A new explicit multisymplectic integrator for the Kawahara-type equation*

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(Received 13 November 2013; revised manuscript received 16 December 2013; published online 27 January 2014)

We derive a new multisymplectic integrator for the Kawahara-type equation which is a fully explicit scheme and thus needs less computation cost. Multisympecticity of such scheme guarantees the long-time numerical behaviors. Numerical experiments are presented to verify the accuracy of this scheme as well as the excellent performance on invariant preservation for three kinds of Kawahara-type equations.

Keywords: Kawahara-type equation, multisymplectic integrator, Euler-box scheme, adjoint scheme

PACS: 02.70.Bf, 03.65.Ge, 45.10.Na, 47.10.Df **DOI:** 10.1088/1674-1056/23/3/030204

1. Introduction

The Kawahara-type equation or the fifth-order KdV-type equation has the general form of

$$2u_t + \alpha u_{xxx} + \beta u_{xxxxx} = f_x(u, u_x, u_{xx}), \quad (x,t) \in [-L, L] \times (0, T], \qquad (1)$$

where α and β are real parameters with $\beta \neq 0$ and $f(u, u_x, u_{xx})$ is a smooth function. Such equation occurs in the theory of magneto-acoustic waves in plasmas[1] and in the theory of shallow water waves with surface tension.[2] Kawahara[1] appears to have been the first to write down the complete equation (1) with $f(u, u_x, u_{xx}) = -3u^2$ being a systematic study and observe that the solitary wave states could have oscillatory tails, and compute examples of such waves numerically. Much attention has been paid to the Kawahara-type equation in the last few years. Various numerical methods have been applied on this equation, including the sinc method and the decomposition series method,[3] the Dual–Petrov–Galerkin method,[4] the multisymplectic Preissmann method,[5] the semi-explicit finite difference scheme,[6] the meshless method,[7] and so on.

The multisymplectic numerical integrators were proposed by Marsden *et al.*[8] as well as Bridges and Reich,[9] which are based on the Lagrangian formalism and Hamiltonian formalism respectively. The corresponding schemes are called multisymplectic integrators which can both automatically preserve the multisymplectic conservation law and show excellent properties on invariant preserving. The multisymplectic integrators have been successfully applied to many important physical and mathematical models, such as the nonlinear wave equation,[10,11] the nonlinear Schrödinger equation,[12,13] the Zakharov system,[14] the Korteweg–de Vries equation,[15,16] the RLW equation,[17] Klein–Gordon–Schrödinger equations,[18,19] the Maxwell's equation,[20] the Ito-type KdV equation,[21] the Camassa–Holm equation,[22] and so on.[23–25]

The classic multisymplectic methods include the Euler-box scheme and the Preissmann scheme which are actually derived by applying the symplectic Runge–Kutta methods on both the spatial and temporal directions. Wu[5] applied the Preissmann scheme on the Kawahara-type equation. However, it is an implicit scheme and thus needs iterations to solve the resulting nonlinear equations. The Euler-box scheme is explicit and easier to implement than the Preissmann scheme. Thus, in the present work, we test the Euler-box scheme on the Kawahara-type equation and verify the accuracy as well as the invariant preserving property.

The rest of this paper is organized as follows. In Section 2, we present the multisymplectic formulation and the corresponding multisymplectic conservation law of the Kawahara-type equation. In Section 3, we apply the Euler-box scheme on the multisymplectic formulation and derive two numerical schemes. Numerical experiments are carried out to test the two schemes in Section 4. In the last section, we draw some concluding remarks.

2. Multi-symplectic formulation of the Kawahara-type equation

In this section, we briefly review the multisymplectic formulation of the Kawahara-type equation.[26] The general procedure of deriving the multisymplectic formulation includes three parts: the existence and formulation of the calculus of variations, covariant Legendre transformations, and the De-Donder Weyl equations.[27] In the first part for the Kawahara-type equation, we have to introduce a potential function $\psi_x =$

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11271195 and 11271196) and the Project of Graduate Education Innovation of Jiangsu Province, China (Grant No. CXZZ12-0385).

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u, and there exits a potential $\mathcal{L}$ subject to the variational derivative of $\mathcal{L}$ yields the Kawahara-type equation (1). Then, following the rest two parts, we can obtain the multisymplectic formulation as

$$\boldsymbol{M}\boldsymbol{z}_t+\boldsymbol{K}\boldsymbol{z}_x=\boldsymbol{\nabla}_z S(z),\ \ \boldsymbol{z}\in\mathbb{R}^6, \tag{2}$$

with

$$\boldsymbol{M}=\begin{pmatrix}0&-1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\quad \boldsymbol{K}=\begin{pmatrix}0&0&0&-1&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&-1\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\end{pmatrix},$$

and

$$\begin{aligned}S(z)\ =\ &\frac{1}{2}\alpha v^2+\frac{1}{2\beta}q^2+wu+pv+F(u,v)\\&+\frac{1}{2\beta}(2q+E(u,v))E(u,v).\end{aligned} \tag{3}$$

Here, $E(u,v)$ and $F(u,v)$ are related to the derivation of the potential of f, subject to

$$\begin{aligned}f(u,v,s)\ =\ &F_u(u,v)-vF_{uv}(u,v)-sF_{vv}(u,v)+2sE_u(u,v)\\&+svE_{uv}(u,v)+v^2E_{uu}(u,v),\end{aligned}$$

where u, v, and s are with respect to u, u_x, and u_{xx}.

The multisymplectic formulation (2) has the following conservation law:

$$\partial_t\omega+\partial_x\kappa=0,$$

where $\omega=\mathrm{d}\boldsymbol{z}\wedge\boldsymbol{M}\mathrm{d}\boldsymbol{z}$, $\kappa=\mathrm{d}\boldsymbol{z}\wedge\boldsymbol{K}\boldsymbol{z}$, which, for the Kawahara-type equation (1), is equivalent to

$$\partial_t(\mathrm{d}u\wedge\mathrm{d}\psi)+\partial_x(\mathrm{d}w\wedge\mathrm{d}\psi+\mathrm{d}p\wedge\mathrm{d}u+\mathrm{d}q\wedge\mathrm{d}v)=0.$$

3. Multisymplectic Euler-box scheme for the Kawahara-type equation

For convenience, we first introduce some notations: $x_i=-L+ih$, $i=0,1,\ldots,N$; $t_j=j\tau$, $j=1,2,\ldots$; $h=2L/N$ and τ are the spatial and temporal step sizes, respectively. u_i^j is an approximation to $u(x_i,t_j)$. We also define the difference operators as

$$\delta_t^+u_i^j=\frac{u_i^{j+1}-u_i^j}{\tau},\quad \delta_t^-u_i^j=\frac{u_i^j-u_i^{j-1}}{\tau},$$
$$\delta_x^+u_i^j=\frac{u_{i+1}^j-u_i^j}{h},\quad \delta_x^-u_i^j=\frac{u_i^j-u_{i-1}^j}{h}.$$

The Euler-box scheme for Eq. (2) is

$$\begin{aligned}&\boldsymbol{M}_+\delta_t^+\boldsymbol{z}_i^j+\boldsymbol{M}_-\delta_t^-\boldsymbol{z}_i^j+\boldsymbol{K}_+\delta_x^+\boldsymbol{z}_i^j+\boldsymbol{K}_-\delta_x^-\boldsymbol{z}_i^j\\&=\boldsymbol{\nabla}_{\boldsymbol{z}}\boldsymbol{S}(\boldsymbol{z}_i^j),\end{aligned} \tag{4}$$

where $\boldsymbol{M}_+$, $\boldsymbol{M}_-$ and $\boldsymbol{K}_+$, $\boldsymbol{K}_-$ are matrix splitting for the symplectic matrices $\boldsymbol{M}$ and $\boldsymbol{K}$, so that,

$$\begin{aligned}&\boldsymbol{M}=\boldsymbol{M}_++\boldsymbol{M}_-,\ \ \boldsymbol{M}_+^{\mathrm{T}}=-\boldsymbol{M}_-,\\&\boldsymbol{K}=\boldsymbol{K}_++\boldsymbol{K}_-,\ \ \boldsymbol{K}_+^{\mathrm{T}}=-\boldsymbol{K}_-.\end{aligned}$$

The Euler-box scheme has the following discrete multisymplectic conservation law:

$$\delta_t^+\omega_i^j+\delta_x^+\kappa_i^j=0,$$

where $\omega_i^j=\mathrm{d}\boldsymbol{z}_i^{j-1}\wedge\boldsymbol{M}\mathrm{d}\boldsymbol{z}_i^j$ and $\kappa_i^j=\mathrm{d}\boldsymbol{z}_{i-1}^j\wedge\boldsymbol{K}\boldsymbol{z}_i^j$.

Note that the matrix splitting (4) is not unique, different splitting methods will lead to different numerical schemes. Here we take the following splitting:

$$\boldsymbol{M}_+=\begin{pmatrix}0&-1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\quad \boldsymbol{K}_+=\begin{pmatrix}0&0&0&-1&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&-1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},$$

for detailed derivation. Substituting the matrices into the Euler-box scheme (4) yields

$$\begin{cases}-\delta_t^+u_i^j-\delta_x^+w_i^j=0,\\ \delta_t^-\psi_i^j-\delta_x^+p_i^j=w_i^j+F_u(u_i^j,v_i^j)\\ \qquad\qquad+\dfrac{1}{\beta}\big(E(u_i^j,v_i^j)+q_i^j\big)E_u(u_i^j,v_i^j),\\ -\delta_x^+q_i^j=p_i^j+\alpha v_i^j+F_v(u_i^j,v_i^j)\\ \qquad\qquad+\dfrac{1}{\beta}\big(E(u_i^j,v_i^j)+q_i^j\big)E_v(u_i^j,v_i^j),\\ \delta_x^-\psi_i^j=u_i^j,\ \ \delta_x^-u_i^j=v_i^j,\\ \delta_x^-v_i^j=\dfrac{1}{\beta}(q_i^j+E(u_i^j,v_i^j)).\end{cases} \tag{5}$$

Eliminating the auxiliary variables ψ,v,w,p,q, we can obtain a new explicit scheme for the Kawahara-type equation (1) as

$$\begin{aligned}&\delta_t^+u_i^j+\delta_t^-u_{i+1}^j+\alpha(\delta_x^+)^2\delta_x^-u_i^j+\beta(\delta_x^+)^3(\delta_x^-)^2u_i^j-\delta_x^+(F_u)_i^j\\&+(\delta_x^+)^2(F_v)_i^j-(\delta_x^+)^3E_i^j-\frac{1}{\beta}\delta_x^+\big[(\delta_x^-)^2u_i^j(E_u)_i^j\big]\\&+\frac{1}{\beta}\delta_x^+\big[(\delta_x^-)^2u_i^j(E_v)_i^j\big]=0.\end{aligned} \tag{6}$$

Here, $(F_u)_i^j$ is the abbreviation of $F_u(u_i^j,v_i^j)$, and so on.

The Euler-box scheme (4) has its adjoint scheme in time, which is

$$\boldsymbol{M}_+\delta_t^-\boldsymbol{z}_i^j+\boldsymbol{M}_-\delta_t^+\boldsymbol{z}_i^j+\boldsymbol{K}_+\delta_x^+\boldsymbol{z}_i^j+\boldsymbol{K}_-\delta_x^-\boldsymbol{z}_i^j=\boldsymbol{\nabla}_{\boldsymbol{z}}\boldsymbol{S}(\boldsymbol{z}_i^j). \tag{7}$$

After eliminating the auxiliary variables, we can also have an explicit scheme which only differs from Eq. (6) in the first two terms,

$$\begin{aligned}&\delta_t^-u_i^j+\delta_t^+u_{i+1}^j+\alpha(\delta_x^+)^2\delta_x^-u_i^j+\beta(\delta_x^+)^3(\delta_x^-)^2u_i^j-\delta_x^+(F_u)_i^j\\&+(\delta_x^+)^2(F_v)_i^j-(\delta_x^+)^3E_i^j-\frac{1}{\beta}\delta_x^+\big[(\delta_x^-)^2u_i^j(E_u)_i^j\big]\\&+\frac{1}{\beta}\delta_x^+\big[(\delta_x^-)^2u_i^j(E_v)_i^j\big]=0.\end{aligned} \tag{8}$$

We have also tried all the rest splitting methods, however, there result just the above two kinds of schemes. In the following discussion, we will investigate these two schemes by the numerical experiments.

4. Numerical experiments

In this section, we consider numerical approximations of solitary wave solutions for the Kawahara equation, the modified Kawahara equation, and the KdV–Kawahara equation given, respectively, by

i) Kawahara equation

$$u_t + uu_x + u_{xxx} - u_{xxxxx} = 0, \tag{9}$$

with $\alpha = 2$, $\beta = -2$, $E(u,v) = 0$, and $F(u,v) = -\frac{1}{3}u^3$.

ii) Modified Kawahara equation

$$u_t + u_x + u^2u_x + u_{xxx} - u_{xxxxx} = 0, \tag{10}$$

with $\alpha = 2$, $\beta = -2$, $E(u,v) = 0$, and $F(u,v) = -u^2 - \frac{1}{6}u^4$.

iii) KdV–Kawahara equation

$$u_t + u_x + uu_x + u_{xxx} - u_{xxxxx} = 0, \tag{11}$$

with $\alpha = 2$, $\beta = -2$, $E(u,v) = 0$, and $F(u,v) = -u^2 - \frac{1}{3}u^3$.

In order to check the accuracy of the numerical method, we need to introduce the norm definition, $||e^j||_\infty = \max_{0\le i\le N}|e_i^j|$, where e^j is the error between the numerical solution and the analytic solution. We also consider the two schemes on the invariant preservation. Under some proper boundary conditions, such as the periodic boundary condition and the zero boundary condition, the Kawahara-type equation admits the energy conservation law (ECL) and the momentum conservation law (MCL), i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathcal{E}(x,t)\,\mathrm{d}x = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\int \mathcal{M}(x,t)\,\mathrm{d}x = 0,$$

where $\mathcal{E} = \frac{\alpha}{2}u_x - \frac{\beta}{2}u_{xx}^2 + F + \frac{1}{2\beta}E^2$ and $\mathcal{M} = u^2$. The corresponding discrete forms of ECL and MCL are

$$\mathbb{E} = h\Big(\sum_{i=0}^{N}\frac{\alpha}{2}\delta_x^+u_i^j - \frac{\beta}{2}(\delta_x^+\delta_x^-u_i^j)^2 + F_i^j + \frac{1}{2\beta}(E_i^j)^2\Big),$$

$$\mathbb{M} = h\sum_{i=0}^{N}(u_i^j)^2.$$

Example 4.1 According to Ref. [28], the Kawahara equation has the travelling wave solution as

$$u(x,t) = \frac{105}{169}\mathrm{sech}^4\Big[\frac{1}{2\sqrt{13}}\Big(x + \frac{36}{169}t\Big)\Big], \tag{12}$$

subjected to the initial condition

$$u(x,0) = \frac{105}{169}\mathrm{sech}^4\Big(\frac{1}{2\sqrt{13}}x\Big).$$

We set $L = 20$, $h = 0.4$, and $\tau = 10^{-4}$. Figure 1 shows the development of the waveform of solution (12) from $t = 0$ to $t = 50$ by scheme (6). We also test the scheme (8) for this case. However, no matter how small the time step is, the numerical solution soon blows up. This is probably due to the up-wind phenomenon. Such phenomenon occurs in the rest of numerical examples, so we only present the results of the scheme (6) in the following. The accuracy test in space is listed in Table 1, which verifies the second-order accuracy of the Euler-box scheme (6). For the temporal order, due to the limitation of the CFL condition, the error is dominated by the spatial direction, thus we can hardly obtain the order by the numerical calculation. However, by the simple Taylor expansion, we can easily derive the second-order accuracy. Thus, the scheme (6) is of two orders both in space and time.

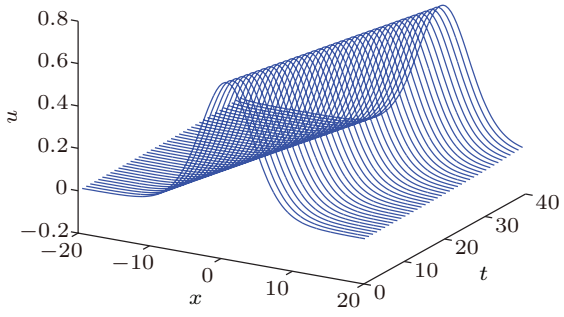


Fig. 1. Waveform of the numerical solution for the Kawahara equation (9).

Table 1. Convergence order in space with $\tau = 10^{-5}$ at $t = 1$.

N	$\|\|e\|\|_\infty$	Order
25	0.0027	
50	6.2742×10^{-4}	2.1055
100	1.4949×10^{-4}	2.0694
150	6.9027×10^{-5}	1.9058

Figure 2 presents the errors in the global energy and momentum conservation law, which shows excellent performance even if the computing time is up to 100 s.

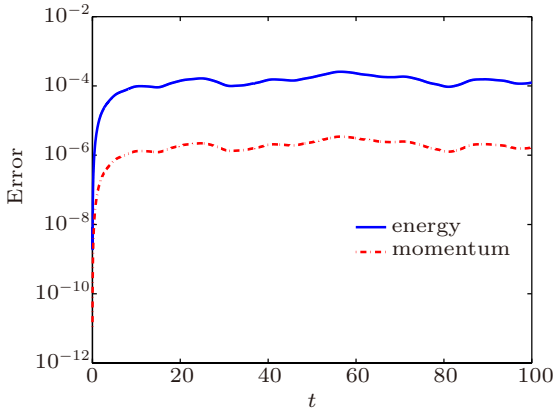


Fig. 2. Errors in the energy and momentum invariants.

Example 4.2 The modified Kawahara equation (10) also has the travelling wave solution

$$u(x,t) = \frac{3}{\sqrt{10}}\mathrm{sech}^2\Big[\frac{1}{2\sqrt{5}}\Big(x - \frac{29}{25}t\Big)\Big], \tag{13}$$

subjected to the initial condition

$$u(x,0) = \frac{3}{\sqrt{10}}\mathrm{sech}^2\Big(\frac{1}{2\sqrt{5}}x\Big).$$

Here, the parameters L, h, and τ are chosen as the above example. Figure 3 shows the waveform of the numerical solution as

well as the errors in two invariants. We can see clearly that the wave shape is still captured very well. The errors in the invariants are also bounded, which guarantees the long-time simulations. Furthermore, since the Euler-box scheme is explicit, the computation cost is rather small. In Fig. 4, we present the time cost in the simulation for the Kawahara equation (9) and the modified Kawahara equation (10) with different space steps at $t = 50$. The computation environment is CPU core i5 (2.3 GHz), 4 GB memory, Mac OS, and MATLAB (R2012b). The result shows that the scheme (6) computes extremely fast for both cases.

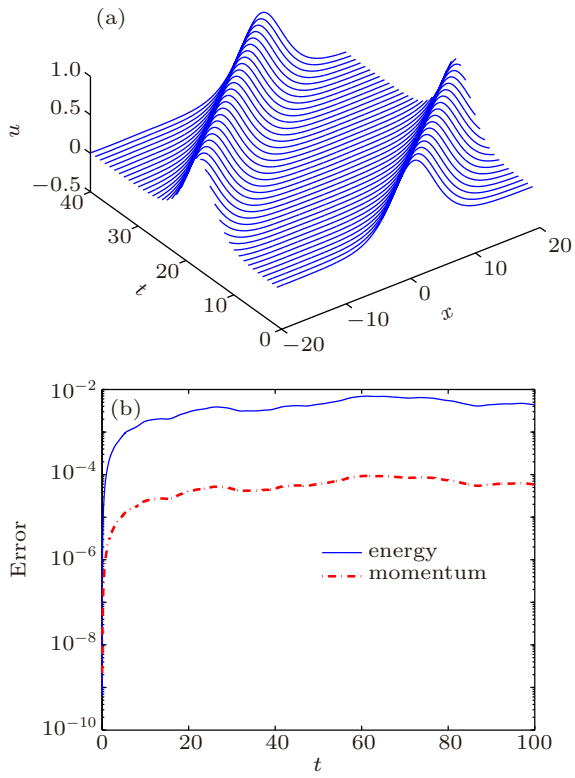


Fig. 3. (a) Waveforms and (b) errors in the energy and momentum invariants.

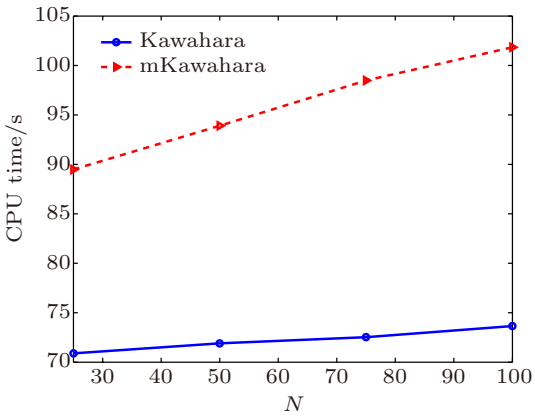


Fig. 4. Time cost with different N. τ is fixed as 10^{-4}.

Example 4.3 The following numerical test is an intersection of two solitons by the KdV–Kawahara equation with the initial condition

$$u(x,0) = \frac{105}{169}\left\{\operatorname{sech}^4\left(\frac{1}{2\sqrt{13}}(x+20)\right) + \frac{1}{4}\operatorname{sech}^4\left(\frac{1}{2\sqrt{13}}x\right)\right\}.$$

The computation domain is $[-50, 100]$, $h = 1$, and $\tau = 10^{-4}$. The three-dimensional plot is presented in Fig. 5 which has been tested similarly in Ref. [6]. The two invariants are again preserved very well.

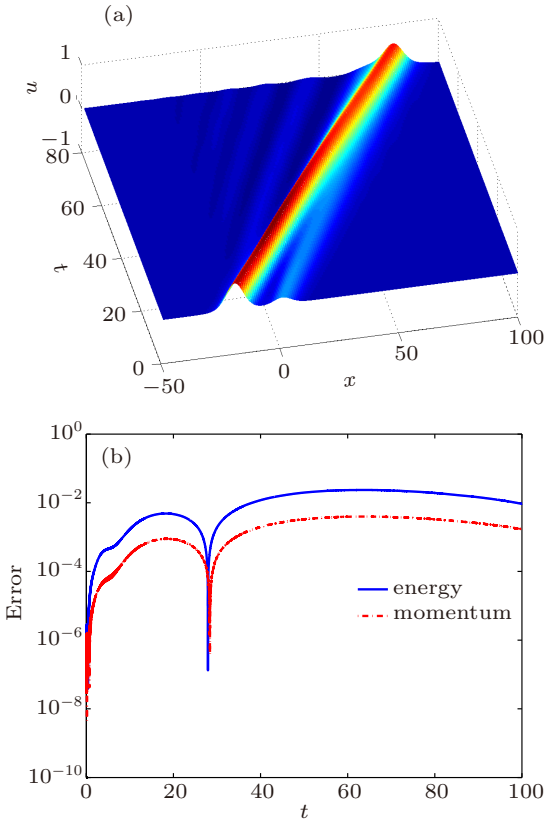


Fig. 5. (a) Waveform and (b) errors in the energy and momentum invariants.

5. Concluding remarks

In summary, we apply the multisymplectic Euler-box scheme on the Kawahara-type equation and obtain two multisymplectic schemes, which are both explicit and adjoint to each other. However, from the numerical experiments, only one scheme can work while the other soon blows up regardless of the choice of the time steps. Thus, we should be careful when using the Euler-box schemes. Three numerical examples show that the Euler-box scheme has good performance in the soliton simulation, not only for the waveform capturing, but also the invariant preserving.

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