Bound state solutions of the Dirac equation with the Deng–Fan potential including a Coulomb tensor interaction

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Approximate analytical solutions of the Dirac equation in the case of pseudospin and spin symmetry limits are investigated under the Deng–Fan potential by applying the asymptotic iteration method for the arbitrary quantum numbers n and κ. Some of the numerical results are also represented in both pseudospin symmetry and spin symmetry limits.

Keywords: Dirac equation, Deng–Fan potential, pseudospin and spin symmetries, tensor interaction

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1. Introduction

In order to understand the pseudospin and spin symmetry concepts, the motion of the nucleons should be taken into account. The Dirac equation is therefore considered. The solutions to this relativistic equation are very significant in describing spin-1/2 particles. The relation between the pseudospin and the Dirac equation was established by Ginocchio,[1] who indicated that in the presence of vector potential $V(r)$ and scalar potential $S(r)$, pseudospin symmetry occurs when $\Sigma(r) = V(r) + S(r) = 0$. On the other hand, spin symmetry, which is relevant to mesons, appears in the Dirac equation when $\Delta(r) = V(r) - S(r) = 0$.[2–4] Subsequently, Meng *et al.*[5–7] showed that pseudospin symmetry occurs under a more general condition, $\Sigma(r) = S(r) + V(r) =$ const. or $\mathrm{d}\Sigma(r)/\mathrm{d}r = 0$, and that spin symmetry occurs when $\Delta(r) = S(r) - V(r) =$ const. or $\mathrm{d}\Delta(r)/\mathrm{d}r = 0$.

The Dirac equations under pseudospin or spin symmetry with various-type potentials have been studied by many authors in recent years.[8–18] Moreover, the tensor interactions on the pseudospin and spin doublets have also been studied.[18–27] In this paper, we obtain the pseudospin and spin symmetry solutions for the Dirac equation with the Deng–Fan (DF) potential, including a Coulomb-like tensor interaction. The DF potential (or the generalized Morse potential) is one of the exponential potential models, and is used to describe the energy spectrum of diatomic molecules and electromagnetic transitions. Deng and Fan[28] proposed a suitable potential of the form

$$V(r) = D_0\left(1 - \frac{b}{\mathrm{e}^{\alpha r} - 1}\right)^2, \quad b = \mathrm{e}^{\alpha r_\mathrm{e}} - 1, \tag{1}$$

where $r \in (0, \infty)$, and parameters D_0, α, and r_e denote the dissociation energy, the range of the potential well, and the nuclear separation, respectively. The above potential model is a special case of the five-parameter exponential-type potential.[29] It is noted that the DF potential has suitable physical boundary conditions at the origin and at infinity, unlike the Morse potential. The DF potential is plotted in Fig. 1 for various potential parameters.

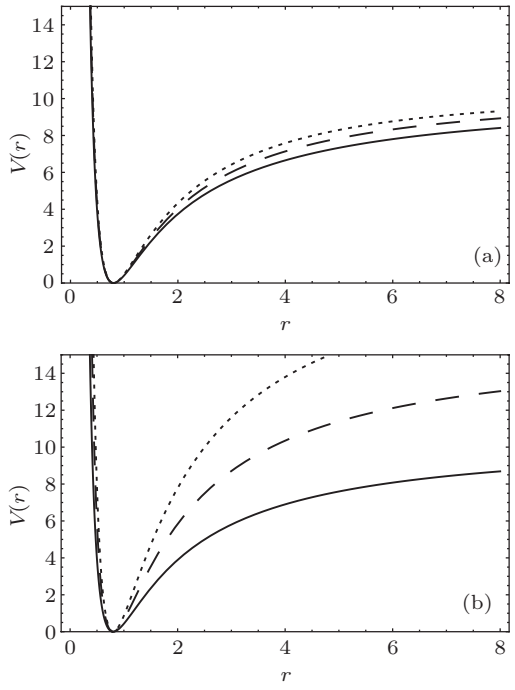


Fig. 1. The behaviors of the DF potential with nuclear separation $r_\mathrm{e} = 0.8$ fm (a) for $\alpha = 0.05\,\mathrm{fm}^{-1}$ (solid line), $\alpha = 0.15$ fm^{-1} (dashed line), $\alpha = 0.25$ fm^{-1} (dotted line), with $D_0 = 10$ fm^{-1}; and (b) for various dissociation energies $D_0 = 10$ fm^{-1} (solid line), $D_0 = 15$ fm^{-1} (dashed line), and $D_0 = 20$ fm^{-1} (dotted line), with $\alpha = 0.10$ fm^{-1}.

In this paper, our aim is to obtain the energy eigenvalues and the corresponding eigenfunctions for the Dirac equa-

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tion under the DF potential in the case of pseudospin symmetry and spin symmetry limits using the asymptotic iteration method (AIM) within the improved approximation. The AIM reproduces the exact solutions to many solvable differential equations, and these equations can be related to the Schrödinger equation. It also gives accurate results for non-solvable Schrödinger equations, such as the sextic oscillator, the cubic oscillator, and the deformed Coulomb potential, which are important in many problems in physics. This method has been widely applied in solving many eigenvalue problems in both relativistic and non-relativistic quantum mechanics.

The paper is organized as follows. In the next section, the AIM is introduced. In Section 3, we review the Dirac equation in the presence of a tensor interaction and give the solution of the Dirac equation in the case of pseudospin and spin symmetry limits. Finally, Section 4 is devoted to the conclusion.

2. Asymptotic iteration method

The AIM was proposed by Ciftci *et al.*[30–32] to solve the second-order homogeneous linear differential equations. The asymptotic iteration method is applied to solve the equation

$$y'' = \lambda_0(x)y' + s_0(x)y, \tag{2}$$

where $\lambda_0(x) \neq 0$, and the functions $s_0(x)$ and $\lambda_0(x)$ are many times differentiable and must be sufficiently differentiable. Equation (2) has a general solution as follows:

$$\begin{aligned} y_n(x) &= \exp\left(-\int^x \alpha(x_1)\,\mathrm{d}x_1\right)\left[C_2\right. \\ &\quad \left. + C_1 \int^x \exp\left(\int^{x_1} [\lambda_0(x_2) + 2\alpha(x_2)]\,\mathrm{d}x_2\right)\mathrm{d}x_1\right]. \end{aligned} \tag{3}$$

For sufficiently large n,

$$\frac{s_n}{\lambda_n} = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)} = \alpha(x), \tag{4}$$

where

$$\begin{aligned} \lambda_n(x) &= \lambda'_{n-1}(x) + s_{n-1}(x) + \lambda_0\lambda_{n-1}(x), \\ s_n(x) &= s'_{n-1}(x) + s_0(x)\lambda_{n-1}(x). \end{aligned} \tag{5}$$

For a given potential such as the exponential-type potential, the Schrödinger-type equation is converted to the form of Eq. (2). Then, $\lambda_0(x)$ and $s_0(x)$ are obtained and $\lambda_n(x)$ and $s_n(x)$ are calculated. The energy eigenvalues are found from the termination condition given by

$$\Delta_n(x) = \begin{vmatrix} \lambda_n(x) & \lambda_{n-1}(x) \\ s_n(x) & s_{n-1}(x) \end{vmatrix} = 0. \tag{6}$$

In this study, we obtain the solutions of the Schrödinger-like equation, the relevant second-order differential equation that takes Eq. (2). The exact eigenfunctions are obtained from the following wave function generator:

$$y_n(x) = C_2 \exp\left(-\int^x \alpha_k(x')\,\mathrm{d}x'\right), \tag{7}$$

where n represents the radial quantum number and $k \geq n$ is the iteration step number.

3. Pseudospin and spin symmetric solutions

3.1. The Dirac equation theory

The Dirac equation under an attractive scalar potential $S(r)$, a repulsive vector potential $V(r)$, and a tensor potential $U(r)$ in the relativistic unit ($\hbar = c = 1$) is[23–25]

$$\begin{aligned} &\left[\boldsymbol{\alpha}\cdot\boldsymbol{p} + \boldsymbol{\beta}\big(\mu + S(r)\big) - \mathrm{i}\boldsymbol{\beta}\boldsymbol{\alpha}\cdot\hat{\boldsymbol{r}}U(r)\right]\boldsymbol{\Psi}(\boldsymbol{r}) \\ &= [E - V(r)]\Psi(\boldsymbol{r}), \end{aligned} \tag{8}$$

where E is the relativistic energy of the system, $\boldsymbol{p} = -\mathrm{i}\boldsymbol{\nabla}$ is the three-dimensional momentum operator, and μ denotes the rest mass of the fermionic particle. Here, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are 4×4 Dirac matrices given by

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} 0 & \boldsymbol{I} \\ \boldsymbol{I} & 0 \end{pmatrix}, \tag{9}$$

where $\boldsymbol{I}$ is the 2×2 unitary matrix, and the Pauli three-vector matrices in $\boldsymbol{\sigma}$ are

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{10}$$

The eigenvalues of the spin–orbit coupling operator are $\kappa = -(j+1/2) < 0$ and $\kappa = (j+1/2) > 0$ for aligned spin $j = \ell + 1/2$ and unaligned spin $j = \ell - 1/2$, respectively. The set $(H,\ K,\ J^2,\ J_z)$ forms a complete set of conserved quantities. Thus, we can write the spinors

$$\boldsymbol{\Psi}_{n\kappa}(\boldsymbol{r}) = \begin{pmatrix} f_{n\kappa}(\boldsymbol{r}) \\ g_{n\kappa}(\boldsymbol{r}) \end{pmatrix} = \begin{pmatrix} \frac{F_{n\kappa}(r)}{r} & \mathrm{Y}^{\ell}_{jm}(\theta,\varphi) \\ \mathrm{i}\frac{G_{n\kappa}(r)}{r} & \mathrm{Y}^{\tilde{\ell}}_{jm}(\theta,\varphi) \end{pmatrix}, \tag{11}$$

where $f_{n\kappa}(\boldsymbol{r})$ is the upper component, $g_{n\kappa}(\boldsymbol{r})$ is the lower component of the Dirac spinors, $\mathrm{Y}^{\ell}_{jm}(\theta,\varphi)$ and $\mathrm{Y}^{\tilde{\ell}}_{jm}(\theta,\varphi)$ are the spin and the pseudospin spherical harmonics, respectively, and m is the projection of the angular momentum along the z axis. With other known identities

$$(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B}) = \boldsymbol{A}\cdot\boldsymbol{B} + \mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{B}), \tag{12a}$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{p} = \boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}\left(\hat{\boldsymbol{r}}\cdot\boldsymbol{p} + \mathrm{i}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{L}}{r}\right), \tag{12b}$$

as well as

$$(\boldsymbol{\sigma}\cdot\boldsymbol{L})\mathrm{Y}^{\tilde{\ell}}_{jm}(\theta,\varphi) = (\kappa-1)\mathrm{Y}^{\tilde{\ell}}_{jm}(\theta,\varphi), \tag{13a}$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{L})\mathrm{Y}^{\ell}_{jm}(\theta,\varphi) = -(\kappa-1)\mathrm{Y}^{\ell}_{jm}(\theta,\varphi), \tag{13b}$$

$$(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\mathrm{Y}^{\tilde{\ell}}_{jm}(\theta,\varphi) = -\mathrm{Y}^{\ell}_{jm}(\theta,\varphi), \tag{13c}$$

$$(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\mathrm{Y}^{\ell}_{jm}(\theta,\varphi) = -\mathrm{Y}^{\tilde{\ell}}_{jm}(\theta,\varphi), \tag{13d}$$

we find the following two coupled first-order Dirac equations

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\right)F_{n\kappa}(r) = \big(\mu+E_{n\kappa}-\Delta(r)\big)G_{n\kappa}(r), \tag{14a}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\right)G_{n\kappa}(r) = \big(\mu-E_{n\kappa}+\Sigma(r)\big)F_{n\kappa}(r), \tag{14b}$$

where $\Delta(r) = V(r) - S(r)$ and $\Sigma(r) = V(r) + S(r)$. From Eqs. (14a) and (14b), we obtain the second-order Schrödinger-like equation as

$$\begin{aligned}&\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa+1)}{r^2}+\frac{2\kappa}{r}U(r)-\frac{\mathrm{d}U(r)}{\mathrm{d}r}-U^2(r)\right.\\&\left.+\frac{\mathrm{d}\Delta(r)/\mathrm{d}r}{\mu+E_{n\kappa}-\Delta(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\right)\right]F_{n\kappa}(r)\\&=\big[\big(\mu+E_{n\kappa}-\Delta(r)\big)\big(\mu-E_{n\kappa}+\Sigma(r)\big)\big]F_{n\kappa}(r),\end{aligned} \tag{15a}$$

$$\begin{aligned}&\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa-1)}{r^2}+\frac{2\kappa}{r}U(r)+\frac{\mathrm{d}U(r)}{\mathrm{d}r}-U^2(r)\right.\\&\left.-\frac{\mathrm{d}\Sigma(r)/\mathrm{d}r}{\mu-E_{n\kappa}+\Sigma(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\right)\right]G_{n\kappa}(r)\\&=\big[\big(\mu+E_{n\kappa}-\Delta(r)\big)\big(\mu-E_{n\kappa}+\Sigma(r)\big)\big]G_{n\kappa}(r),\end{aligned} \tag{15b}$$

where $\kappa(\kappa+1) = \tilde{\ell}(\tilde{\ell}+1)$ and $\kappa(\kappa-1) = \ell(\ell+1)$.

3.2. Pseudospin symmetry limit

The pseudospin symmetry occurs when $\mathrm{d}\Sigma(r)/\mathrm{d}r = 0$ or equivalently $\Sigma(r) = C_{\mathrm{ps}} = \text{const}$. The potential $\Delta(r)$ is taken as the DF potential

$$\Delta(r) = D_0\left(1-\frac{b}{\mathrm{e}^{\alpha r}-1}\right)^2, \tag{16}$$

and the Coulomb tensor potential is

$$U(r) = -\frac{U_0}{r}. \tag{17}$$

By substituting Eqs. (16) and (17) into Eq. (15b), the Schrödinger-like equation becomes

$$\begin{aligned}&\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{T}{r^2}+\big(E_{n\kappa}-\mu-C_{\mathrm{ps}}\big)\left(E_{n\kappa}+\mu-D_0\right.\right.\\&\left.\left.-D_1\frac{\mathrm{e}^{-\alpha r}}{1-\mathrm{e}^{-\alpha r}}-D_2\frac{\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-\alpha r})^2}\right)\right]G_{n\kappa}(r)=0,\end{aligned} \tag{18}$$

where $D_1 = -2D_0b$, $D_2 = D_0b^2$, and $T = (\kappa+U_0)(\kappa+U_0-1)$. We consider the approximation[33]

$$\frac{1}{r^2}\approx\alpha^2\left(c_0+\frac{1}{\mathrm{e}^{\alpha r}-1}+\frac{1}{(\mathrm{e}^{\alpha r}-1)^2}\right),\quad c_0=\frac{1}{12}. \tag{19}$$

Therefore, we have

$$\begin{aligned}&\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-T\alpha^2\left(c_0+\frac{\mathrm{e}^{-\alpha r}}{1-\mathrm{e}^{-\alpha r}}+\frac{\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-\alpha r})^2}\right)\right.\\&+\left(E_{n\kappa}+\mu-D_0-D_1\frac{\mathrm{e}^{-\alpha r}}{1-\mathrm{e}^{-\alpha r}}-D_2\frac{\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-\alpha r})^2}\right)\\&\left.\times\big(E_{n\kappa}-\mu-C_{\mathrm{ps}}\big)\right]G_{n\kappa}(r)=0.\end{aligned} \tag{20}$$

The behavior of the improved approximation is plotted in Fig. 2.

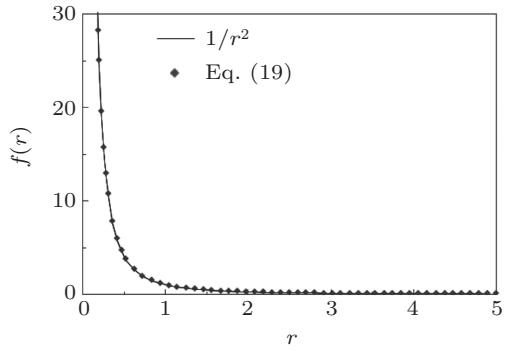


Fig. 2. The behavior of $1/r^2$ and its approximation for $\alpha = 0.05$ fm^{-1}.

If we use a new variable of the form $x = 1 - \mathrm{e}^{-\alpha r}$, we obtain

$$\begin{aligned}&\left[x(1-x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}-x\frac{\mathrm{d}}{\mathrm{d}x}+Q_{\mathrm{ps}}\left(\frac{x}{1-x}\right)\right.\\&\left.-L_{\mathrm{ps}}\left(\frac{1-x}{x}\right)-K_{\mathrm{ps}}\right]G_{n\kappa}(x)=0,\end{aligned} \tag{21}$$

where

$$\begin{aligned}Q_{\mathrm{ps}}&=\frac{(E_{n\kappa}-\mu-C_{\mathrm{ps}})}{\alpha^2}(E_{n\kappa}+\mu-D_0)-Tc_0,\\K_{\mathrm{ps}}&=\frac{(E_{n\kappa}-\mu-C_{\mathrm{ps}})D_1}{\alpha^2}+T,\\L_{\mathrm{ps}}&=\frac{(E_{n\kappa}-\mu-C_{\mathrm{ps}})D_2}{\alpha^2}+T.\end{aligned} \tag{22}$$

Owing to the asymptotic behavior of Eq. (21), we can write the ansatz solution as follows:

$$G_{n\kappa}(x) = x^{\gamma}(1-x)^{\Lambda}f(x), \tag{23}$$

where

$$\gamma=\frac{1}{2}+\sqrt{\frac{1}{4}+L_{\mathrm{ps}}},\quad \Lambda^2=-Q_{\mathrm{ps}}. \tag{24}$$

Hence we have

$$\begin{aligned}\frac{\mathrm{d}^2f(x)}{\mathrm{d}x^2}&=\left[\frac{2\gamma-(1+2\gamma+2\Lambda)x}{x^2-x}\right]\frac{\mathrm{d}f(x)}{\mathrm{d}x}\\&+\left[\frac{\varepsilon+(\Lambda+\gamma)^2}{x-x^2}\right]f(x),\end{aligned} \tag{25}$$

where $\varepsilon = Q_{\mathrm{ps}} + K_{\mathrm{ps}} - L_{\mathrm{ps}}$.

3.2.1. Iterative solutions: the energy spectrum

To apply the AIM, we need to compare Eq. (2) with Eq. (25). Therefore, the functions $\lambda_0(x)$ and $s_0(x)$ are given as

$$\lambda_0(x)=\frac{2\gamma-(1+2\gamma+2\Lambda)x}{x^2-x},\quad s_0(x)=\frac{\varepsilon+(\Lambda+\gamma)^2}{x-x^2}. \tag{26}$$

If we consider Eq. (5), we obtain

$$\begin{aligned}
&\lambda_1(x)=\lambda_0'(x)+s_0(x)+\lambda_0^2(x),\\
&s_1(x)=s_0'(x)+s_0(x)\lambda_0(x),\\
&\lambda_2(x)=\lambda_1'(x)+s_1(x)+\lambda_0\lambda_1(x),\\
&s_2(x)=s_1'(x)+s_0(x)\lambda_1(x),\\
&\lambda_3(x)=\lambda_2'(x)+s_2(x)+\lambda_0\lambda_2(x),\\
&s_3(x)=s_2'(x)+s_0(x)\lambda_2(x).
\end{aligned} \tag{27}$$

The energy eigenvalues are found from Eq. (6), and we have

$$\begin{aligned}
&\frac{\lambda_0}{s_0}-\frac{\lambda_1}{s_1}=0,\quad \varepsilon_1=-1-2(\Lambda+\gamma)-(\Lambda+\gamma)^2,\\
&\frac{\lambda_1}{s_1}-\frac{\lambda_2}{s_2}=0,\quad \varepsilon_2=-4-4(\Lambda+\gamma)-(\Lambda+\gamma)^2,\\
&\frac{\lambda_2}{s_2}-\frac{\lambda_3}{s_3}=0,\quad \varepsilon_3=-9-6(\Lambda+\gamma)-(\Lambda+\gamma)^2,\\
&\cdots.
\end{aligned} \tag{28}$$

The expression ε as a function of n is determined as

$$\varepsilon_n=-n^2-2n(\Lambda+\gamma)-(\Lambda+\gamma)^2,\quad n=0,1,2,3,\ldots \tag{29}$$

Thus, we find that

$$\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+L_{\rm ps}}+\sqrt{-Q_{\rm ps}}\right)^2+Q_{\rm ps}+K_{\rm ps}-L_{\rm ps}=0. \tag{30}$$

Putting Eq. (22) to Eq. (30), the energy eigenvalue equation is obtained as

$$\begin{aligned}
&\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{D_2(E_{n\kappa}-\mu-C_{\rm ps})}{\alpha^2}+T}\right.\\
&\left.+\sqrt{\frac{(\mu-E_{n\kappa}+C_{\rm ps})}{\alpha^2}(E_{n\kappa}+\mu-D_0)+Tc_0}\right)^2\\
&=\frac{(\mu-E_{n\kappa}+C_{\rm ps})}{\alpha^2}(E_{n\kappa}+\mu-D_0+D_1-D_2)+Tc_0.
\end{aligned} \tag{31}$$

The numerical values for the relativistic energy eigenvalues (in units of fm^{-1}) of pseudospin symmetry are given in Tables 1–4. When $U_0=0$ (in the absence of the tensor interaction), the degeneracy between members of pseudospin doublets is observed. In Tables 3 and 4, when $U_0\neq 0$ (in the presence of the tensor interaction), the degeneracy is removed. The numerical results are compared with the obtained energy spectrum by using the Nikiforov–Uvarov (NU) method.[17]

Table 1. The energy eigenvalues (in units of fm^{-1}) of the exact pseudospin symmetry (i.e., $C_{\rm ps}=0$) for several n and κ with parameters $\mu=1$ fm^{-1}, $D_0=15$ fm^{-1}, $\alpha=0.1$ fm^{-1}, and $r_{\rm e}=0.4$ fm in the absence of tensor interaction ($U_0=0$).

$\tilde{\ell}$	$n,\kappa<0$	(ℓ,j)	$E_{n,\kappa<0}$ Present work	$E_{n,\kappa<0}$ NU method[17]	$n-1,\kappa>0$	$(\ell+2,j+1)$	$E_{n-1,\kappa>0}$ Present work	$E_{n-1,\kappa>0}$ NU method[17]
1	$1,-1$	$1s_{1/2}$	7.197600315	7.1975980	0,2	$0d_{3/2}$	7.197600315	7.1975980
2	$1,-2$	$1p_{3/2}$	7.918501897	7.9184956	0,3	$0f_{5/2}$	7.918501897	7.9184956
3	$1,-3$	$1d_{5/2}$	8.677411014	8.6773995	0,4	$0g_{7/2}$	8.677411014	8.6773995
4	$1,-4$	$1f_{7/2}$	9.388774480	9.3887565	0,5	$0h_{9/2}$	9.388774480	9.3887565
1	$2,-1$	$2s_{1/2}$	8.705473085	8.7054710	1,2	$1d_{3/2}$	8.705473085	8.7054710
2	$2,-2$	$2p_{3/2}$	9.176903051	9.1768973	1,3	$1f_{5/2}$	9.176903051	9.1768973
3	$2,-3$	$2d_{5/2}$	9.710114666	9.7101038	1,4	$1g_{7/2}$	9.710114666	9.7101038
4	$2,-4$	$2f_{7/2}$	10.23527470	10.2352574	1,5	$1h_{9/2}$	10.23527470	10.2352574

Table 2. The energy eigenvalues (in units of fm^{-1}) of the exact pseudospin symmetry (i.e., $C_{\rm ps}=0$) for several n and κ with parameters $\mu=1$ fm^{-1}, $D_0=15$ fm^{-1}, $\alpha=0.1$ fm^{-1}, and $r_{\rm e}=0.4$ fm in the presence of tensor interaction ($U_0=0.75$).

$\tilde{\ell}$	$n,\kappa<0$	(ℓ,j)	$E_{n,\kappa<0}$	$n-1,\kappa>0$	$(\ell+2,j+1)$	$E_{n-1,\kappa>0}$
1	$1,-1$	$1s_{1/2}$	6.794829113	0,2	$0d_{3/2}$	7.729362578
2	$1,-2$	$1p_{3/2}$	7.365885334	0,3	$0f_{5/2}$	8.490137490
3	$1,-3$	$1d_{5/2}$	8.109454094	0,4	$0g_{7/2}$	9.217533655
4	$1,-4$	$1f_{7/2}$	8.861456346	0,5	$0h_{9/2}$	9.872778657
1	$2,-1$	$2s_{1/2}$	8.461851255	1,2	$1d_{3/2}$	9.049434354
2	$2,-2$	$2p_{3/2}$	8.811822984	1,3	$1f_{5/2}$	9.575632993
3	$2,-3$	$2d_{5/2}$	9.307966390	1,4	$1g_{7/2}$	10.10695911
4	$2,-4$	$2f_{7/2}$	9.843908908	1,5	$1h_{9/2}$	10.60367003

Table 3. The energy eigenvalues (in units of fm^{-1}) for the pseudospin symmetry limit with parameters $D_0 = 15\ \mathrm{fm}^{-1}$, $\mu = 1\ \mathrm{fm}^{-1}$, $r_e = 0.4$ fm, and $C_{ps} = 0$.

α/fm^{-1}	$E_{n\kappa}(U_0=0)$		$E_{n\kappa}(U_0=0.75)$	
	$1f_{7/2}$	$0h_{9/2}$	$1f_{7/2}$	$0h_{9/2}$
0.05	9.285086305	9.285086305	8.767118547	9.759175624
0.10	9.388774480	9.388774480	8.861456346	9.872778657
0.15	9.490166716	9.490166716	8.953901369	9.983660471
0.20	9.589389363	9.589389363	9.044553059	10.09197834
0.25	9.686558550	9.686558550	9.133503117	10.19787634
0.30	9.781781572	9.781781572	9.220836547	10.30148714
0.35	9.875157932	9.875157932	9.306632434	10.40293338
0.40	9.966780008	9.966780008	9.390964375	10.50232867

Table 4. The energy eigenvalues (in units of fm^{-1}) for the pseudospin symmetry limit with parameters $D_0 = 15\ \mathrm{fm}^{-1}$, $\alpha = 0.1\ \mathrm{fm}^{-1}$, $r_e = 0.4$ fm, and $C_{ps} = 0$.

μ/fm^{-1}	$E_{n\kappa}(U_0=0)$		$E_{n\kappa}(U_0=0.75)$	
	$1f_{7/2}$	$0h_{9/2}$	$1f_{7/2}$	$0h_{9/2}$
0.2	9.778009987	9.778009987	9.231181388	10.28418479
0.4	9.677634295	9.677634295	9.135105353	10.17875788
0.6	9.579272085	9.579272085	9.041417193	10.07502462
0.8	9.482969796	9.482969796	8.950179640	9.973019813
1.0	9.388774480	9.388774480	8.861456346	9.872778657
1.2	9.296733747	9.296733747	8.775311801	9.774336746
1.4	9.206895714	9.206895714	8.691811232	9.677730016
1.6	9.119308935	9.119308935	8.611020494	9.582994711

3.2.2. Iterative solutions: wave functions

In order to obtain the upper- and lower-spinor wave functions, we consider the special case of Eq. (2). The second-order homogeneous linear differential equation is given in the form[31]

$$y'' = 2\left(\frac{c_1 x^{N+1}}{1-c_2 x^{N+2}} - \frac{c_3+1}{x}\right)y' - \left(\frac{c_4 x^N}{1-c_2 x^{N+2}}\right)y, \quad (32)$$

where c_1, c_2, c_3, and c_4 are constants, $N = -1, 0, 1, 2, 3, \ldots$ The solution of Eq. (32) is the hypergeometric function

$$y_n(x) = \mathcal{N}\,{}_2\mathrm{F}_1\left(-n,\ p+n,\ q;\ c_2 x^{N+2}\right),$$
$$n = 0, 1, 2, 3, \ldots, \quad (33)$$

where $\mathcal{N} = (-1)^n C_n (q)_n (N+2)^n$ is the normalization constant, $(q)_n = \Gamma(q+n)/\Gamma(n)$ is the Pochhammer symbol, C_n is a constant, $p = \frac{((2c_3+1)c_2+2c_1)}{(N+2)c_2}$, and $q = \frac{(2c_3+N+3)}{(N+2)}$. Comparing Eq. (25) with Eq. (32), we determine all the constants for the pseudospin symmetric equation. All these constants and the Dirac-wave functions are given in Table 5.

Table 5. The constants in the iterative solutions for the pseudospin symmetry.

$c_1 = (2\Lambda+1)/2$	$p = 2\gamma + 2\Lambda$
$c_2 = 1$	$q = 2\gamma$
$c_3 = \gamma - 1$	$f(x) = \mathcal{N}\,{}_2\mathrm{F}_1\,(-n,\ 2\gamma+2\Lambda+n,\ 2\gamma;\ x)$
$c_4 = -\varepsilon - (\Lambda+\gamma)^2$	$G_{n\kappa}(x) = \mathcal{N} x^{\gamma}(1-x)^{\Lambda}$ $\times {}_2\mathrm{F}_1\,(-n,\ 2\gamma+2\Lambda+n,\ 2\gamma;\ x)$
$N = 1$	$G_{n\kappa}(r) = \mathcal{N}(1-\mathrm{e}^{-\alpha r})^{\gamma}\mathrm{e}^{-\Lambda\alpha r}$ $\times {}_2\mathrm{F}_1\left(-n,\ 2\gamma+2\Lambda+n,\ 2\gamma;\ 1-\mathrm{e}^{-\alpha r}\right)$

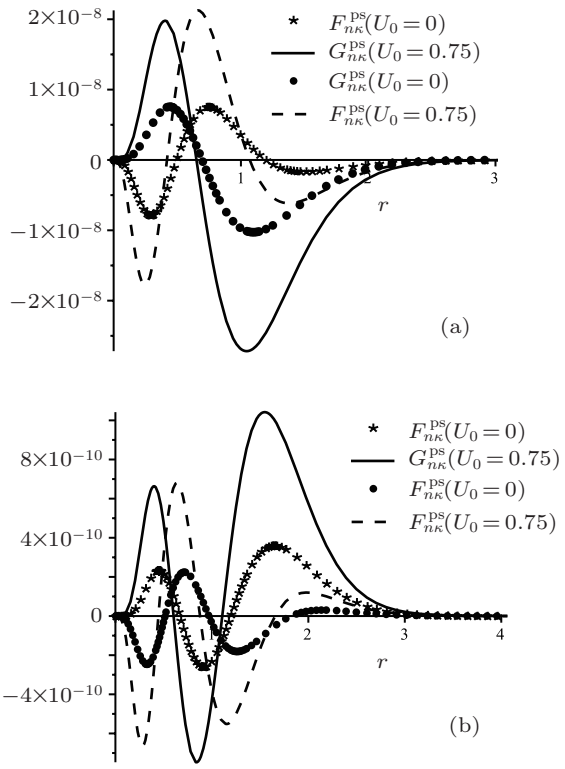


Fig. 3. The upper and lower spinor wave functions of the pseudospin symmetry for the (a) $1s_{1/2}$ and (b) $2_{p3/2}$ states.

Moreover, the upper-spinor wave function is obtained as

$$F_{n\kappa}(r) = \frac{1}{(\mu - E_{n\kappa} + C_{ps})}\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r} + U(r)\right)G_{n\kappa}(r). \quad (34)$$

The effects of the tensor interaction on the upper and the lower components of the Dirac equation for the pseudospin symme-

try can be seen. In Fig. 3, we have plotted the upper and lower spinor wave functions for the $1s_{1/2}$ and $2p_{3/2}$ states.

3.3. The spin symmetry limit

The spin symmetry occurs when $\mathrm{d}\Delta(r)/\mathrm{d}r = 0$, i.e., $\Delta(r) = C_\mathrm{s} = \text{const}$. Then, the sum potential $\Sigma(r)$ is taken as the DF potential

$$\Sigma(r) = D_0\left(1 - \frac{b}{\mathrm{e}^{\alpha r} - 1}\right)^2. \tag{35}$$

By substituting the potentials given in Eqs. (35) and (17) into Eq. (15a), the Schrödinger-like equation is obtained as

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\tilde{T}}{r^2} + (E_{n\kappa} + \mu - C_\mathrm{s})\left(E_{n\kappa} - \mu - D_0 - D_1\frac{\mathrm{e}^{-\alpha r}}{1-\mathrm{e}^{-\alpha r}} - D_2\frac{\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-\alpha r})^2}\right)\right]F_{n\kappa}(r) = 0, \tag{36}$$

where $D_1 = -2D_0 b$, $D_2 = D_0 b^2$, and $\tilde{T} = (\kappa + U_0)(\kappa + U_0 + 1)$. The above equation is similar to Eq. (18). Then, we can apply the same procedure as that in the previous section. Hence, the Schrödinger-like equation for the spin symmetry limit is obtained as

$$\left[x(1-x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x\frac{\mathrm{d}}{\mathrm{d}x} + Q_\mathrm{s}\left(\frac{x}{1-x}\right) - L_\mathrm{s}\left(\frac{1-x}{x}\right) - K_\mathrm{s}\right]F_{n\kappa}(x) = 0, \tag{37}$$

where

$$\begin{aligned} Q_\mathrm{s} &= \frac{(E_{n\kappa} + \mu - C_\mathrm{s})}{\alpha^2}(E_{n\kappa} - \mu - D_0) - \tilde{T}c_0, \\ K_\mathrm{s} &= \frac{(E_{n\kappa} + \mu - C_\mathrm{s})D_1}{\alpha^2} + \tilde{T}, \\ L_\mathrm{s} &= \frac{(E_{n\kappa} + \mu - C_\mathrm{s})D_2}{\alpha^2} + \tilde{T}. \end{aligned} \tag{38}$$

If we use the same procedure as that in the previous section, we find

$$\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + L_\mathrm{s}} + \sqrt{-Q_\mathrm{s}}\right)^2 + Q_\mathrm{s} + K_\mathrm{s} - L_\mathrm{s} = 0. \tag{39}$$

Putting Eq. (38) into Eq. (39), the energy eigenvalue equation is obtained as

$$\begin{aligned} &\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{D_2(E_{n\kappa} + \mu - C_\mathrm{s})}{\alpha^2} + \tilde{T}} + \sqrt{\frac{(E_{n\kappa} + \mu - C_\mathrm{s})}{\alpha^2}(\mu - E_{n\kappa} + D_0) + \tilde{T}c_0}\right)^2 \\ &= \frac{(E_{n\kappa} + \mu - C_\mathrm{s})}{\alpha^2}(\mu - E_{n\kappa} + D_0 - D_1 + D_2) + \tilde{T}c_0. \end{aligned} \tag{40}$$

In Tables 6–9, we give the numerical results for the spin symmetric energy eigenvalues (in units of fm^{-1}). In the absence of the tensor interaction ($U_0 = 0$), the degeneracy between spin doublets is observed. When $U_0 = 0.75$ (in the presence of the tensor interaction), the degeneracy is removed.

Table 6. The energy eigenvalues (in units of fm^{-1}) of the exact spin symmetry (i.e., $C_\mathrm{s} = 0$) for several n and κ with parameters $\mu = 1\ \mathrm{fm}^{-1}$, $D_0 = 15\ \mathrm{fm}^{-1}$, $\alpha = 0.1\ \mathrm{fm}^{-1}$, and $r_\mathrm{e} = 0.4$ fm in the absence of the tensor interaction ($U_0 = 0$).

$\tilde{\ell}$	$n, \kappa < 0$	(ℓ, j)	$E_{n,\kappa<0}$ Present work	$E_{n,\kappa<0}$ NU method[17]	$n, \kappa > 0$	(ℓ, j)	$E_{n,\kappa>0}$ Present work	$E_{n,\kappa>0}$ NU method[17]
1	$0,-2$	$0p_{3/2}$	5.579109950	5.5791076	$0,1$	$0p_{1/2}$	5.579109950	5.5791076
2	$0,-3$	$0d_{5/2}$	6.811866241	6.8118605	$0,2$	$0d_{3/2}$	6.811866241	6.8118605
3	$0,-4$	$0f_{7/2}$	8.017117230	8.0171073	$0,3$	$0f_{5/2}$	8.017117230	8.0171073
4	$0,-5$	$0g_{9/2}$	9.102532476	9.1025175	$0,4$	$0g_{7/2}$	9.102532476	9.1025175
1	$1,-2$	$1p_{3/2}$	8.182369492	8.1823677	$1,1$	$1p_{1/2}$	8.182369492	8.1823677
2	$1,-3$	$1d_{5/2}$	8.881538842	8.8815340	$1,2$	$1d_{3/2}$	8.881538842	8.8815340
3	$1,-4$	$1f_{7/2}$	9.660319511	9.6603105	$1,3$	$1f_{5/2}$	9.660319511	9.6603105
4	$1,-5$	$1g_{9/2}$	10.42003368	10.4200196	$1,4$	$1g_{7/2}$	10.42003368	10.4200196

Table 7. The energy eigenvalues (in units of fm^{-1}) of the exact spin symmetry (i.e., $C_\mathrm{s} = 0$) for several n and κ with parameters $\mu = 1\ \mathrm{fm}^{-1}$, $D_0 = 15\ \mathrm{fm}^{-1}$, $\alpha = 0.1\ \mathrm{fm}^{-1}$, and $r_\mathrm{e} = 0.4$ fm in the presence of the tensor interaction ($U_0 = 0.75$).

$\tilde{\ell}$	$n, \kappa < 0$	(ℓ, j)	$E_{n,\kappa<0}$	$n, \kappa > 0$	(ℓ, j)	$E_{n,\kappa>0}$
1	$0,-2$	$0p_{3/2}$	4.814248855	$0,1$	$0p_{1/2}$	6.499493635
2	$0,-3$	$0d_{5/2}$	5.878652972	$0,2$	$0d_{3/2}$	7.725365497
3	$0,-4$	$0f_{7/2}$	7.121531438	$0,3$	$0f_{5/2}$	8.843987082
4	$0,-5$	$0g_{9/2}$	8.301074711	$0,4$	$0g_{7/2}$	9.826262681
1	$1,-2$	$1p_{3/2}$	7.814998689	$1,1$	$1p_{1/2}$	8.693712060
2	$1,-3$	$1d_{5/2}$	8.341272121	$1,2$	$1d_{3/2}$	9.464775655
3	$1,-4$	$1f_{7/2}$	9.073904535	$1,3$	$1f_{5/2}$	10.23487831
4	$1,-5$	$1g_{9/2}$	9.854403434	$1,4$	$1g_{7/2}$	10.95042371

Table 8. The energy eigenvalues (in units of fm^{-1}) for the spin symmetry limit with parameters $D_0 = 15\ \mathrm{fm}^{-1}$, $\mu = 1\ \mathrm{fm}^{-1}$, $r_\mathrm{e} = 0.4$ fm, and $C_\mathrm{s} = 0$.

μ/fm^{-1}	$E_{n\kappa}(U_0 = 0)$		$E_{n\kappa}(U_0 = 0.75)$	
	$0\mathrm{p}_{3/2}$	$0\mathrm{p}_{1/2}$	$0\mathrm{p}_{3/2}$	$0\mathrm{p}_{1/2}$
0.05	5.546472389	5.546472389	4.782701192	6.462391495
0.10	5.579109950	5.579109950	4.814248855	6.499493635
0.15	5.611562210	5.611562210	4.845726109	6.536225058
0.20	5.643836603	5.643836603	4.877136093	6.572599376
0.25	5.675940161	5.675940161	4.908481709	6.608629541
0.30	5.707879687	5.707879687	4.939765758	6.644328069
0.35	5.739661833	5.739661833	4.970990989	6.679707143
0.40	5.771292951	5.771292951	5.002159986	6.714778444

Table 9. The energy eigenvalues (in units of fm^{-1}) for the spin symmetry limit with parameters $D_0 = 15\ \mathrm{fm}^{-1}$, $\alpha = 0.1\ \mathrm{fm}^{-1}$, $r_\mathrm{e} = 0.4$ fm, and $C_\mathrm{s} = 0$.

α/fm^{-1}	$E_{n\kappa}(U_0 = 0)$		$E_{n\kappa}(U_0 = 0.75)$	
	$0\mathrm{p}_{3/2}$	$0\mathrm{p}_{1/2}$	$0\mathrm{p}_{3/2}$	$0\mathrm{p}_{1/2}$
0.2	5.239930558	5.239930558	4.427833130	6.179347196
0.4	5.316471084	5.316471084	4.515193193	6.252620284
0.6	5.398695732	5.398695732	4.609004438	6.330510473
0.8	5.486331422	5.486331422	4.708828141	6.412855538
1.0	5.579109950	5.579109950	4.814248855	6.499493635
1.2	5.676769778	5.676769778	4.924875729	6.590264184
1.4	5.779057412	5.779057412	5.040342986	6.685008638
1.6	5.885728454	5.885728454	5.160309750	6.783571143

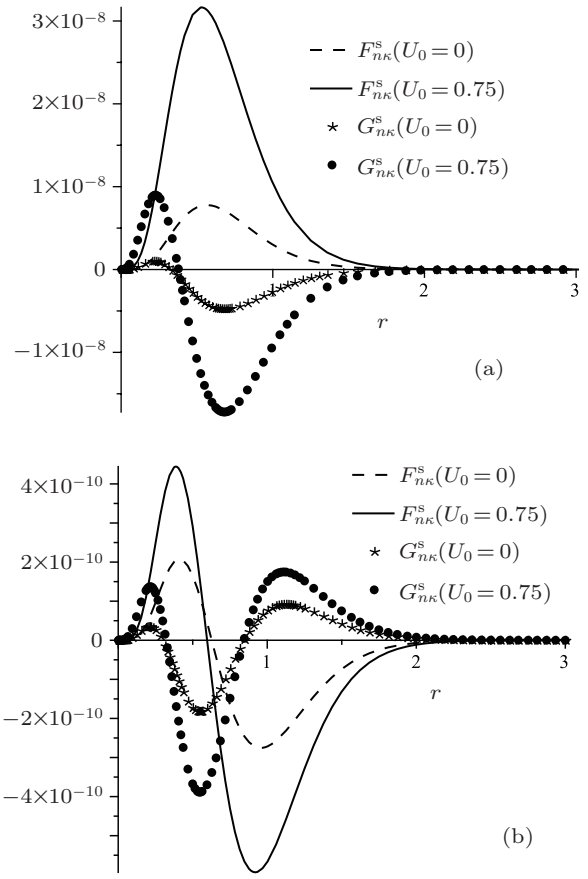


Fig. 4. The upper and lower spinor wave functions of the spin symmetry for the (a) $0\mathrm{p}_{3/2}$ and (b) $1_{\mathrm{p}3/2}$ states.

If we use the same procedure as that in the previous section, the upper-spinor wave function for the spin symmetry limit is obtained as

$$F_{n\kappa}(r) = \tilde{\mathcal{N}}(1-\mathrm{e}^{-\alpha r})^{\tilde{\gamma}}\mathrm{e}^{-\tilde{\Lambda}\alpha r} \times {}_2\mathrm{F}_1\left(-n,\ 2\tilde{\gamma}+2\tilde{\Lambda}+n,\ 2\tilde{\gamma};\ 1-\mathrm{e}^{-\alpha r}\right), \quad (41)$$

where $\tilde{\gamma} = 1/2 + \sqrt{1/4 + L_\mathrm{s}}$ and $\tilde{\Lambda}^2 = -Q_\mathrm{s}$. From Eq. (14a), the lower-spinor wave function is obtained as

$$G_{n\kappa}(r) = \frac{1}{(\mu + E_{n\kappa} - C_\mathrm{s})}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r} - U(r)\right)F_{n\kappa}(r). \quad (42)$$

We can see the effect of the tensor field on the spinor wave functions for the spin symmetry. The wave functions of the $0\mathrm{p}_{3/2}$ and $1\mathrm{p}_{3/2}$ states are plotted in Fig. 4.

4. Conclusion

In this paper, we investigated the approximate analytical solutions of the Dirac equation for the DF potential with a Coulomb-like tensor interaction under pseudospin and spin symmetry limits. The bound-state energy eigenvalues and the corresponding wave functions are obtained. Some of the numerical results of the effects of the potential parameters on the energy eigenvalues and the upper and lower spinor wave functions were investigated. It is shown that the degeneracies between pseudospin and spin doublets are removed by the tensor effect.

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