$\phi_t = -\frac{1}{6}h_2 v_{xx}\phi - \frac{1}{3}\left(h_2 v_x + 3h_3 + h_1\lambda\right)\phi_x. \tag{68b}$$

Especially, when $h_2 = 6h_1$, v satisfies the same form of Eq. (66)

$$v_t = -h_1(v_{xxx} + 3v_x^2) - h_3 v_x. \tag{69}$$

In other words, by taking $h_2 = 6h_1$ into Eqs. (67) and (68), the auto-Bäcklund transformation and the Lax pair of Eq. (69) or (66) can be obtained. When $h_1 = 1$, $h_2 = 6$, $h_3 = 0$, the auto-Bäcklund transformation[19] and the Lax pair of the classical KdV equation can be derived.

Similar to the procedures of Eqs. (12)–(20), infinite numbers of conservation laws of the above three cases can be derived through direct calculations, which are omitted here for conciseness.

5. Discussion and conclusion

The nonisospectral and the generalized vc-KdV equations, although hard to be handled, have attracted the attention of several scholars.[12–14,30] These equations are of current interest for explaining the soliton breaking phenomena. In this paper, some important integrable properties including the Bäcklund transformations, the Lax pairs, and infinite numbers of conservation laws of these equations are given, and some earlier results are just the special cases of our conclusions. Our work provides a new tool for the further study of these two equations, which are presently under investigation as follows.

1) Can the Darboux transformations and the Painlevé analysis of these two equations be obtained from the Riccati-type Bäcklund transformations? Can the N-soliton solutions be obtained from these Bäcklund transformations?

2) The N-order equation of the Riccati chain[31] is given by

$$L^N v(x) + \sum_{j=1}^{N} \alpha_j(x)\left(L^{j-1}v(x)\right) + \alpha_0(x) = 0, \tag{70}$$

where N is an integer representing the order of the Riccati equation, and L is the differential operator given as

$$L = \frac{\mathrm{d}}{\mathrm{d}x} + cv(x). \tag{71}$$

Therefore, the equation is the higher-order Riccati equations as the Bäcklund transformations of these equations. Especially, the Bäcklund transformation of the generalized vc-KdV equation is derived under constrain condition (29), does there exist a specific-order Riccati equation as the Bäcklund transformation of this equation to make the constrain condition much more generalized?

3) Recently, Lou[32] obtained some interesting results from the nonlocal symmetries related to the Bäcklund transformation. Our Bäcklund transformations are very analogous to the Bäcklund transformation of the KdV equation in the form. In fact, we have obtained the nonlocal symmetries from the Bäcklund transformations (9) and (33), and the related results will be reported in our future research work.

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