Mutual transformations between the P–Q, Q–P, and generalized Weyl ordering of operators*

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Based on the generalized Weyl quantization scheme, which relies on the generalized Wigner operator $\Omega_k(p,q)$ with a real k parameter and can unify the P–Q, Q–P, and Weyl ordering of operators in $k=1,-1,0$, respectively, we find the mutual transformations between $\delta(p-P)\delta(q-Q)$, $\delta(q-Q)\delta(p-P)$, and $\Omega_k(p,q)$, which are, respectively, the integration kernels of the P–Q, Q–P, and generalized Weyl quantization schemes. The mutual transformations provide us with a new approach to deriving the Wigner function of quantum states. The $\mathfrak{P}-$ and $\mathfrak{Q}-$ ordered forms of $\Omega_k(p,q)$ are also derived, which helps us to put the operators into their $\mathfrak{P}-$ and $\mathfrak{Q}-$ ordering, respectively.

Keywords: generalized Wigner operator, generalized Weyl quantization scheme, different operator ordering rules, mutual transformation

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1. Introduction

Since the birth of quantum mechanics, operator ordering and the classical correspondence of operators have been intensively discussed.[1–3] They are also useful in quantum optics and quantum statistics.[4–7] In a recent paper,[8] a generalized Weyl–Wigner quantization scheme that can unify the P–Q ordering (all the Ps stand on the left of all the Qs, denoted as $\mathfrak{P}-$), the Q–P ordering (all the Qs stand on the left of all the Ps, denoted as $\mathfrak{Q}-$), and the Weyl ordering of operators was proposed, where Q and P are the coordinate and the momentum operators, respectively, $[Q,P]=\mathrm{i}$ ($\hbar=1$). In that scheme, an operator H corresponds to its classical function $h_k(p,q)$ through the following integration:

$$H=\iint_{-\infty}^{\infty}\mathrm{d}q\mathrm{d}p\Omega_k(p,q)h_k(p,q), \tag{1}$$

where Ω_k composes a family of generalized Wigner operators (GWOs) for different real k parameters

$$\Omega_k(p,q)\equiv\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{d}u\mathrm{e}^{\mathrm{i}u(q-Q)+\mathrm{i}y(p-P)+\mathrm{i}\frac{k}{2}yu}. \tag{2}$$

When $k=0$, $\Omega_k(p,q)$ reduces to the usual Wigner operator.[9,10] As each quantization scheme accompanies operator ordering, say, the Weyl–Wigner correspondence accompanies the Weyl ordering of the operator, so Eqs. (1) and (2) assign a new operator ordering rule. Using the coordinate eigenstate $|q\rangle$ and $|q\rangle\langle q|=\delta(q-Q)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}u\mathrm{e}^{-\mathrm{i}u(q-Q)}$, one can express $\Omega_k(p,q)$ in the coordinate representation as

$$\begin{aligned}&\Omega_k(p,q)\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{e}^{-\mathrm{i}py}\mathrm{e}^{\mathrm{i}\frac{k+1}{2}yP}|q\rangle\langle q|\mathrm{e}^{\mathrm{i}\frac{1-k}{2}yP}\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{e}^{-\mathrm{i}py}\left|q-\frac{k+1}{2}y\right\rangle\left\langle q+\frac{1-k}{2}y\right|.\end{aligned} \tag{3}$$

Denoting the normal ordering as : :, by virtue of the integration method within normally ordered operators,[11,12] we can obtain the normally ordered expansion of $\Omega_k(p,q)$

$$\begin{aligned}&\Omega_k(p,q)\\&=\frac{1}{4\pi^2}:\iint_{-\infty}^{\infty}\mathrm{d}y\mathrm{d}u\mathrm{e}^{-\frac{1}{4}\left(y^2+u^2\right)+\mathrm{i}u(q-Q)+\mathrm{i}y(p-P)+\mathrm{i}\frac{k}{2}yu}:\\&=\frac{1}{2\pi\sqrt{\pi}}:\int\mathrm{d}u\mathrm{e}^{-\frac{1}{4}u^2+\mathrm{i}u(q-Q)-\left(p-P+\frac{k}{2}u\right)^2}:\\&=\frac{1}{2\pi\sqrt{\pi}}:\int\mathrm{d}u\mathrm{e}^{-\frac{1}{4}u^2\left(1+k^2\right)+\mathrm{i}u[q-Q+\mathrm{i}k(p-P)]-(p-P)^2}:\\&=\frac{1}{\pi\sqrt{(1+k^2)}}:\exp\left\{-\frac{1}{1+k^2}[(q-Q)^2\right.\\&\quad\left.+2\mathrm{i}k(q-Q)(p-P)+(p-P)^2]\right\}:.\end{aligned} \tag{4}$$

An interesting question thus naturally arises: what are the $\mathfrak{P}-$ ordered and $\mathfrak{Q}-$ ordered forms of GWO $\Omega_k(p,q)$ and what are their uses? What are the mutual transformations between the P–Q ordering, Q–P ordering and generalized Weyl ordering of the operators? We shall answer these questions in the following sections. The mutual transformations provide us with

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a new approach to deriving the Wigner function of quantum states. The $\mathfrak{P}-$ ordered and $\mathfrak{Q}-$ ordered forms of $\Omega_k(p,q)$ help us to put the operators into $\mathfrak{P}-$ ordering and $\mathfrak{Q}-$ ordering, respectively.

2. The $\mathfrak{Q}-$ ordered form of $\Omega_k(p,q)$

From Eq. (4), we see that when $k=0$,

$$\Omega_{k=0}(p,q)=\frac{1}{\pi}:\mathrm{e}^{-(q-Q)^2-(p-P)^2}:, \tag{5}$$

which is just the normal ordering of the Wigner operator $\Delta(p,q)$. When $k=1$, using $\langle y|\,p\rangle=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}py}$, from Eq. (3) and $\int_{-\infty}^{\infty}\mathrm{d}y\,|y\rangle\langle y|=1$ of the coordinate representation, we obtain

$$\begin{aligned}\Omega_{k=1}(p,q)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{e}^{-\mathrm{i}py}|q-y\rangle\langle q|\\&=\mathrm{e}^{-\mathrm{i}Pq}\int\frac{\mathrm{d}y}{2\pi}\mathrm{e}^{-\mathrm{i}py}|-y\rangle\langle q|\\&=\mathrm{e}^{-\mathrm{i}Pq}\int\frac{\mathrm{d}y}{\sqrt{2\pi}}|y\rangle\langle y|\,p\rangle\langle q|=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}Pq}|p\rangle\langle q|\\&=\frac{1}{\sqrt{2\pi}}|p\rangle\langle q|\,\mathrm{e}^{-\mathrm{i}pq}=\delta(p-P)\delta(q-Q),\end{aligned} \tag{6}$$

so $\Omega_{k=1}(p,q)$ represents the P–Q ordering ($\delta(p-P)$ and stands on the left of $\delta(q-Q)$, denoted as $\mathfrak{P}-$). When $k=-1$,

$$\begin{aligned}\Omega_{k=-1}(p,q)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{e}^{-\mathrm{i}py}|q\rangle\langle q+y|\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}y|q\rangle\langle p|\,y\rangle\langle y|\,\mathrm{e}^{\mathrm{i}Pq}\\&=\frac{1}{\sqrt{2\pi}}|q\rangle\langle p|\,\mathrm{e}^{\mathrm{i}pq}=|q\rangle\langle q\,|p\rangle\langle p|\\&=\delta(q-Q)\delta(p-P),\end{aligned} \tag{7}$$

where $\delta(q-Q)$ stands on the left of $\delta(p-P)$, so $\Omega_{k=-1}(p,q)$ represents the Q–P ordering form (denoted as $\mathfrak{Q}-$).

Using the momentum eigenstate's completeness relation

$$\int_{-\infty}^{\infty}\mathrm{d}p\,|p\rangle\langle p|=1 \tag{8}$$

and Eq. (3), we have

$$\begin{aligned}\Omega_k(p,q)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{e}^{-\mathrm{i}py}\\&\quad\times\left|q-\frac{k+1}{2}y\right\rangle\left\langle q+\frac{1-k}{2}y\right|\int_{-\infty}^{\infty}\mathrm{d}p'\,|p'\rangle\langle p'|\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}p'\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{e}^{-\mathrm{i}py}\\&\quad\times\left|q-\frac{k+1}{2}y\right\rangle\langle p'|\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}\left(q+\frac{1-k}{2}y\right)p'}.\end{aligned} \tag{9}$$

Then, using Eq. (7), we obtain

$$\begin{aligned}&\left|q-\frac{k+1}{2}y\right\rangle\langle p'|\\&=\delta\left(q-\frac{1+k}{2}y-Q\right)\delta(p'-P)\mathrm{e}^{-\mathrm{i}\left(q-\frac{1+k}{2}y\right)p'}.\end{aligned} \tag{10}$$

Substituting Eq. (10) into Eq. (9) yields the $\mathfrak{Q}-$ ordering form of $\Omega_k(p,q)$

$$\begin{aligned}\Omega_k(p,q)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}p'\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{e}^{-\mathrm{i}(p-p')y}\\&\quad\times\delta\left(q-\frac{1+k}{2}y-Q\right)\delta(p'-P)\\&=\frac{1}{\pi(k+1)}\mathfrak{Q}\left[\mathrm{e}^{-\frac{2\mathrm{i}}{k+1}(q-Q)(p-P)}\right].\end{aligned} \tag{11}$$

In particular, when $k=0$, $\Omega_{k=0}(p,q)=\Delta(p,q)$, so

$$\Delta(p,q)=\frac{1}{\pi}\mathfrak{Q}\left[\mathrm{e}^{-\mathrm{i}2(q-Q)(p-P)}\right]. \tag{12}$$

When $k=1$,

$$\Omega_{k=1}(p,q)=\frac{1}{2\pi}\mathfrak{Q}\left[\mathrm{e}^{-\mathrm{i}(q-Q)(p-P)}\right]. \tag{13}$$

Comparing with Eq. (6), we have

$$\delta(p-P)\delta(q-Q)=\frac{1}{2\pi}\mathfrak{Q}\left[\mathrm{e}^{-\mathrm{i}(q-Q)(p-P)}\right]. \tag{14}$$

Here, the left is in $\mathfrak{P}-$ ordering, while the right is in $\mathfrak{Q}-$ ordering. When $k=-1$, from Eqs. (11) and (7), we obtain

$$\begin{aligned}\Omega_{k=-1}(p,q)&=\lim_{\varepsilon\to0}\frac{1}{\pi\varepsilon}\mathfrak{Q}\left[\mathrm{e}^{-\mathrm{i}\frac{2}{\varepsilon}(q-Q)(p-P)}\right]\\&=\delta(q-Q)\delta(p-P),\end{aligned} \tag{15}$$

which embodies a new formula for the Dirac δ function.

3. The $\mathfrak{P}-$ ordered form of $\Omega_k(p,q)$

Similar to Eq. (9), we derive

$$\begin{aligned}\Omega_k(p,q)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}p'\,|p'\rangle\langle p'|\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{e}^{-\mathrm{i}py}\\&\quad\times\left|q-\frac{k+1}{2}y\right\rangle\left\langle q+\frac{1-k}{2}y\right|\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{e}^{-\mathrm{i}py}|p'\rangle\left\langle q+\frac{1-k}{2}y\right|\\&\quad\times\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}\left(q-\frac{k+1}{2}y\right)p'}.\end{aligned} \tag{16}$$

Using Eq. (6), we know

$$\begin{aligned}&\frac{1}{\sqrt{2\pi}}|p'\rangle\left\langle q+\frac{1-k}{2}y\right|\\&=\delta(p'-P)\delta\left(q+\frac{1-k}{2}y-Q\right)\mathrm{e}^{\mathrm{i}p'\left(q+\frac{1-k}{2}y\right)}.\end{aligned} \tag{17}$$

Substituting Eq. (17) into Eq. (16) leads to the $\mathfrak{P}-$ ordered form of $\Omega_k(p,q)$

$$\begin{aligned}&\Omega_k(p,q)\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{e}^{\mathrm{i}(p'-p)y}\delta(p'-P)\delta\left(q+\frac{1-k}{2}y-Q\right)\\&=\frac{1}{\pi(1-k)}\mathfrak{P}\left[\mathrm{e}^{\frac{2\mathrm{i}}{1-k}(q-Q)(p-P)}\right].\end{aligned} \tag{18}$$

In particular, when $k=0$, $\Omega_{k=0}(p,q)=\Delta(p,q)$, so Eq. (18) reduces to

$$\Delta(p,q)=\frac{1}{\pi}\mathfrak{P}\left[\mathrm{e}^{\mathrm{i}2(q-Q)(p-P)}\right], \tag{19}$$

which differs from Eq. (11). When $k=-1$, equation (18) becomes

$$\Omega_{k=-1}(p,q)=\frac{1}{2\pi}\mathfrak{P}\left[\mathrm{e}^{\mathrm{i}(q-Q)(p-P)}\right]. \tag{20}$$

Comparing with Eq. (7) leads to the identity

$$\delta(q-Q)\delta(p-P)=\frac{1}{2\pi}\mathfrak{P}\left[\mathrm{e}^{\mathrm{i}(q-Q)(p-P)}\right]. \tag{21}$$

Here, the left is in $\mathfrak{Q}-$ ordering, while the right is in $\mathfrak{P}-$ ordering. When $k=1$, equation (18) presents

$$\begin{aligned}\Omega_{k=1}(p,q) &= \lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\mathfrak{P}\left[\mathrm{e}^{\frac{2\mathrm{i}}{\varepsilon}(q-Q)(p-P)}\right]\\ &= \delta(p-P)\delta(q-Q).\end{aligned} \tag{22}$$

4. The mutual transformation between $\delta(p-P)\delta(q-Q)$ and $\Omega_k(p,q)$

In Ref. [7], it has been proved that the classical correspondence of operator H, defined through the k-parameter quantization scheme in Eq. (1), is

$$h_k(p,q)=2\pi\,\mathrm{tr}\left[\Omega_{-k}(p,q)H\right], \tag{23}$$

so the classical correspondence of $\delta(p'-P)\delta(q'-Q)$ is given by

$$\begin{aligned}&2\pi\,\mathrm{tr}\left[\Omega_{-k}(p,q)\,\delta(p'-P)\delta(q'-Q)\right]\\ &=\mathrm{tr}\left[\int\mathrm{d}y\,\mathrm{e}^{-\mathrm{i}py}\left|q-\frac{1-k}{2}y\right\rangle\left\langle q+\frac{1+k}{2}y\right|\right.\\ &\quad\left.\times\frac{1}{\sqrt{2\pi}}\left|p'\right\rangle\left\langle q'\right|\mathrm{e}^{-\mathrm{i}p'q'}\right]\\ &=\frac{1}{2\pi}\int\mathrm{d}y\,\mathrm{e}^{-\mathrm{i}py}\,\mathrm{e}^{\mathrm{i}p'\left(q+\frac{1+k}{2}y\right)}\delta\left(q-\frac{1-k}{2}y-q'\right)\mathrm{e}^{-\mathrm{i}p'q'}\\ &=\frac{1}{2\pi}\int\mathrm{d}y\,\mathrm{e}^{\mathrm{i}\left(p'\frac{1+k}{2}-p\right)y}\delta\left(q-\frac{1-k}{2}y-q'\right)\mathrm{e}^{-\mathrm{i}p'(q'-q)}\\ &=\frac{1}{\pi(1-k)}\mathrm{e}^{-\mathrm{i}p'(q'-q)}\,\mathrm{e}^{\mathrm{i}\left(p'\frac{1+k}{2}-p\right)\frac{2}{1-k}(q-q')}\\ &=\frac{1}{\pi(1-k)}\mathrm{e}^{\left(\frac{1+k}{1-k}+1\right)\mathrm{i}p'(q-q')-\frac{2\mathrm{i}p}{1-k}(q-q')}\\ &=\frac{1}{\pi(1-k)}\mathrm{e}^{\frac{2\mathrm{i}p'}{1-k}(q-q')-\frac{2\mathrm{i}p}{1-k}(q-q')}\\ &=\frac{1}{\pi(1-k)}\mathrm{e}^{\frac{-2\mathrm{i}}{1-k}(p'-p)(q'-q)}.\end{aligned} \tag{24}$$

Substituting Eq. (24) into Eq. (1) gives

$$\begin{aligned}&\delta(p'-P)\delta(q'-Q)\\ &=\frac{1}{\pi(1-k)}\int\mathrm{d}q\mathrm{d}p\Omega_k(p,q)\,\mathrm{e}^{\frac{-2\mathrm{i}}{1-k}(p'-p)(q'-q)}.\end{aligned} \tag{25}$$

Conversely, due to Eq. (18), we know

$$\begin{aligned}&\frac{1}{\pi(1-k)}\iint\mathrm{d}q'\mathrm{d}p'\,\mathrm{e}^{\frac{2\mathrm{i}}{1-k}(p'-p)(q'-q)}\delta(p'-P)\delta(q'-Q)\\ &=\frac{1}{\pi(1-k)}\mathfrak{P}\left[\mathrm{e}^{\frac{2\mathrm{i}}{1-k}(p-P)(q-Q)}\right]=\Omega_k(p,q),\end{aligned} \tag{26}$$

this is the integration transformation for converting $\delta(p'-P)\delta(q'-Q)$ into $\Omega_k(p,q)$, so it is the reciprocal integration of Eq. (25).

5. The mutual transformation between $\delta(q-Q)\delta(p-P)$ and $\Omega_k(p,q)$

We now turn to search for the classical correspondence of $\delta(q'-Q)\delta(p'-P)$ through Eq. (23)

$$\begin{aligned}&2\pi\,\mathrm{tr}\left[\Omega_{-k}(p,q)\,\delta(q'-Q)\delta(p'-P)\right]\\ &=\mathrm{tr}\left[\int\mathrm{d}y\mathrm{e}^{-\mathrm{i}py}\left|q-\frac{1-k}{2}y\right\rangle\left\langle q+\frac{1+k}{2}y\right|\right.\\ &\quad\left.\times\frac{1}{\sqrt{2\pi}}\left|q'\right\rangle\left\langle p'\right|\mathrm{e}^{\mathrm{i}p'q'}\right]\\ &=\frac{1}{2\pi}\int\mathrm{d}y\mathrm{e}^{-\mathrm{i}py}\delta\left(q+\frac{1+k}{2}y-q'\right)\mathrm{e}^{-\mathrm{i}p'\left(q-\frac{1-k}{2}y\right)}\mathrm{e}^{\mathrm{i}p'q'}\\ &=\frac{1}{\pi(1+k)}\mathrm{e}^{\mathrm{i}p'(q'-q)}\,\mathrm{e}^{\mathrm{i}\left(p'\frac{1-k}{2}-p\right)\frac{2}{1+k}(q'-q)}\\ &=\frac{1}{\pi(1+k)}\mathrm{e}^{\left(\frac{1-k}{1+k}+1\right)\mathrm{i}p'(q'-q)-\frac{2\mathrm{i}p}{1+k}(q'-q)}\\ &=\frac{1}{\pi(1+k)}\mathrm{e}^{\frac{2\mathrm{i}}{1+k}(p'-p)(q'-q)}.\end{aligned} \tag{27}$$

Substituting Eq. (27) into Eq. (1), we have

$$\begin{aligned}&\delta(q'-Q)\delta(p'-P)\\ &=\frac{1}{\pi(1+k)}\iint\mathrm{d}q\mathrm{d}p\Omega_k(p,q)\,\mathrm{e}^{\frac{2\mathrm{i}}{1+k}(p'-p)(q'-q)},\end{aligned} \tag{28}$$

this is the integration transformation for converting $\delta(q'-Q)\delta(p'-P)$ into $\Omega_k(p,q)$, and its reciprocal transformation is just the the result in Eq. (11)

$$\begin{aligned}&\frac{1}{\pi(1+k)}\iint\mathrm{d}q'\mathrm{d}p'\,\mathrm{e}^{\frac{-2\mathrm{i}}{1+k}(p'-p)(q'-q)}\delta(q'-Q)\delta(p'-P)\\ &=\frac{1}{\pi(k+1)}\mathfrak{Q}\left[\mathrm{e}^{-\frac{2\mathrm{i}}{k+1}(q-Q)(p-P)}\right]=\Omega_k(p,q).\end{aligned} \tag{29}$$

For $k=0$, we obtain the mutual relation between $\delta(q'-Q)\delta(p'-P)$ and the Wigner operator

$$\begin{aligned}&\frac{1}{\pi}\iint\mathrm{d}q'\mathrm{d}p'\,\mathrm{e}^{-2\mathrm{i}(p'-p)(q'-q)}\delta(q'-Q)\delta(p'-P)\\ &=\Delta(p,q),\end{aligned} \tag{30}$$

$$\begin{aligned}&\delta(q'-Q)\delta(p'-P)\\ &=\frac{1}{\pi(1+k)}\iint\mathrm{d}q\mathrm{d}p\Delta(p,q)\,\mathrm{e}^{2\mathrm{i}(p'-p)(q'-q)},\end{aligned} \tag{31}$$

which provides us with a new approach to deriving the Wigner function of quantum states.

6. The generalized Weyl correspondence of the squeezing operator

We now aim to derive the P–Q, Q–P, and Weyl ordering of the squeezing operator, which is in ket–bra form

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{\sqrt{\mu}}\left|q/\mu\right\rangle\left\langle q\right|\equiv S_1. \tag{32}$$

This form indicates that the classical dilation $q\to q/\mu$ maps onto the squeezing operator in the Hilbert space. As the first step, we derive S_1's classical correspondence by using Eqs. (3) and (23), and the result is

$$\begin{aligned}
&2\pi\,\mathrm{tr}\left[\Omega_{-k}(p,q)S_1\right]\\
&=\int\mathrm{d}y\mathrm{e}^{-\mathrm{i}py}\left\langle q+\frac{1+k}{2}y\right|\int_{-\infty}^{\infty}\frac{\mathrm{d}q'}{\sqrt{\mu}}\left|q'/\mu\right\rangle\left\langle q'\right|\left|q-\frac{1-k}{2}y\right\rangle\\
&=\int\mathrm{d}y\mathrm{e}^{-\mathrm{i}py}\int_{-\infty}^{\infty}\frac{\mathrm{d}q'}{\sqrt{\mu}}\delta\left(q+\frac{1+k}{2}y-\frac{q'}{\mu}\right)\\
&\quad\times\delta\left(q'-q+\frac{1-k}{2}y\right)\\
&=\frac{1}{\sqrt{\mu}}\int\mathrm{d}y\mathrm{e}^{-\mathrm{i}py}\delta\left(q+\frac{1+k}{2}y-\frac{q-\frac{1-k}{2}y}{\mu}\right)\\
&=2\sqrt{\mu}\int\mathrm{d}y\mathrm{e}^{-\mathrm{i}py}\delta\left[(y+y\mu-ky+ky\mu-2q+2q\mu)\right]\\
&=\frac{2\sqrt{\mu}}{1+\mu+k\mu-k}\int\mathrm{d}y\mathrm{e}^{-\mathrm{i}py}\delta\left(y+\frac{2q\mu-2q}{1+\mu+k\mu-k}\right)\\
&=\frac{2\sqrt{\mu}}{1+\mu+k\mu-k}\mathrm{e}^{2\mathrm{i}pq\frac{\mu-1}{1+\mu+k\mu-k}}.
\end{aligned}\tag{33}$$

Due to Eqs. (1) and (33), we have

$$S_1=\frac{2\sqrt{\mu}}{1+\mu+k\mu-k}\iint\mathrm{d}q\mathrm{d}p\mathrm{e}^{\mathrm{i}2pq\frac{\mu-1}{1+\mu+k\mu-k}}\Omega_k(p,q); \tag{34}$$

for $k=1$ or $\mathfrak{P}-$ ordering

$$S_1=\frac{1}{\sqrt{\mu}}\mathfrak{P}\mathrm{e}^{\mathrm{i}PQ\frac{\mu-1}{\mu}}, \tag{35}$$

for $k=-1$ or $\mathfrak{Q}-$ ordering

$$S_1=\sqrt{\mu}\mathfrak{Q}\mathrm{e}^{\mathrm{i}QP(\mu-1)}, \tag{36}$$

for $k=0$ or Weyl ordering

$$S_1=\frac{2\sqrt{\mu}}{1+\mu}:\mathrm{e}^{\mathrm{i}2PQ\frac{\mu-1}{1+\mu}}:, \tag{37}$$

where $:\ :$ is the Weyl ordering symbol.[13,14]

7. Conclusion

In summary, based on the k-parameterized generalized Wigner operator Ω_k and the quantization scheme unifying P–Q ordering, Q–P ordering, and Weyl ordering of operators, we found the mutual transformations between the P–Q, Q–P, and generalized Weyl ordering of the operators. The mutual transformations provide us with a new approach to deriving the Wigner function of quantum states. The $\mathfrak{P}-$ ordered and $\mathfrak{Q}-$ ordered forms of $\Omega_k(p,q)$ derived in this paper help us to put the operators into their $\mathfrak{P}-$ ordering and $\mathfrak{Q}-$ ordering, respectively.

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