Evolution law of a negative binomial state in an amplitude dissipative channel*

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(Received 8 July 2013; revised manuscript received 20 August 2013; published online 20 February 2014)

For the first time we derive the evolution law of the negative binomial state $\sum_{n=0}^{\infty}\binom{n+s}{n}\gamma^{s+1}(1-\gamma)^n|n\rangle\langle n|$ in an amplitude dissipative channel with a damping constant κ. We find that after passing through the channel, the final state is still a negative binomial state, however the parameter γ evolves into γ', where $\gamma'=\gamma/(\mathrm{e}^{-2\kappa t}(1-\gamma)+\gamma)$. The decay law of the average photon number is also obtained.

Keywords: negative binomial state, thermal entangled state representation, master equation for damping, Kraus operator

PACS: 03.65.–w, 42.50.–p **DOI:** 10.1088/1674-1056/23/3/030304

1. Introduction

In nature, the systems we are concerned with are usually surrounded by thermo reservoirs. The amplitude dissipation is an important topic in the fields of quantum optics,[1–3] quantum computing, and quantum information processing.[4,5] The amplitude dissipation of a quantum optical field can be judged by photocounting its final state and comparing with its initial state.[6–8] In this paper, we discuss the time dissipation of a negative binomial state[9–12] in a dissipative reservoir. The negative binomial state can be implemented in a process of absorbing several photons from a thermal photon field. The negative binomial distribution in statistics is

$$\binom{n+s}{n}\gamma^{s+1}(1-\gamma)^n, \quad \gamma<1. \tag{1}$$

Correspondingly, the density operator of the negative binomial state[13] is

$$\rho_{0s}=\sum_{n=0}^{\infty}\binom{n+s}{n}\gamma^{s+1}(1-\gamma)^n|n\rangle\langle n|, \tag{2}$$

which in this paper is taken as an input of the amplitude dissipative channel, here $|n\rangle$ is the Fock state, $|n\rangle=a^{\dagger n}|0\rangle/\sqrt{n!}$, and the physical meaning of γ (the negative binomial parameter) will be shown later in Eqs. (22)–(24). The master equation describing the dissipative channel is[14]

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}=\kappa\left(2a\rho a^{\dagger}-a^{\dagger}a\rho-\rho a^{\dagger}a\right), \tag{3}$$

where κ is the rate of dissipation, and $\left[a,a^{\dagger}\right]=1$. We hope to find the output state $\rho(t)$ of the damping channel when the initial state is ρ_{0s}, i.e., we study how a photon field with a negative binomial distribution (or named a negative binomial state) evolves in an amplitude dissipative channel.

In the following we shall show that the final state is still a negative binomial state, however the parameter γ evolves into γ' (see Eq. (34)). In Section 2, we briefly review how to use the thermo entangled state representation to solve the master equation to obtain the operator-sum-representation of $\rho(t)$. In Section 3, after explaining the summation technique within an antinormally ordered product of operators, we employ it to derive the relation between γ' and γ, thus the exact evolution law of the negative binomial state in the amplitude dissipative channel is obtained.

2. Converting ρ's master equations into pure state's evolution equations

In order to solve Eq. (3), instead of converting the operator equation into its c-number equation by using the Wigner function approach of P-representation,[6] we introduce the thermo entangled state representation[15,16]

$$|\tau\rangle=\exp\left(-\frac{1}{2}|\tau|^2+\tau a^{\dagger}-\tau^{*}\tilde{a}^{\dagger}+a^{\dagger}\tilde{a}^{\dagger}\right)|0,\tilde{0}\rangle, \tag{4}$$

where $\tilde{a}^{\dagger}$ is a fictitious mode accompanying the real photon creation operator $a^{\dagger}$, $|\tilde{0}\rangle$ is annihilated by $\tilde{a}$, $\left[\tilde{a},\tilde{a}^{\dagger}\right]=1$, $|n,\tilde{n}\rangle=\left(a^{\dagger}\tilde{a}^{\dagger}\right)^n|0,\tilde{0}\rangle/n!$, $\left[a,\tilde{a}^{\dagger}\right]=0$. The $|\tau\rangle$ is complete in the sense $\int\mathrm{d}^2\tau|\tau\rangle\langle\tau|/\pi=1$. Operating to ρ on $|\tau=0\rangle$, we make up the pure state $|\rho\rangle$

$$|\rho\rangle=\rho|\tau=0\rangle, \ |\tau=0\rangle=\exp\left(a^{\dagger}\tilde{a}^{\dagger}\right)|0,\tilde{0}\rangle=\sum_{n=0}^{\infty}|n,\tilde{n}\rangle. \tag{5}$$

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11175113 and 11247009).
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From Eq. (5), it is easily shown that

$$a|\tau=0\rangle=\tilde{a}^{\dagger}|\tau=0\rangle,\quad a^{\dagger}|\tau=0\rangle=\tilde{a}|\tau=0\rangle,$$
$$\left(a^{\dagger}a\right)^{n}|\tau=0\rangle=\left(\tilde{a}^{\dagger}\tilde{a}\right)^{n}|\tau=0\rangle. \tag{6}$$

Acting both sides of Eq. (3) on state $|\tau=0\rangle$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho\rangle=\kappa\left(2a\rho a^{\dagger}-a^{\dagger}a\rho-\rho a^{\dagger}a\right)|\tau=0\rangle. \tag{7}$$

Using Eq. (6) and noticing that ρ is defined in the real space and is commutative with the fictitious operators, we can convert Eq. (7) into

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho\rangle=\kappa\left(2a\tilde{a}-a^{\dagger}a-\tilde{a}^{\dagger}\tilde{a}\right)|\rho\rangle. \tag{8}$$

Its formal solution is

$$|\rho\rangle=\exp\{\kappa t\left(2a\tilde{a}-a^{\dagger}a-\tilde{a}^{\dagger}\tilde{a}\right)\}|\rho_0\rangle, \tag{9}$$

where

$$|\rho_0\rangle\equiv\rho_0|\tau=0\rangle. \tag{10}$$

Thus by introducing a fictitious mode $\tilde{a}^{\dagger}$ to be a counterpart of the system mode, we can convert density operators' master equations into evolution equations of pure states, which leads to the formal solution (9).

Since

$$\left[\frac{a^{\dagger}a+\tilde{a}^{\dagger}\tilde{a}}{2},a\tilde{a}\right]=-a\tilde{a}, \tag{11}$$

we can use the operator identity

$$\mathrm{e}^{\lambda(A+\sigma B)}=\mathrm{e}^{\lambda A}\exp\left[\sigma\left(1-\mathrm{e}^{-\lambda\tau}\right)B/\tau\right], \tag{12}$$

which is valid for $[A,B]=\tau B$, to decompose

$$\exp\left[-2\kappa t\left(\frac{a^{\dagger}a+\tilde{a}^{\dagger}\tilde{a}}{2}-a\tilde{a}\right)\right]$$
$$=\exp\left[-\kappa t\left(a^{\dagger}a+\tilde{a}^{\dagger}\tilde{a}\right)\right]\exp\left[Ta\tilde{a}\right], \tag{13}$$

where $T=1-\mathrm{e}^{-2\kappa t}$. Substituting Eq. (13) into Eq. (9) and using Eq. (10) yield

$$\begin{aligned}|\rho(t)\rangle&=\rho(t)|\tau=0\rangle\\&=\exp\left[-\kappa t\left(a^{\dagger}a+\tilde{a}^{\dagger}\tilde{a}\right)\right]\exp\left[Ta\tilde{a}\right]|\rho_0\rangle\\&=\exp\left[-\kappa t\left(a^{\dagger}a+\tilde{a}^{\dagger}\tilde{a}\right)\right]\sum_{n=0}^{\infty}\frac{T^n}{n!}a^n\tilde{a}^n\rho_0|\tau=0\rangle.\end{aligned} \tag{14}$$

Due to

$$\tilde{a}^n\rho_0|\tau=0\rangle=\rho_0\tilde{a}^n|\tau=0\rangle=\rho_0a^{\dagger n}|\tau=0\rangle, \tag{15}$$

$$\begin{aligned}&\exp\left[-\kappa t\tilde{a}^{\dagger}\tilde{a}\right]a^n\rho_0a^{\dagger n}|\tau=0\rangle\\&=a^n\rho_0a^{\dagger n}\exp\left[-\kappa t\tilde{a}^{\dagger}\tilde{a}\right]|\tau=0\rangle\\&=a^n\rho_0a^{\dagger n}\exp\left[-\kappa ta^{\dagger}a\right]|\tau=0\rangle,\end{aligned} \tag{16}$$

we have

$$\begin{aligned}&\rho(t)|\tau=0\rangle\\&=\exp\left[-\kappa ta^{\dagger}a\right]\sum_{n=0}^{\infty}\frac{T^n}{n!}a^n\rho_0a^{\dagger n}\exp\left[-\kappa t\tilde{a}^{\dagger}\tilde{a}\right]|\tau=0\rangle\\&=\sum_{n=0}^{\infty}\frac{T^n}{n!}\exp\left[-\kappa ta^{\dagger}a\right]a^n\rho_0a^{\dagger n}\exp\left[-\kappa ta^{\dagger}a\right]|\tau=0\rangle,\end{aligned} \tag{17}$$

so

$$\rho(t)=\sum_{n=0}^{\infty}\frac{T^n}{n!}\exp\left[-\kappa ta^{\dagger}a\right]a^n\rho_0a^{\dagger n}\exp\left[-\kappa ta^{\dagger}a\right]. \tag{18}$$

Comparing with the standard form of the operator-sum representation[17]

$$\rho(t)=\sum_{m=0}^{\infty}M_m\rho_0M_m^{\dagger}, \tag{19}$$

we can see that the Kraus operator M_m for this damping process is

$$M_m=\sqrt{\frac{T^m}{m!}}\,\mathrm{e}^{-\kappa ta^{\dagger}a}a^m, \tag{20}$$

and we can prove the unitarity

$$\sum_{m=0}^{\infty}M_m^{\dagger}M_m=1. \tag{21}$$

3. Physical meaning of γ in the negative binomial state

By observing that, directly substituting $\rho_{0s}=\sum_{n=0}^{\infty}\binom{n+s}{n}\gamma^{s+1}(1-\gamma)^n|n\rangle\langle n|$ into Eq. (18) is not convenient for further calculation, we use $a|n\rangle=\sqrt{n}|n-1\rangle$ and $a^s|n\rangle=\sqrt{\frac{n!}{(n-s)!}}|n-s\rangle$ to convert ρ_{0s} to

$$\begin{aligned}\rho_{0s}&=\frac{\gamma^{s+1}}{(1-\gamma)^s s!}a^s\sum_{n=0}^{\infty}(1-\gamma)^n|n\rangle\langle n|a^{\dagger s}\\&=\frac{1}{s!\left(n_{\mathrm{c}}\right)^s}a^s\rho_{\mathrm{c}}a^{\dagger s},\end{aligned} \tag{22}$$

where

$$\rho_{\mathrm{c}}=\sum_{n=0}^{\infty}\gamma(1-\gamma)^n|n\rangle\langle n|=\gamma(1-\gamma)^{a^{\dagger}a}=\gamma\mathrm{e}^{a^{\dagger}a\ln(1-\gamma)} \tag{23}$$

represents a thermo field light (chaotic light), and

$$n_{\mathrm{c}}=\frac{1}{\gamma}-1=\frac{1-\gamma}{\gamma}. \tag{24}$$

We can prove that n_{c} is just the average photon number of the chaotic light field, indeed, using the photon number operator $N=a^{\dagger}a$, we have

$$\begin{aligned}\mathrm{Tr}\left(\rho_{\mathrm{c}}N\right)&=\sum_{n=0}^{\infty}\gamma(1-\gamma)^n\langle n|a^{\dagger}a|n\rangle\\&=\gamma\sum_{n=0}^{\infty}n(1-\gamma)^n=\frac{1}{\gamma}-1=n_{\mathrm{c}},\end{aligned} \tag{25}$$

thus $\gamma = 1/(n_c+1)$. Similarly,

$$\begin{aligned}\mathrm{Tr}(\rho_{0s}N) &= \sum_{n=0}^{\infty} n\gamma^{s+1}\binom{n+s}{n}(1-\gamma)^n \\ &= \sum_{n=0}^{\infty}\gamma^{s+1}\frac{(n+s)!}{(n-1)!(s)!}(1-\gamma)^n \\ &= \sum_{n'=0}^{\infty}\gamma^{s+1}\frac{(n'+s+1)!}{(n')!(s)!}(1-\gamma)^{n'+1} \\ &= \frac{1-\gamma}{\gamma}\sum_{n'=0}^{\infty}\gamma^{s+2}\binom{n'+s+1}{n'}(1-\gamma)^{n'} \\ &= \frac{1-\gamma}{\gamma} = n_c, \end{aligned} \tag{26}$$

so the average photon numbers for ρ_{0s} and ρ_c are the same.

4. Damping formula of the negative-binomial state in a dissipative channel

Now from Eq. (23) and the antinormally ordered expansion of the operator

$$e^{\lambda a^\dagger a} = e^{-\lambda}{:}\exp\left[\left(1-e^{-\lambda}\right)aa^\dagger\right]{:}, \tag{27}$$

or

$$\frac{1}{1-\lambda}e^{a^\dagger a\ln\frac{1}{1-\lambda}} = {:}e^{\lambda aa^\dagger}{:},$$

where $\vdots\,\vdots$ denotes the antinormal ordering,[18,19] we reform

$$\rho_c = \gamma e^{a^\dagger a\ln(1-\gamma)}, \tag{28}$$

or

$$\rho_c = \frac{\gamma}{1-\gamma}{:}e^{\frac{\gamma}{\gamma-1}aa^\dagger}{:}. \tag{29}$$

Substituting $\rho_{0s} = \frac{1}{s!(n_c)^s}a^s\rho_c a^{\dagger s}$ into Eq. (18) and using Eq. (27) as well as

$$e^{fa^\dagger a}ae^{-fa^\dagger a} = ae^{-f}, \tag{30}$$

we obtain

$$\begin{aligned} s!(n_c)^s\rho(t) &= \sum_{m=0}^{\infty}\frac{T^m}{m!}e^{-\kappa t a^\dagger a}a^m a^s\rho_c a^{\dagger s}a^{\dagger m}e^{-\kappa t a^\dagger a} \\ &= \sum_{m=0}^{\infty}\frac{T^m}{m!}e^{2\kappa t(m+s)}a^{m+s}e^{-\kappa t a^\dagger a}\rho_c e^{-\kappa t a^\dagger a}a^{\dagger m+s} \\ &= \gamma\sum_{m=0}^{\infty}\frac{T^m}{m!}e^{2\kappa t(m+s)}a^{m+s}e^{(\lambda-2\kappa t)a^\dagger a}a^{\dagger m+s}, \\ \lambda &= \ln(1-\gamma). \end{aligned} \tag{31}$$

Here we encounter an obstacle in doing the summation over m, the difficulty lies in that there is a term $e^{(\lambda-2\kappa t)a^\dagger a}$ standing in between a^{m+s} and $a^{\dagger m+s}$. Therefore, we use Eq. (27) to express Eq. (31) as

$$\begin{aligned} s!(n_c)^s\rho(t) &= \gamma e^{-(\lambda-2\kappa t)}\sum_{m=0}^{\infty}\frac{T^m}{m!}e^{2\kappa t(m+s)}{:}a^{m+s} \\ &\quad\times\exp\left[\left(1-e^{2\kappa t-\lambda}\right)aa^\dagger\right]a^{\dagger m+s}{:}. \end{aligned} \tag{32}$$

Since a and $a^\dagger$ are commutable within the antinormal ordering symbol $\vdots\,\vdots$, now we can do the summation over m

$$\begin{aligned} & s!(n_c)^s\rho(t) \\ &= \gamma e^{-(\lambda-2\kappa t)}e^{2\kappa t s}a^s{:}\exp\{\left[Te^{2\kappa t}+\left(1-e^{2\kappa t-\lambda}\right)\right]aa^\dagger\}{:}a^{\dagger s} \\ &= \gamma e^{-(\lambda-2\kappa t)}e^{2\kappa t s}a^s{:}\exp\{e^{2\kappa t}\left(1-e^{-\lambda}\right)aa^\dagger\}{:}a^{\dagger s} \\ &= \frac{\gamma e^{2\kappa t}}{1-\gamma}e^{2\kappa t s}a^s{:}\exp\{\frac{\gamma e^{2\kappa t}}{\gamma-1}aa^\dagger\}{:}a^{\dagger s}, \end{aligned} \tag{33}$$

in which

$$\begin{aligned} & \frac{\gamma e^{2\kappa t}}{1-\gamma}{:}\exp\left\{\frac{\gamma e^{2\kappa t}}{\gamma-1}aa^\dagger\right\}{:} \\ &= \frac{\gamma e^{2\kappa t}}{1-\gamma(1-e^{2\kappa t})}\exp\left\{aa^\dagger\ln\frac{\gamma-1}{\gamma(1-e^{2\kappa t})-1}\right\} \\ &= \frac{\gamma e^{2\kappa t}}{1-\gamma(1-e^{2\kappa t})}\exp\left\{aa^\dagger\ln\left[1-\frac{\gamma e^{2\kappa t}}{1-\gamma(1-e^{2\kappa t})}\right]\right\} \\ &= \gamma' e^{a^\dagger a\ln(1-\gamma')}, \end{aligned} \tag{34}$$

where

$$\gamma' = \frac{\gamma e^{2\kappa t}}{1-\gamma(1-e^{2\kappa t})} = \frac{1}{e^{-2\kappa t}(1-\gamma)+\gamma}\gamma < \gamma, \tag{35}$$

so equation (33) is simplified as

$$\begin{aligned} \rho(t) &= \frac{1}{s!(n_c e^{-2\kappa t})^s}a^s\gamma' e^{a^\dagger a\ln(1-\gamma')}a^{\dagger s} \\ &= \frac{1}{s!(n_c e^{-2\kappa t})^s}a^s\rho_c' a^{\dagger s} \\ &= \frac{1}{s!(n_c e^{-2\kappa t})^s}\sum_{n=0}^{\infty}\gamma'(1-\gamma')^n a^s|n\rangle\langle n|a^{\dagger s} \\ &= \frac{1}{s!(n_c e^{-2\kappa t})^s}\sum_{n=0}^{\infty}\gamma'(1-\gamma')^n\frac{n!}{(n-s)!}|n-s\rangle\langle n-s| \\ &= \frac{(1-\gamma')^s}{\gamma'^s}\frac{1}{\left(\frac{1-\gamma}{\gamma}e^{-2\kappa t}\right)^s} \\ &\quad\times\sum_{n'=0}^{\infty}\gamma'^{s+1}(1-\gamma')^{n'}\binom{n'+s}{n'}|n'\rangle\langle n'| \\ &= \sum_{n=0}^{\infty}\gamma'^{s+1}(1-\gamma')^n\binom{n+s}{n}|n\rangle\langle n|, \end{aligned} \tag{36}$$

in the last step we have used the relation

$$\frac{1-\gamma'}{1-\gamma}\frac{\gamma}{\gamma'} = e^{-2\kappa t}. \tag{37}$$

5. Conclusion

We reach the conclusion that after passing through the channel, the final state is still a negative binomial state, however the parameter γ evolves into γ', $\gamma' < 1$.

Therefore we see $\mathrm{Tr}\rho(t) = 1$, and the average photon numbers for $\rho(t)$ is

$$\mathrm{Tr}\left[\rho(t)a^{\dagger}a\right] = \frac{1-\gamma'}{\gamma'} = \frac{1-\gamma}{\gamma}\mathrm{e}^{-2\kappa t} = n_{\mathrm{c}}\mathrm{e}^{-2\kappa t}, \quad (38)$$

so the average photon number n_{c} of a negative bimomial state decays exponentially in the damping process.

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