Averaging in SU(2) open quantum random walk

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We study the average position and the symmetry of the distribution in the SU(2) open quantum random walk (OQRW). We show that the average position in the central limit theorem (CLT) is non-uniform compared with the average position in the non-CLT. The symmetry of distribution is shown to be even in the CLT.

Keywords: open quantum random walk, central limit theorem, average position, SU(2) operator

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1. Introduction

The quantum walk (QW) is regarded as the analogue of the classical random walk. The walk is intensively investigated in the literature due to its connection with quantum computing. Like the classical random walk, the quantum walk has two parts, the discrete-time quantum walk and the continuous-time quantum walk. The most notable difference between the classical random walk and the quantum random walk is the nature of the probability distribution. For the classical random walk, the probability distribution has the form of a binomial distribution, with a width that spreads like $\sqrt{t}$; the variance grows linearly with time increasing. The variance in the quantum walk, by contrast, grows quadratically with time increasing, and the distribution has a complicated, oscillatory form. Both of these are effects of interference between the possible paths of the walker. For the recent most comprehensive review on the quantum walk, the readers can refer to Ref. [1].

Recently, a new model of the quantum walk has appeared in the literature,[2,3] the open quantum random walk (OQRW). The walk arose in an effort to understand the role quantum transport plays in biological systems. In order to investigate the role quantum transport plays in a biological system, it is necessary to find a framework for OWs in an open environment. The OQRW is based on the non-unitary dynamics induced by the environment. The walk is dealt with by density matrices rather than pure states. For a review of the OQRW, the readers can refer to Refs. [1]–[7].

In this paper, we study the average position and the symmetry of distribution in the SU(2) OQRW. In the QW there are known results, see Refs. [8]–[10] for examples. However, in the OQRW the results are unknown. The SU(2) operator is the operator in Ref. [8], defined by

$$U^S_{\alpha,\beta,\gamma} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha}\cos\beta & -\mathrm{e}^{-\mathrm{i}\gamma}\sin\beta \\ \mathrm{e}^{\mathrm{i}\gamma}\sin\beta & \mathrm{e}^{-\mathrm{i}\alpha}\cos\beta \end{pmatrix}.$$

The operator is related to the U(2) operator in the following way: $U_{\alpha,\beta,\gamma,\theta} = \mathrm{e}^{\mathrm{i}\theta}U^S_{\alpha,\beta,\gamma}$. The SU(2) operator is related to the Hadamard gate in the following way: $H = \mathrm{e}^{-\mathrm{i}\pi/2}U^S_{\pi/2,\pi/4,\pi/2}$.

The rest of this paper is organized as follows. In Section 2 we define the SU(2) OQRW. In Section 3 we examine the average position of the SU(2) OQRW. In particular, we obtain the limit theorems of the SU(2) OQRW and show in the central limit theorem (CLT) the average position is non-uniform compared with the average position in the non-CLT. In Section 4 we study the symmetry of the distribution of the SU(2) OQRW and show that in the CLT the distribution is even. Section 5 is devoted to the conclusions.

2. Definition of the SU(2) OQRW

In this section, we treat the OQRW in one-dimensional (1D) space Z. Recall that the Hilbert space is given by $H = H_C \otimes H_S$, where $H_C = \mathrm{span}\{|L\rangle, |R\rangle\}$, and $H_S = \mathrm{span}\{|x\rangle : x \in Z\}$. Let B, C be two matrices in H such that $B^*B + C^*C = I$. Define a complete positive map of the density of matrices of H_C by $L(\rho) = B\rho B^* + C\rho C^*$, and consider density matrices in the Hilbert space H of the type $\rho = \sum\limits_{x\in Z}\rho_x \otimes |x\rangle\langle x|$, where each ρ_x is a positive matrix and satisfies $\sum\limits_{x\in Z}\mathrm{Tr}(\rho_x) = 1$. Let $\tilde{L}$ be an operator of the density matrices of H of the type immediately above, defined by $\tilde{L}(\rho) = \sum\limits_{x\in Z}(B\rho_{x+1}B^* + C\rho_{x-1}C^*) \otimes |x\rangle\langle x|$. The evolution of the OQRW is determined by the iteration of the map $\tilde{L}$. We may suppose that the initial state of the OQRW is given by the following density matrix in the Hilbert space $H : \rho^{(0)} = \rho_0 \otimes |0\rangle\langle 0|$, then the evolution of the OQRW in 1D space Z generated by B and C is given by

$$\rho^{(n)} = \sum_x \rho_x^{(n)} \otimes |x\rangle\langle x|,$$

$$\rho_x^{(n+1)} = B\rho_{x+1}^{(n)}B^* + C\rho_{x-1}^{(n)}C^*, \quad x \in Z, \quad n \in Z_+.$$

The probability to find the walker at site $x \in Z$ at time $n \in Z_+$ is given by $p_x^{(n)} = \mathrm{Tr}(\rho_x^{(n)})$.

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If we let $\xi(H_C)$ be the space of density matrices in H_C, such that the distribution of space component of the process coincides with that of the OQRW, then as a Markov process in $\xi(H_C)\times Z$ the OQRW can be considered as a Markov chain $(\rho_n, X_n)_{n\in N}$ with values in $\xi(H_C)\times Z$ according to the transition rule: from any position (ρ, X) it jumps to one of two states: $(B\rho B^*/p_B, X-1)$ with probability $p_B := \mathrm{Tr}(B\rho B^*)$ or to $(C\rho C^*/p_C, X+1)$ with probability $p_C := \mathrm{Tr}(C\rho C^*)$.

The SU(2) OQRW is defined by taking

$$B \equiv P^S_{\alpha,\beta,\gamma} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha}\cos\beta & -\mathrm{e}^{-\mathrm{i}\gamma}\sin\beta \\ 0 & 0 \end{pmatrix},$$

$$C \equiv Q^S_{\alpha,\beta,\gamma} = \begin{pmatrix} 0 & 0 \\ \mathrm{e}^{\mathrm{i}\gamma}\sin\beta & \mathrm{e}^{-\mathrm{i}\alpha}\cos\beta \end{pmatrix}.$$

Note that $P^S_{\alpha,\beta,\gamma} + Q^S_{\alpha,\beta,\gamma} = U^S_{\alpha,\beta,\gamma}$.[7] In particular we can check that $\left(P^S_{\alpha,\beta,\gamma}\right)^* P^S_{\alpha,\beta,\gamma} + \left(Q^S_{\alpha,\beta,\gamma}\right)^* Q^S_{\alpha,\beta,\gamma} = I_2$. From now on, we study the OQRW generated by $\left\{P^S_{\alpha,\beta,\gamma},\, Q^S_{\alpha,\beta,\gamma}\right\}$.

3. Average position of the SU(2) OQRW

We first examine the nature of the distribution of the SU(2) OQRW. From Theorem 2.3 in Ref. [5], we obtain the following Corollary.

Corollary 1

$$p_x^{(n)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}kx}\mathrm{Tr}(\rho_0 Y_n(k))\,\mathrm{d}k,$$

where

$$Y_n(k) = \left(\mathrm{e}^{\mathrm{i}k} L_{(P^S_{\alpha,\beta,\gamma})^*} R_{P^S_{\alpha,\beta,\gamma}} + \mathrm{e}^{-\mathrm{i}k} L_{(Q^S_{\alpha,\beta,\gamma})^*} R_{Q^S_{\alpha,\beta,\gamma}}\right)^{(n)},$$

$\rho^{(0)} = \rho_0 \otimes |0\rangle\langle 0|$ is the initial state, $L_{J^*}R_J(A) = J^*AJ$ for any matrix A in the set consisting of the algebra of 2×2 matrices, and $J \in \{P^S_{\alpha,\beta,\gamma},\, Q^S_{\alpha,\beta,\gamma}\}$.

Let A be the one-matrix, fix $\rho_0 = I_2/2$, and $\alpha = \gamma = 0$. We show the distribution for $x \in (-100, 100)$ and $\beta \in [0, 2\pi]$ after one step $(n = 1)$ in Fig. 1.

Next, we examine the probability distribution of the SU(2) OQRW in the central limit theorem (CLT). First, we compute $L(\rho) = P^S_{\alpha,\beta,\gamma}\rho(P^S_{\alpha,\beta,\gamma})^* + Q^S_{\alpha,\beta,\gamma}\rho(Q^S_{\alpha,\beta,\gamma})^*$ that has a unique invariant state $\rho_\infty = I_2/2$. Therefore, from Theorem 2.1 in Ref. [5] we obtain the CLT as follows.

Corollary 2 The distribution of the SU(2) OQRW in the CLT is

$$N(0, 4\cos^2\beta\cos^2(\alpha+\gamma) + \cot^2\beta).$$

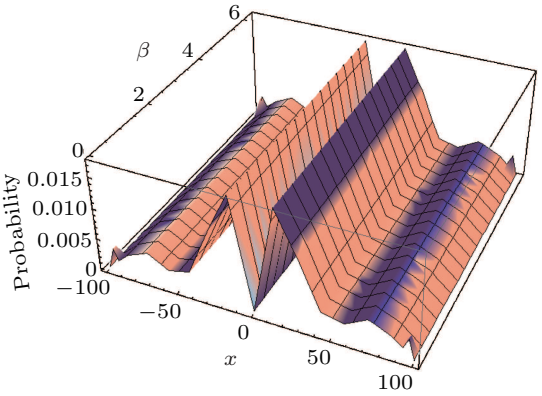


Fig. 1. (color online) Probability distribution of SU(2) OQRW.

Proof From Theorem 2.1 in Ref. [5], the mean is zero by definition. Alternatively, from the definition of $P^S_{\alpha,\beta,\gamma}$, $Q^S_{\alpha,\beta,\gamma}$, ρ_∞, one can check that

$$m = \mathrm{Tr}\left[Q^S_{\alpha,\beta,\gamma}\rho_\infty\left(Q^S_{\alpha,\beta,\gamma}\right)^*\right] - \mathrm{Tr}\left[P^S_{\alpha,\beta,\gamma}\rho_\infty\left(P^S_{\alpha,\beta,\gamma}\right)^*\right] = 0.$$

Let

$$R = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}.$$

Since $m = 0$, we can check that the solution to $R - L^*(R) = \left(Q^S_{\alpha,\beta,\gamma}\right)^* Q^S_{\alpha,\beta,\gamma} - \left(P^S_{\alpha,\beta,\gamma}\right)^* P^S_{\alpha,\beta,\gamma}$ is given by

$$\begin{aligned} y_2 &= y_3 = \sin(2\beta)\cos(\alpha+\gamma), \\ y_4 &= \frac{-1}{4}\{-1-2y_1+2(-2+y_1)\cos(2\beta)+\cos(4\beta) \\ &\quad + [-1+\cos(4\beta)]\cos[2(\alpha+\gamma)]\}\csc^2\beta. \end{aligned}$$

Let $y_1 = 0$, then from Theorem 2.1 in Ref. [5], we see that

$$\begin{aligned} \sigma^2 &= \mathrm{Tr}\left[P^S_{\alpha,\beta,\gamma}\rho_\infty\left(P^S_{\alpha,\beta,\gamma}\right)^* + Q^S_{\alpha,\beta,\gamma}\rho\left(Q^S_{\alpha,\beta,\gamma}\right)^*\right] \\ &\quad + 2\mathrm{Tr}\left\{\left[Q^S_{\alpha,\beta,\gamma}\rho\left(Q^S_{\alpha,\beta,\gamma}\right)^* - P^S_{\alpha,\beta,\gamma}\rho_\infty\left(P^S_{\alpha,\beta,\gamma}\right)^*\right]R\right\} \\ &= 4\cos^2\beta\cos^2(\alpha+\gamma) + \cot^2\beta, \end{aligned}$$

and the proof is finished.

Fixing $\alpha = \gamma = 0$, figure 2 shows the distribution in the CLT for $x \in [-100, 100]$ and $\beta \in [0, 2\pi]$.

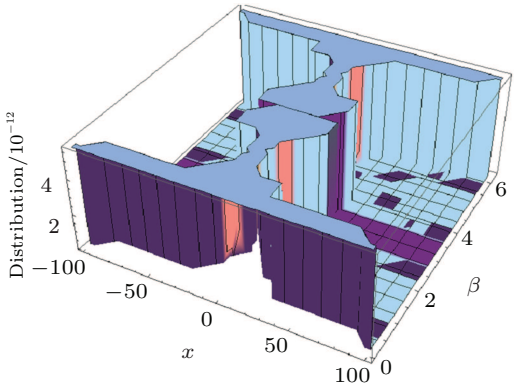


Fig. 2. (color online) SU(2) OQRW in the CLT.

Definition 1 (Average position of OQRW) Let $\langle x\rangle$ denote the average position in the OQRW, then $\langle x\rangle = \sum_x x p_x^{(n)}$, where $p_x^{(n)}$ is the probability distribution of the OQRW.

Write $S_k = \sum_{x=-k}^{k} x p_x^{(n)}$, $k \in Z_+$. From Corollary 2, for $\alpha = \gamma = 0$, and $\beta = \pi/4$, the distribution of S_k for increasing k is highly oscillatory as shown in Fig. 3.

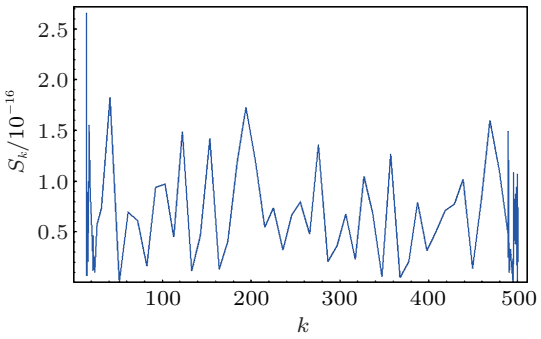


Fig. 3. (color online) Average position of SU(2) OQRW (CLT).

On the other hand, from Corollary 1, letting A be the one-matrix, and fixing $\rho_0 = I_2/2$, $\alpha = \gamma = 0$, and $\beta = \pi/4$, we see from Fig. 4 for increasing k, the distribution of S_k is uniform, even when we fix $n = 1$.

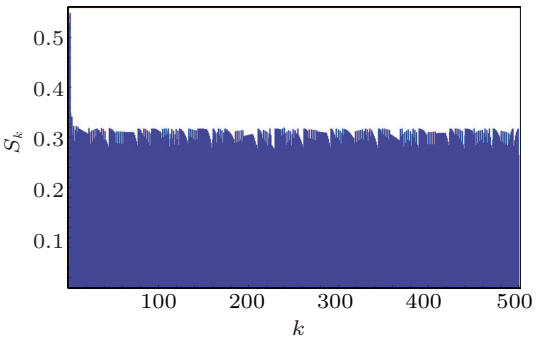


Fig. 4. (color online) Average position of SU(2) OQRW (non-CLT).

It appears that the average position is very stable in the non-CLT compared with the average position in the CLT. Clearly, the behavior of the average position should differ, since the probability distribution of the SU(2) OQRW in the CLT is not the same as that obtained from Corollary 1. We should remark that in the CLT the average position is independent of the step of walking, but only depends on the parameters α, β, and γ.

4. Symmetrization in the SU(2) OQRW

In this section, we investigate the SU(2) OQRW for symmetry relations. We obtain the following result in the CLT.

Theorem 1 The SU(2) OQRW in the CLT is even.

Proof Let $f(m,\sigma,x)$ denote the normal distribution with mean m, and standard deviation given by σ, then we will see that

$$f(m,\sigma,x) = \frac{\mathrm{e}^{-(x-m)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}.$$

In the CLT, the distribution of the SU(2) OQRW has $m = 0$, and $\sigma = [4\cos^2\beta\cos^2(\alpha+\gamma)+\cot^2\beta]^{1/2}$. It is easy to check that $f(0,\sigma,x) = f(0,\sigma,-x)$ for any of α, β, and γ, and the result follows.

5. Concluding remarks

In this paper, we study the average position in the SU(2) OQRW. In the CLT, the average position is highly oscillatory compared with the average position in the non-central limit theorem, which appears to be stable even after the first step of walking. We note that the average position in the CLT is independent of the step of walking, which is responsible for its non-uniform behavior, but only depends on the parameters α, β, and γ. We also study the symmetric nature of the SU(2) OQRW. In the CLT, we find out that the distribution is even.

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