Master equation describing the diffusion process for a coherent state*

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The evolution of a pure coherent state into a chaotic state is described very well by a master equation, as is validated via an examination of the coherent state's evolution during the diffusion process, fully utilizing the technique of integration within an ordered product (IWOP) of operators. The same equation also describes a limitation that maintains the coherence in a weak diffusion process, i.e., when the dissipation is very weak and the initial average photon number is large. This equation is $\mathrm{d}\rho/\mathrm{d}t = -\kappa[a^+a\rho - a^+\rho a - a\rho a^+ + \rho aa^+]$. The physical difference between this diffusion equation and the better-known amplitude damping master equation is pointed out.

Keywords: master equation describing diffusion, coherent state evolution, chaotic state, integration within an ordered product (IWOP) of operators

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1. Introduction

In nature, most systems are immersed in a thermal environment, and the evolution of a system is described by a master equation. Master equations are set up for better understanding how quantum decoherence is processed while keeping the unitary character in the dissipation of systems. Usually quantum amplitude damping accompanies the decoherence. The simplest master equation describing an amplitude dissipation is[1–3]

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = \kappa\left[2a\rho a^+ - a^+a\rho - \rho a^+a\right], \tag{1}$$

where κ is the dissipation constant. By using the thermal entangled state representation,[4] we show here that the continuous operator sum representation of the solution for Eq. (1) is[5]

$$\rho(t) = \sum_{n=0}^{\infty}\frac{T^n}{n!}\mathrm{e}^{-\kappa t a^+a}a^n\rho_0 a^{+n}\mathrm{e}^{-\kappa t a^+a}, \tag{2}$$

where ρ_0 is the initial density operator. Note that this evolution is trace-preserving. When ρ_0 is a pure coherent state, $\rho_0 = |z\rangle\langle z|$, $|z\rangle = \exp[-|z|^2 + za^+]|0\rangle$, so

$$\begin{aligned}\rho(t) &= \sum_{n=0}^{\infty}\frac{T^n}{n!}\mathrm{e}^{-\kappa t a^+a}a^n|z\rangle\langle z|a^{+n}\mathrm{e}^{-\kappa t a^+a}\\ &= \sum_{n=0}^{\infty}\frac{T^n|z|^{2n}}{n!}\mathrm{e}^{-\kappa t a^+a}|z\rangle\langle z|\mathrm{e}^{-\kappa t a^+a}\\ &= |z\mathrm{e}^{-\kappa t}\rangle\langle z\mathrm{e}^{-\kappa t}|.\end{aligned} \tag{3}$$

This indicates that the amplitude exponentially damps as $|z\rangle \to |z\mathrm{e}^{-\kappa t}\rangle$, however, the pure coherent state remains coherent.

An interesting question thus arises: which quantum master equation can govern a pure coherent state evolving into a mixed state (a chaotic field) and maintain the coherence when the dissipation constant is very small and the initial photon number is large? In other words, we seek a master equation that can describe a diffusion process with decoherence for a coherent state and show that that state remains approximately coherent in the case with a small-κ limitation.

In Section 2, we shall show that the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -\kappa\left[a^+a\rho - a^+\rho a - a\rho a^+ + \rho aa^+\right] \tag{4}$$

accounts for such a diffusion process, and we will employ the thermo entangled state representation to solve this equation. In Section 3, we demonstrate how a pure coherent state evolves under the influence of Eq. (4), and what happens under the limitation of a small κ.

2. The solution to Eq. (4)

By introducing the thermo entangled state representation[4]

$$|\eta\rangle = \exp\left(-\frac{|\eta|^2}{2} + \eta a^+ - \eta^*\tilde{a} + a^+\tilde{a}^+\right)|0\tilde{0}\rangle, \tag{5}$$

where $a|0\tilde{0}\rangle = 0$, $\tilde{a}|0\tilde{0}\rangle = 0$, $[\tilde{a},\tilde{a}^+] = 1$, we can see that $|\eta\rangle$ satisfies the eigenvalue equations

$$\left(a - \tilde{a}^+\right)|\eta\rangle = \eta|\eta\rangle, \quad \left(a^+ - \tilde{a}\right)|\eta\rangle = \eta^*|\eta\rangle, \tag{6}$$

$$\langle\eta|\left(a^+ - \tilde{a}\right) = \eta^*\langle\eta|, \quad \langle\eta|\left(a - \tilde{a}^+\right) = \eta\langle\eta|. \tag{7}$$

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Using the technique of integration within an ordered product (IWOP) of operators, we can show the completeness relation[6–9]

$$\begin{aligned}&\int \frac{\mathrm{d}^2\eta}{\pi}\,|\eta\rangle\langle\eta| \\ &= \int \frac{\mathrm{d}^2\eta}{\pi}:\exp[-|\eta|^2+\eta\left(a^+-\tilde{a}\right)+\eta^*\left(a-\tilde{a}^+\right) \\ &\quad +a^+\tilde{a}^++\tilde{a}a-a^+a-\tilde{a}^+\tilde{a}]:=1, \end{aligned} \tag{8}$$

where : : denotes the normal ordering. Since

$$|\eta=0\rangle=\exp\left(a^+\tilde{a}^+\right)|0\tilde{0}\rangle\equiv|I\rangle, \tag{9}$$

$$\begin{aligned}a|\eta=0\rangle&=\left[a,\mathrm{e}^{a^+\tilde{a}^+}\right]|0\tilde{0}\rangle=\tilde{a}^+\exp\left(a^+\tilde{a}^+\right)|0\tilde{0}\rangle \\ &=\tilde{a}^+|I\rangle, \end{aligned} \tag{10}$$

$$\tilde{a}|\eta=0\rangle=a^+|\eta=0\rangle=a^+|I\rangle, \tag{11}$$

it then follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\rho\rangle &= -\kappa\left[a^+a\rho-a^+\rho a-a\rho a^++\rho aa^+\right]|I\rangle \\ &= -\kappa\left(a^+-\tilde{a}\right)\left(a-\tilde{a}^+\right)|\rho\rangle. \end{aligned} \tag{12}$$

Here we have introduced $|\rho\rangle=\rho|I\rangle$. The formal solution to Eq. (12) is

$$|\rho\rangle=\exp[-\kappa t(a^+-\tilde{a})(a-\tilde{a}^+)]|\rho_0\rangle, \tag{13}$$

where $|\rho_0\rangle=\rho_0|I\rangle$, and ρ_0 is the initial density operator of the system, so it commutes with $\tilde{a}$ and $\tilde{a}^+$. By projecting state (13) onto the thermo entangled state representation and using Eqs. (6) and (7), we have

$$\begin{aligned}\langle\eta|\rho\rangle &= \langle\eta|\exp[-\kappa t(a^+-\tilde{a})(a-\tilde{a}^+)]|\rho_0\rangle \\ &= \mathrm{e}^{-\kappa t|\eta|^2}\langle\eta|\rho_0\rangle. \end{aligned} \tag{14}$$

Then we employ the completeness relation (8) and the technique of integration, and find

$$\begin{aligned}|\rho\rangle &= \int \frac{\mathrm{d}^2\eta}{\pi}\,\mathrm{e}^{-\kappa t|\eta|^2}|\eta\rangle\langle\eta|\rho_0\rangle \\ &= \int \frac{\mathrm{d}^2\eta}{\pi}:\exp[-(1+\kappa t)|\eta|^2+\eta\left(a^+-\tilde{a}\right) \\ &\quad +\eta^*\left(a-\tilde{a}^+\right)+a^+\tilde{a}^++\tilde{a}a-a^+a-\tilde{a}^+\tilde{a}]:|\rho_0\rangle \\ &= \frac{1}{1+\kappa t}:\exp\left[\frac{\kappa t}{1+\kappa t}\left(a^+\tilde{a}^++\tilde{a}a-\tilde{a}^+\tilde{a}-a^+a\right)\right]:|\rho_0\rangle \\ &= \frac{1}{1+\kappa t}\,\mathrm{e}^{\frac{\kappa t}{1+\kappa t}a^+\tilde{a}^+}:\exp\left[\frac{\kappa t}{1+\kappa t}\left(-\tilde{a}^+\tilde{a}-a^+a\right)\right]: \\ &\quad \times\,\mathrm{e}^{\frac{\kappa t}{1+\kappa t}a\tilde{a}}|\rho_0\rangle. \end{aligned} \tag{15}$$

Using

$$\mathrm{e}^{\lambda\left(a^+a+\tilde{a}^+\tilde{a}\right)}=:\mathrm{e}^{\left(\mathrm{e}^{\lambda}-1\right)\left(a^+a+\tilde{a}^+\tilde{a}\right)}:, \tag{16}$$

we can reform Eq. (15) as

$$|\rho\rangle=\frac{1}{1+\kappa t}\,\mathrm{e}^{\frac{\kappa t}{1+\kappa t}a^+\tilde{a}^+}\left(\frac{1}{1+\kappa t}\right)^{\tilde{a}^+\tilde{a}+a^+a}\mathrm{e}^{\frac{\kappa t}{1+\kappa t}a\tilde{a}}|\rho_0\rangle. \tag{17}$$

Further, by noticing that ρ_0 commutes with $\tilde{a}$ and $\tilde{a}^+$ and using Eqs. (11) and (12), we derive

$$\begin{aligned}\mathrm{e}^{\frac{\kappa t}{1+\kappa t}a\tilde{a}}|\rho_0\rangle &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\kappa t}{1+\kappa t}a\right)^n\tilde{a}^n\rho_0|I\rangle \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\kappa t}{1+\kappa t}a\right)^n\rho_0a^{+n}|I\rangle. \end{aligned} \tag{18}$$

Substituting Eq. (18) into Eq. (17) and using Eqs. (10) and (11) yields

$$\begin{aligned}|\rho\rangle &= \mathrm{e}^{\frac{\kappa t}{1+\kappa t}a^+\tilde{a}^+}\left(\frac{1}{1+\kappa t}\right)^{a^+a+1}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\kappa t}{1+\kappa t}a\right)^n \\ &\quad \times\rho_0a^{+n}\left(\frac{1}{1+\kappa t}\right)^{a^+a}|I\rangle \\ &= \sum_{m,n=0}^{\infty}\frac{1}{m!n!}\frac{(\kappa t)^{m+n}}{(1+\kappa t)^{m+n+1}}a^{+m}\left(\frac{1}{1+\kappa t}\right)^{a^+a}a^n \\ &\quad \times\rho_0a^{+n}\left(\frac{1}{1+\kappa t}\right)^{a^+a}a^m|I\rangle. \end{aligned} \tag{19}$$

Therefore

$$\begin{aligned}\rho(t) &= \sum_{m,n=0}^{\infty}\frac{1}{m!n!}\frac{(\kappa t)^{m+n}}{(1+\kappa t)^{m+n+1}}a^{+m}\left(\frac{1}{1+\kappa t}\right)^{a^+a}a^n \\ &\quad \times\rho_0a^{+n}\left(\frac{1}{1+\kappa t}\right)^{a^+a}a^m \\ &= \sum_{m,n=0}^{\infty}M_{m,n}\rho_0M_{m,n}^+, \end{aligned} \tag{20}$$

where $M_{m,n}$ is known as the Kraus operator, which in infinitive sum representation is

$$M_{m,n}=\sqrt{\frac{1}{m!n!}\frac{(\kappa t)^{m+n}}{(1+\kappa t)^{m+n+1}}a^{+m}\left(\frac{1}{1+\kappa t}\right)^{a^+a}a^n}. \tag{21}$$

We can confirm that

$$\sum_{m,n=0}^{\infty}M_{m,n}^+M_{m,n}=1. \tag{22}$$

3. Diffusion for a coherent state

We now explore how an initial state $|z\rangle\langle z|$ evolves according to the master equation (4). Substituting the state into the summation part over n-number in Eq. (20) and using

$$\begin{aligned}&\mathrm{e}^{\lambda a^+a}a^+\,\mathrm{e}^{-\lambda a^+a}=a^+\,\mathrm{e}^{\lambda}, \\ &\mathrm{e}^{\lambda a^+a}|z\rangle=\mathrm{e}^{-|z|^2/2}||z\mathrm{e}^{\lambda}\rangle, \end{aligned} \tag{23}$$

where $||z\mathrm{e}^{\lambda}\rangle$ is an un-normalized coherent state, we have

$$\sum_{n=0}^{\infty}\frac{1}{n!}\frac{(\kappa t)^n}{(1+\kappa t)^n}\left(\frac{1}{1+\kappa t}\right)^{\frac{a^+a}{2}}a^n|z\rangle\langle z|a^{+n}\left(\frac{1}{1+\kappa t}\right)^{\frac{a^+a}{2}}$$

$$= \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!} \frac{(\kappa t)^n}{(1+\kappa t)^n} \mathrm{e}^{-\frac{1}{2}a^+ a \ln(1+\kappa t)} |z\rangle \ \langle z| \ \mathrm{e}^{-\frac{1}{2}a^+ a \ln(1+\kappa t)}$$
$$= \mathrm{e}^{\frac{\kappa t}{1+\kappa t}|z|^2 - |z|^2} \left| z\frac{1}{\sqrt{1+\kappa t}} \right\rangle \left\langle z\frac{1}{\sqrt{1+\kappa t}} \right|$$
$$= \left| z\frac{1}{\sqrt{1+\kappa t}} \right\rangle \left\langle z\frac{1}{\sqrt{1+\kappa t}} \right|, \tag{24}$$

where we note that

$$\mathrm{e}^{\frac{\kappa t}{1+\kappa t}|z|^2 - |z|^2} = \mathrm{e}^{-|z|^2 \frac{1}{1+\kappa t}} = \mathrm{e}^{-\left|\frac{|z|}{\sqrt{1+\kappa t}}\right|^2}. \tag{25}$$

Then, inserting Eq. (24) into Eq. (20) and using

$$|0\rangle\langle 0| =: \mathrm{e}^{-a^+ a} :, \tag{26}$$

we obtain

$$\begin{aligned}
\rho(t) &= \sum_{m=0}^{\infty} \frac{1}{m!} \frac{(\kappa t)^m}{(1+\kappa t)^{m+1}} a^{+m} \left(\frac{1}{\sqrt{1+\kappa t}}\right)^{a^+ a} \\
&\quad \times \left| z\frac{1}{\sqrt{1+\kappa t}} \right\rangle \left\langle z\frac{1}{\sqrt{1+\kappa t}} \right| \left(\frac{1}{\sqrt{1+\kappa t}}\right)^{a^+ a} a^m \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \frac{(\kappa t)^m}{(1+\kappa t)^{m+1}} a^{+m} \left| z\frac{1}{\sqrt{1+\kappa t}} \right\rangle \left\langle z\frac{1}{\sqrt{1+\kappa t}} \right| \\
&\quad \times \mathrm{e}^{-\frac{|z|^2}{1+\kappa t}} a^m \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \frac{(\kappa t)^m}{(1+\kappa t)^{m+1}} a^{+m} \\
&\quad \times :\exp\left(\frac{z}{1+\kappa t} a^+ + \frac{z^*}{1+\kappa t} a - a^+ a - \frac{|z|^2}{1+\kappa t} \right) : a^m \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \frac{(\kappa t)^m}{(1+\kappa t)^{m+1}} a^{+m} a^m \\
&\quad \times :\exp\left(\frac{z}{1+\kappa t} a^+ + \frac{z^*}{1+\kappa t} a - a^+ a - \frac{|z|^2}{1+\kappa t} \right) : \\
&= \frac{1}{1+\kappa t} :\exp\left(\frac{\kappa t}{1+\kappa t} a^+ a + \frac{z}{1+\kappa t} a^+ \right. \\
&\quad \left. + \frac{z^*}{1+\kappa t} a - a^+ a - \frac{|z|^2}{1+\kappa t} \right) : .
\end{aligned} \tag{27}$$

This is the density operator at time t. By using Eq. (16), it becomes

$$\begin{aligned}
\rho(t) &= \frac{1}{1+\kappa t} \exp\left(-\frac{|z|^2}{1+\kappa t} + \frac{z}{1+\kappa t} a^+ \right) \\
&\quad \times \exp\left(a^+ a \ln \frac{\kappa t}{1+\kappa t} \right) \exp\left(\frac{z^*}{1+\kappa t} a \right),
\end{aligned} \tag{28}$$

which represents a mixed state and is sharply in contrast to the solution (2) of Eq. (1). We can check whether the density operator in Eq. (27) obeys $\mathrm{Tr}\rho(t) = 1$. For this aim, we introduce the completeness relation of coherent state $\int (\mathrm{d}^2\alpha/\pi) |\alpha\rangle\langle\alpha| = 1$; then we have

$$\begin{aligned}
\mathrm{Tr}\rho(t) &= \int \frac{\mathrm{d}^2\alpha}{\pi} \langle\alpha|\rho|\alpha\rangle \\
&= \frac{1}{1+\kappa t} \int \frac{\mathrm{d}^2\alpha}{\pi} \exp\left(-\left(1 - \frac{\kappa t}{1+\kappa t}\right) |\alpha|^2 \right. \\
&\quad \left. + \frac{z}{1+\kappa t}\alpha^* + \frac{z^*}{1+\kappa t}\alpha - \frac{|z|^2}{1+\kappa t} \right) \\
&= 1,
\end{aligned} \tag{29}$$

which indicates that the density operator in Eq. (27) or (28) is correct and trace-preserving in time evolution.

4. Discussion and conclusion

From Eq. (27) we can see that when κ approaches zero, it reduces to

$$\rho(t) \to :\exp(za^+ + z^* a - a^+ a - |z|^2) := |z\rangle\langle z|, \tag{30}$$

and the density operator remains approximately coherent. This explains that the dissipation governed by Eq. (4) is a slow process and can be considered as diffusion, and this is quite different from the fast amplitude exponential damping $|z\rangle \to |z\mathrm{e}^{-\kappa t}\rangle$ of Eq. (2). On the other hand, when much time has passed, t becomes larger, however since κ is very small (the dissipation is very weak and this damping process means diffusion), κt is still small, $1+\kappa t \to 1$, then Eq. (27) is approximately equal to

$$\rho(t) \to \exp(-|z|^2 + za^+) \exp(a^+ a \ln \kappa t) \exp(z^* a), \tag{31}$$

which represents a chaotic state (a thermo field). Thus we can see that for a diffusion process, a pure coherent state evolves into a mixed state. Therefore we conclude that the solution to master equation (4) is quite different from the solution to master equation (1): while Eq. (4) describes diffusion, Eq. (1) describes amplitude damping.

In summary, we have shown that Eq. (4) has the following merits: according to its solution, a pure coherent state will evolve into a mixed state (a chaotic field), and it will remain approximately coherent when κ is very small, so Eq. (4) mainly describes a diffusion process.

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