Lie group analysis, numerical and non-traveling wave solutions for the (2+1)-dimensional diffusion–advection equation with variable coefficients

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In this paper, the variable-coefficient diffusion–advection (DA) equation, which arises in modeling various physical phenomena, is studied by the Lie symmetry approach. The similarity reductions are derived by determining the complete sets of point symmetries of this equation, and then exact and numerical solutions are reported for the reduced second-order nonlinear ordinary differential equations. Further, an extended (G'/G)-expansion method is applied to the DA equation to construct some new non-traveling wave solutions.

Keywords: diffusion–advection equation, Lie group analysis, numerical solutions, extended (G'/G)-expansion method

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1. Introduction

Many phenomena in physics and other fields are described by nonlinear evolution equations (NEEs). When we want to understand the physical mechanisms of these phenomena described by NEEs, exact, numerical, and non-traveling wave solutions have to be explored. In this direction, many effective methods for determining the exact solutions of NEEs have been established and developed during the past few decades. Some of the well-known methods include the Lie symmetry approach, the inverse scattering method, Hirota's bilinear method, the homogeneous balance method, and the (G'/G)-expansion method.

Among the various different methods, the Lie symmetry method, also called the Lie group method, is one of the most powerful in finding the solutions of NEEs. The fundamental basis of this method is that when a differential equation is invariant under a Lie group of transformations,[1–3] a reduction transformation exists. For partial differential equations (PDEs) with two independent variables, a single group reduction transforms the PDEs into ordinary differential equations (ODEs), which are generally easier to solve. In the recent past, there have been considerable developments in symmetry methods for differential equations, as is evident by the number of research papers, books, and new symbolic software devoted to the subject.[4–13]

Wang *et al.*[14] introduced the (G'/G)-expansion method[15,16] to look for traveling wave solutions of NEEs. This method is based on the assumption that the traveling wave solutions can be expressed by a polynomial in (G'/G), where G satisfies the second-order linear ordinary differential equation. The degree of the polynomial can be determined by the homogeneous balance between the highest-order derivative and the nonlinear terms appearing in the given NEE. The coefficients of the polynomial can be obtained by solving a set of algebraic equations.

Recently, Li and Ma[17] established a connection between the (G'/G)-expansion method and the soliton structure excitation. By extending the linear traveling wave transformation to the nonlinear transformation with the method, non-traveling wave solutions with variable separation can be constructed for NEEs. For the non-traveling wave solutions including arbitrary functions of independent variables, it is convenient to excite abundant soliton structures.

The (2+1)-dimensional diffusion–advection (DA) equation in the form

$$u_t = \alpha u_{yy} - \beta \left[K(u)\right]_x \tag{1}$$

was firstly investigated by Alexander and Lebowitz,[18] where α and β are arbitrary constants. Many special cases of this class of equations have been successfully used to model problems in mathematical physics, chemistry, and biology. In fact, when $K(u) = u(1-u)$ represents the flow of particles in a lattice fluid past an impenetrable obstacle,[18,19] $u(x,y,t)$ is the particle concentration moving by diffusion and advection or drift, and α and β are the constants that represent the diffusion coefficient and the drift velocity, respectively.

In Refs. [20] and [21], the classical Lie group method was used to study another model of a driven diffusive flow whose behavior is quite similar to that of the rod in a lattice fluid given in Refs. [18] and [19]. Following Refs. [18]–[21] and us-

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ing the classical symmetry reduction technique, the fourth order Runge–Kutta method, and the extended (G'/G)-expansion method, in this paper we study the variable-coefficient version of the DA equation

$$u_t = \alpha(t)\, u_{yy} - \beta(t)\,[K(u)]_x \tag{2}$$

for exact, numerical, and non-traveling solutions, where $K(u)$ is an arbitrary function of u, and $\alpha(t)$ and $\beta(t)$ are arbitrary functions of t.

The outline of this paper is as follows. In Section 2, the classical Lie method is utilized to obtain the optimal system of sub-algebras for Eq. (2). In Section 3, we derive some explicit exact and numerical solutions of the nonlinear equation (2). In Section 4, we apply the extended (G'/G)-expansion method to the DA equation to construct some new non-traveling wave solutions. In the last section, the conclusion is drawn.

2. Lie group analysis for the DA equation

First, let us consider a one-parameter Lie group of infinitesimal transformations

$$\begin{aligned} x^* &= x + \varepsilon X(x,y,t,u) + \mathrm{O}(\varepsilon^2),\\ t^* &= t + \varepsilon T(x,y,t,u) + \mathrm{O}(\varepsilon^2),\\ y^* &= y + \varepsilon Y(x,y,t,u) + \mathrm{O}(\varepsilon^2),\\ u^* &= u + \varepsilon U(x,y,t,u) + \mathrm{O}(\varepsilon^2), \end{aligned} \tag{3}$$

with a small parameter ε. The vector field associated with the above group of transformations can be written as

$$\begin{aligned} V \;=\; & X(x,y,t,u)\frac{\partial}{\partial x} + T(x,y,t,u)\frac{\partial}{\partial t}\\ & + Y(x,y,t,u)\frac{\partial}{\partial y} + U(x,y,t,u)\frac{\partial}{\partial u}. \end{aligned} \tag{4}$$

The condition that the above transformation (3) leaves invariant Eq. (2) under consideration yields an overdetermined system of linear equations for the infinitesimals $X(x,y,t,u)$, $T(x,y,t,u)$, $Y(x,y,t,u)$, and $U(x,y,t,u)$. The infinitesimals for different forms of $K(u)$ are given in Table 1. The relations between $\alpha(t)$ and $\beta(t)$ for different forms of $K(u)$ are given in Table 2.

Table 1. Infinitesimals corresponding to different forms of $K(u)$.

$K(u)$	X	Y	T	U
$u^m, m \neq 0,1,2$	k_2x+k_3	k_4y+k_5	$\dfrac{2k_4\int\alpha(t)\,\mathrm{d}t+k_6}{\alpha(t)}$	k_1u
e^{mu}	k_2x+k_3	k_4y+k_5	$\dfrac{2k_4\int\alpha(t)\,\mathrm{d}t+k_6}{\alpha(t)}$	k_1
$\log(u)$	k_2x+k_3	k_4y+k_5	$\dfrac{2k_4\int\alpha(t)\,\mathrm{d}t+k_6}{\alpha(t)}$	k_1u

Table 2. The relation between $\alpha(t)$ and $\beta(t)$.

$K(u)$	Relation between $\alpha(t)$ and $\beta(t)$
$u^m, m \neq 0,1,2$	$2k_4 - T\dfrac{\alpha'(t)}{\alpha(t)} + (m-1)k_1 - k_2 + \dfrac{\beta'(t)}{\beta(t)}T = 0$
e^{mu}	$2k_4 - T\dfrac{\alpha'(t)}{\alpha(t)} + mk_1 - k_2 + \dfrac{\beta'(t)}{\beta(t)}T = 0$
$\log(u)$	$2k_4 - T\dfrac{\alpha'(t)}{\alpha(t)} - k_1 - k_2 + \dfrac{\beta'(t)}{\beta(t)}T = 0$

When $K(u) = u^m$, $m \neq 0,1,2$ and $\log(u)$, the Lie algebra of infinitesimal symmetries of Eq. (2) is spanned by the following six linearly independent infinitesimal generators:

$$\begin{aligned} & V_1 = u\frac{\partial}{\partial u}, \quad V_2 = x\frac{\partial}{\partial x}, \quad V_3 = \frac{\partial}{\partial x},\\ & V_4 = \frac{\partial}{\partial y}, \quad V_5 = \frac{1}{\alpha(t)}\frac{\partial}{\partial t},\\ & V_6 = y\frac{\partial}{\partial y} + \frac{2}{\alpha(t)}\int\alpha(t)\,\mathrm{d}t\frac{\partial}{\partial t}. \end{aligned} \tag{5}$$

It is easy to verify that V_j, $j = 1,2,\ldots,6$ are closed under the Lie bracket.

The classification of one-dimensional subalgebras of the whole symmetry algebra (5) is performed using an inductive approach.[1] If $V = \sum_{i=1}^{6} a_iV_i$ is a non-zero vector field, we will simplify as many of the a_i coefficients as possible by acting it on various adjoint transformations to obtain the subalgebras.

Table 3. The similarity variables, similarity functions, and coefficient functions of the DA equation when $K(u) = u^m$ for $m \neq 0,1,2$ and $K(u) = \log(u)$.

Essential vector fields	Similarity variables $(f,\ g)$	Similarity solution u
$\langle V_1 + \lambda V_2 + \mu V_6\rangle$	$x(\int\alpha(t)\,\mathrm{d}t)^{-\lambda/2\mu},\ y(\int\alpha(t)\,\mathrm{d}t)^{-1/2}$	$(\int\alpha(t)\,\mathrm{d}t)^{1/2\mu}H(f,g)$
$\langle V_2 + \rho V_6\rangle$	$x(\int\alpha(t)\,\mathrm{d}t)^{-1/2\rho},\ y(\int\alpha(t)\,\mathrm{d}t)^{-1/2}$	$H(f,g)$
$\langle V_3 + \sigma V_6\rangle$	$x(\int\alpha(t)\,\mathrm{d}t)^{-1/2\sigma},\ y(\int\alpha(t)\,\mathrm{d}t)^{-1/2}$	$H(f,g)$
$\langle V_6\rangle$	$x,\ y(\int\alpha(t)\,\mathrm{d}t)^{-1/2}$	$H(f,g)$

Table 4. The coefficient functions of the DA equation for different forms of $K(u)$.

Essential vector fields	Form of coefficient function when $K(u) = u^m$ for $m \neq 0,1,2$	Form of coefficient function when $K(u) = \log(u)$
$\langle V_1 + \lambda V_2 + \mu V_6\rangle$	$k(\int\alpha(t)\,\mathrm{d}t)^{\lambda-(m-1)-2\mu/2\mu}\alpha(t)$	$k(\int\alpha(t)\,\mathrm{d}t)^{\lambda+1-2\mu/2\mu}\alpha(t)$
$\langle V_2 + \rho V_6\rangle$	$k(\int\alpha(t)\,\mathrm{d}t)^{1/2\rho-1}\alpha(t)$	$k(\int\alpha(t)\,\mathrm{d}t)^{1/2\rho-1}\alpha(t)$
$\langle V_3 + \sigma V_6\rangle$	$k(\int\alpha(t)\,\mathrm{d}t)^{-1}\alpha(t)$	$k(\int\alpha(t)\,\mathrm{d}t)^{-1}\alpha(t)$
$\langle V_6\rangle$	$k(\int\alpha(t)\,\mathrm{d}t)^{-1}\alpha(t)$	$k(\int\alpha(t)\,\mathrm{d}t)^{-1}\alpha(t)$

The types of one-dimensional subalgebras of Eq. (2) are as follows:

$$\begin{aligned}&\text{(i)}\quad \langle V_1+\lambda V_2+\mu V_6\rangle,\\ &\text{(ii)}\quad \langle V_2+\rho V_6\rangle,\\ &\text{(iii)}\quad \langle V_3+\sigma V_6\rangle,\\ &\text{(iv)}\quad \langle V_6\rangle,\end{aligned}\tag{6}$$

where λ,μ,ρ, and σ are arbitrary constants.

Once the subalgebras of Eq. (2) are determined, the symmetry variables for the associated reduction can be found by solving the characteristic equation

$$\frac{\mathrm{d}x}{X(x,y,t,u)}=\frac{\mathrm{d}y}{Y(x,y,t,u)}=\frac{\mathrm{d}t}{T(x,y,t,u)}=\frac{\mathrm{d}u}{U(x,y,t,u)}.\tag{7}$$

In Table 3, we list the Lie ansätze for all the essential fields that comprise the optimal system. The forms of the coefficient functions in Eq. (2) are listed in Table 4.

3. Similarity reductions, exact and numerical solutions when $K(u)=\log(u)$

Now, we deal with the symmetry reductions, and the exact and numerical solutions to Eq. (2). We consider the similarity reductions and the group-invariant solutions. From an optimal system of group-invariant solutions to Eq. (2), every other such solution to the equation can be derived. Using the ansatze in Table 3, for the form of $K(u)=\log(u)$, we can reduce the nonlinear equation (2) to an ODE.

3.1. Subalgebra $\langle V_1+\lambda V_2+\mu V_6\rangle$

Substituting the forms of the similarity solution and the coefficient functions corresponding to this vector field to Eq. (2) yields the following PDE:

$$H-\lambda fH_f-\mu gH_g-2\mu H_{gg}+2\mu kH^{-1}H_f=0.\tag{8}$$

To reduce the PDE, let us assume a transformation of the form

$$H(f,g)=f^{-1}J(z),\quad z=g,\tag{9}$$

which reduces Eq. (8) to the ODE of the form

$$(1+\lambda)J-\mu zJ'-2\mu J''-2\mu k=0.\tag{10}$$

Solving Eq. (10) and reverting back to the original variables, we obtain the following group-invariant solutions of the DA equation (2):

$$\begin{aligned}&u(x,y,t)\\ &=x^{-1}\left(\int\alpha(t)\,\mathrm{d}t\right)^{1+\lambda/2\mu}\exp\left(\frac{-1}{4}y^2\left(\int\alpha(t)\,\mathrm{d}t\right)^{-1}\right)\\ &\times\ \mathrm{KummerM}\left(\frac{1}{2}\frac{2\mu+1+\lambda}{\mu},\frac{3}{2},\frac{1}{4}y^2\left(\int\alpha(t)\,\mathrm{d}t\right)^{-1}\right)y\\ &\times\left(\int\alpha(t)\,\mathrm{d}t\right)^{-1/2}C_1\\ &+x^{-1}\left(\int\alpha(t)\,\mathrm{d}t\right)^{1+\lambda/2\mu}\exp\left(\frac{-1}{4}y^2\left(\int\alpha(t)\,\mathrm{d}t\right)^{-1}\right)\\ &\times\ \mathrm{KummerU}\left(\frac{1}{2}\frac{2\mu+1+\lambda}{\mu},\frac{3}{2},\frac{1}{4}y^2\left(\int\alpha(t)\,\mathrm{d}t\right)^{-1}\right)y\\ &\times\left(\int\alpha(t)\,\mathrm{d}t\right)^{-1/2}C_2+\frac{2\mu k}{1+\lambda},\end{aligned}\tag{11}$$

where λ, μ, k, C_1, and C_2 are arbitrary constants.

3.2. Subalgebra $\langle V_2+\rho V_6\rangle$

For the case under consideration, the reduced PDE is

$$fH_f+\rho gH_g+2\rho H_{gg}-2\rho kH^{-1}H_f=0.\tag{12}$$

Perform the transformation

$$H(f,g)=f^{-1}J(z),\quad z=g.\tag{13}$$

Substituting Eq. (13) into Eq. (12), we obtain an ODE of the form

$$J-\rho zJ'-2\rho J''-2\rho k=0.\tag{14}$$

For the ODE (14), we are unable to obtain the nontrivial exact solutions for the DA equation, so under this subalgebra, we

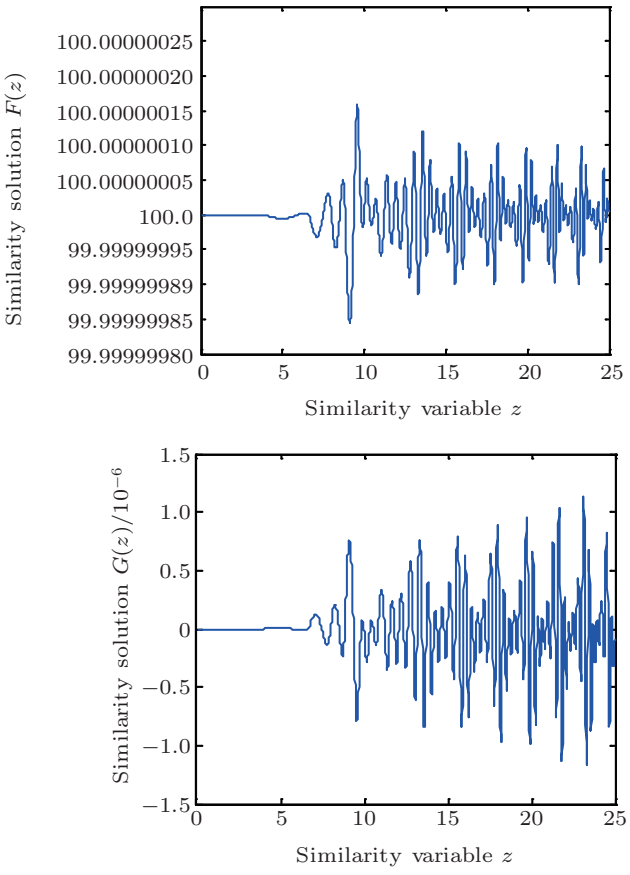


Fig. 1. (color online) The numerical solutions of the reduced IVP (15)–(17) for the DA equation at $h=0.01$ as $z\to 25$.

present the numerical solutions of the main equation (2) by solving the reduced ODE (14) with the help of the classical fourth-order Runge–Kutta (RK4) method. For the numerical solutions, we start with initial values at $z = 0$. The constants are under the restrictions $\rho = 100$, $k = 0.5$. We pose the following initial value problem (IVP) for the ODE (14) after reducing it to the first-order ODEs:

$$\frac{\mathrm{d}F(z)}{\mathrm{d}z} = G(z), \tag{15}$$

$$\frac{\mathrm{d}G(z)}{\mathrm{d}z} = \frac{F(z)}{2\rho} - \frac{z}{2}G(z) - k, \tag{16}$$

$$F(0) = 100, \quad G(0) = -1.019607920 \times 10^{-9}. \tag{17}$$

The numerical solutions to Eqs. (15)–(17) are depicted in Figs. 1 and 2.

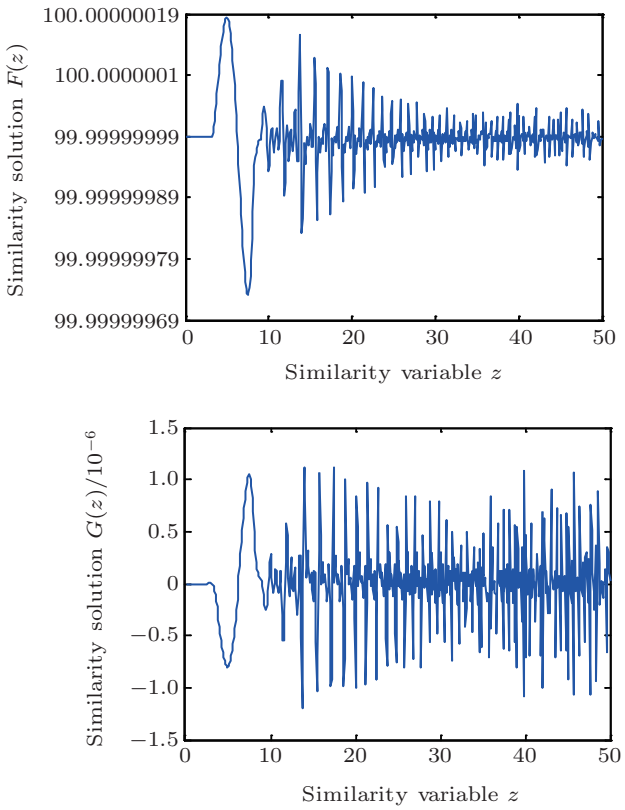


Fig. 2. (color online) The numerical solutions of the reduced IVP (15)–(17) for the DA equation at $h = 0.1$ as $z \to 50$.

3.3. Subalgebra $\langle V_3 + \sigma V_6 \rangle$

For this vector field, equation (2) is transformed into the following second-order nonlinear PDE:

$$H_f + \sigma g H_g + 2\sigma H_{gg} - 2\sigma k H^{-1} H_f = 0. \tag{18}$$

We further transform (18) via $H(f,g) = J(z)$, $z = g$ to the ODE

$$zJ' + 2J'' = 0. \tag{19}$$

Then the solution of the above ODE leads to the following trivial group-invariant solution of Eq. (2):

$$u(x,y,t) = C_1 + \mathrm{erf}\left(\frac{1}{2}y\left(\int \alpha(t)\,\mathrm{d}t\right)^{-1/2}\right)C_2, \tag{20}$$

where C_1 and C_2 are arbitrary constants.

It is noted that subalgebra (iv) $\langle V_6 \rangle$ leads to constant solutions.

4. Application of the extended (G'/G)-expansion method to the DA equation

When $K(u) = u(1-u)$, then equation (2) can be written as

$$u_t = \alpha(t)u_{yy} - \beta(t)[u(1-u)]_x. \tag{21}$$

In this section, we apply the extended (G'/G)-expansion method to the DA equation (21), and some non-traveling wave solutions with parameters are obtained.

The solution of Eq. (21) can be expressed as a polynomial in (G'/G)

$$u = a_0 + \sum_{i=0}^{m} a_i \left[\frac{G'(\theta)}{G(\theta)}\right]^i, \tag{22}$$

where a_i $(i = 1,2,\ldots,m)$ are arbitrary functions of variables x, y, t to be determined, and $G(\theta)$ satisfies the following equation:

$$G''(\theta) + \lambda G'(\theta) + \mu G(\theta) = 0, \tag{23}$$

with $\theta(x,y,t) = f(x,t) + g(y)$, and $f(x,t)$, $g(y)$ being arbitrary functions of the indicated variables.

By balancing the highest order nonlinear term and the highest order partial derivative term of u in Eq. (21), we obtain $m = 1$. In order to search for explicit solutions, we suppose that equation (21) has the following formal solution:

$$u = a_0 + a_1\left(\frac{G'}{G}\right), \quad a_1 \neq 0. \tag{24}$$

By substituting Eq. (24) into Eq. (21), using Eq. (23), collecting all terms of the same order together, the left-hand side of Eq. (21) is converted into a polynomial in $(G'/G)^j, (j = 0,1,2,3)$. Setting each coefficient of this polynomial to zero, we derive the following set of overdetermined differential equations for a_0 and a_1:

$$\left(\frac{G'}{G}\right)^3 : -2a_1(g')^2\alpha(t) + 2\beta(t)a_1^2 f_x = 0, \tag{25}$$

$$\begin{aligned}\left(\frac{G'}{G}\right)^2 : {} & -a_1 f_t - g'\left[3\lambda a_1 g' - 2(a_1)_y\right]\alpha(t) \\ & - a_1 f_x \beta(t) + 2\lambda\beta(t)a_1^2 f_x \\ & + 2\beta(t)a_0 a_1 f_x + a_1 g''\alpha(t) = 0,\end{aligned} \tag{26}$$

$$\begin{aligned}\left(\frac{G'}{G}\right) : {} & (a_1)_t - \lambda a_1 f_t - g'\left[\left(\lambda^2 + 2\mu\right)a_1 g' - 2\lambda(a_1)_y\right]\alpha(t) \\ & + \left[(a_1)_x - \lambda a_1 f_x\right]\beta(t) \\ & - 2a_1\left[(a_0)_x - \mu a_1 f_x\right]\beta(t)\end{aligned}$$

$$-2a_0\left[(a_1)_x-\lambda a_1 f_x\right]\beta(t)$$
$$-\left[(a_1)_{yy}-\lambda a_1 g''\right]\alpha(t)=0, \tag{27}$$

$$\left(\frac{G'}{G}\right)^0:\ (a_0)_t-\mu a_1 f_t+g'\left[\lambda\mu a_1-2\mu(a_1)_y\right]\alpha(t)$$
$$+\left[(a_0)_x-\mu a_1 f_x\right]\beta(t)-2a_0\left[(a_0)_x-\mu a_1 f_x\right]\beta(t)$$
$$-\left[(a_0)_{yy}-\mu a_1 g''\right]\alpha(t)=0. \tag{28}$$

Solving this set of equations, we obtain the following results:

$$a_0=\frac{f_t+\lambda(g')^2\alpha(t)+f_x\beta(t)-5g''\alpha(t)}{2\beta(t)f_x},$$
$$a_1=\frac{\alpha(t)(g')^2}{\beta(t)f_x},\quad f_x\neq 0. \tag{29}$$

Substituting the general solutions of Eq. (23) into Eq. (24) and using Eqs. (25)–(29), we obtain the following three types of exact solutions of Eq. (21).

Type 1 When $\lambda^2-4\mu>0$, we have

$$\frac{G'(\theta)}{G(\theta)}=-\frac{\lambda}{2}+\Lambda_1\frac{A_1\sinh\Lambda_1(f+g)+A_2\cosh\Lambda_1(f+g)}{A_1\cosh\Lambda_1(f+g)+A_2\sinh\Lambda_1(f+g)}, \tag{30}$$

where $\Lambda_1=\sqrt{\lambda^2-4\mu}/2$ and A_1, A_2 are arbitrary constants.

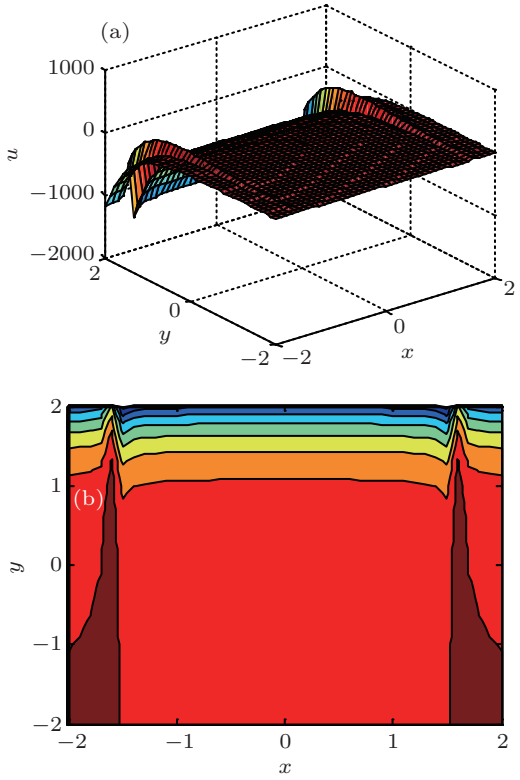


Fig. 3. (color online) (a) The soliton-like solution given by Eq. (31) with the settings $\lambda=3$, $\mu=1$, $f(x,t)=\sin(x)+t$, $g(y)=\exp(y)$, $\alpha(t)=\beta(t)=\cos(t)$ and $x,y\in[-2,2]$ at $t=0$. (b) The projection of the solution on the *xoy* plane.

We can obtain the non-traveling wave solutions with hyperbolic functions for the DA equation (21)

$$u(x,y,t)=\frac{f_t+f_x\beta(t)-5g''\alpha(t)}{2\beta(t)f_x}+\left(\frac{\alpha(t)(g')^2}{\beta(t)f_x}\right)\Lambda_1$$
$$\times\frac{A_1\sinh\Lambda_1(f+g)+A_2\cosh\Lambda_1(f+g)}{A_1\cosh\Lambda_1(f+g)+A_2\sinh\Lambda_1(f+g)}, \tag{31}$$

where the arbitrary function $f(x,t)$ satisfies the conditions that (i) $f_x\neq 0$ and (ii) f_x exists; and $g(y)$ satisfies the conditions that (i) $g'\neq 0$, (ii) g' exists, and (iii) g'' exists. The solution is shown in Fig. 3.

Type 2 When $\lambda^2-4\mu<0$, we have

$$\frac{G'(\theta)}{G(\theta)}=-\frac{\lambda}{2}+\Lambda_2\frac{-A_1\sin\Lambda_2(f+g)+A_2\cos\Lambda_2(f+g)}{A_1\cos\Lambda_2(f+g)+A_2\sin\Lambda_2(f+g)}, \tag{32}$$

where $\Lambda_2=\sqrt{4\mu-\lambda^2}/2$ and A_1, A_2 are arbitrary constants.

We can obtain the non-traveling wave solutions with trigonometric functions for the DA equation (21)

$$u(x,y,t)=\frac{f_t+f_x\beta(t)-5g''\alpha(t)}{2\beta(t)f_x}+\left(\frac{\alpha(t)(g')^2}{\beta(t)f_x}\right)\Lambda_2$$
$$\times\frac{-A_1\sin\Lambda_2(f+g)+A_2\cos\Lambda_2(f+g)}{A_1\cos\Lambda_2(f+g)+A_2\sin\Lambda_2(f+g)}, \tag{33}$$

where the arbitrary function $f(x,t)$ satisfies the conditions that (i) $f_x\neq 0$ and (ii) f_x exists; and $g(y)$ satisfies the conditions that (i) $g'\neq 0$; (ii) g' exists; and (iii) g'' exists. The solution is shown in Fig. 4.

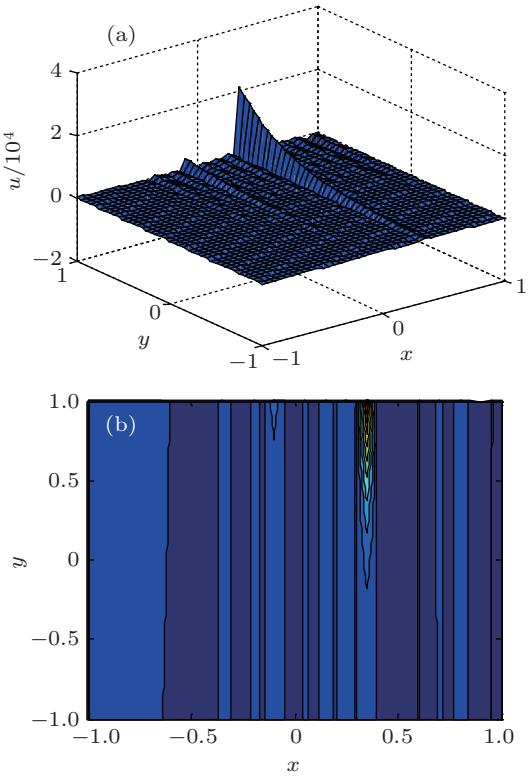


Fig. 4. (color online) (a) The soliton-like solution given by Eq. (33) with the settings $\lambda=\mu=2$, $f(x,t)=\sin(x)+t$, $g(y)=\exp(y)$, $\alpha(t)=\beta(t)=\cos(t)$ and $x,y\in[-1,1]$ at $t=0$. (b) The projection of the solution on the *xoy* plane.

Type 3 When $\lambda^2-4\mu=0$, we have

$$\frac{G'(\theta)}{G(\theta)} = -\frac{\lambda}{2} + \frac{A_2}{A_1 + A_2(f+g)}, \tag{34}$$

where A_1 and A_2 are arbitrary constants.

We can obtain the non-traveling wave solutions with rational functions for the DA equation (21)

$$u(x,y,t) = \frac{f_t + f_x\beta(t) - 5g''\alpha(t)}{2\beta(t)f_x} + \left(\frac{\alpha(t)(g')^2}{\beta(t)f_x}\right) \times \frac{A_2}{A_1 + A_2(f+g)}, \tag{35}$$

where the arbitrary function $f(x,t)$ satisfies the conditions that (i) $f_x \neq 0$ and (ii) f_x exists; and $g(y)$ satisfies the conditions that (i) $g' \neq 0$; (ii) g' exists; and (iii) g'' exists. The solution is shown in Fig. 5.

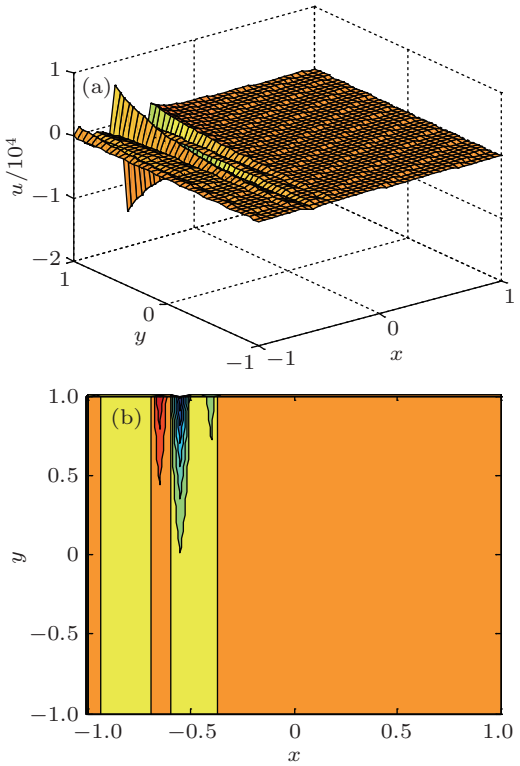


Fig. 5. (color online) (a) The soliton-like solution given by Eq. (35) with the settings $f(x,t) = \sin(x) + t$, $g(y) = \exp(y)$, $\alpha(t) = \beta(t) = \cos(t)$ and $x, y \in [-1, 1]$ at $t = 0$. (b) The projection of the solution on the *xoy* plane.

5. Discussion and concluding remarks

The Lie symmetry method was utilized to investigate the symmetries and the invariant solutions of the DA equations. By determining the transformation group under which a given system is invariant, we obtained information about the invariants and the symmetries of the equation. This information, in turn, was used to determine similarity variables that reduced the number of independent variables. The vector fields of the optimal system led to the reduction of the nonlinear PDEs to ODEs. The infinitesimal generators in the optimal system were used for reductions and exact solutions. By using these vector fields, we have found a group-invariant solution under subalgebra (3.1). Then, under subalgebra (3.2), the initial value problems were posed to Eq. (27), and the numerical solutions were shown with the help of Figs. 1 and 2 for a fairly finite value of z. Further, we successfully obtained non-traveling solutions with arbitrary parameters to DA equation (21) via the extended (G'/G)-expansion method, and in all the cases, we chose the arbitrary function $f(x,t)$ or $g(y)$ along with various other arbitrary parameters, in a suitable manner, to simulate the physical behavior of the solutions with the help of Figs. 3–5. Thus, we found some new exact, numerical, and non-traveling wave solutions that might prove to be potentially useful for applications in mathematical physics and applied mathematics.

References

[1] Olver P J 1993 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)

[2] Bluman G W and Kumei S 1989 *Symmetries and Differential Equations* (New York: Springer)

[3] Ovsyannikov L V 1982 *Group Analysis of Differential Equations* (New York: Academic)

[4] Demetriou E, Christou M A and Sophocleous C 2007 *Appl. Math. Comput.* **187** 1333

[5] Singh K and Gupta R K 2006 *Int. J. Eng. Sci.* **44** 241

[6] Kumar S, Singh K and Gupta R K 2012 *Commun. Nonlinear Sci. Numer. Simul.* **17** 1529

[7] Gupta R K and Singh K 2011 *Commun. Nonlinear Sci. Numer. Simul.* **16** 4189

[8] Goyal N and Gupta R K 2012 *Phys. Scr.* **85** 015004

[9] Kumar V, Gupta R K and Jiwari R 2013 *Chin. Phys. B* **22** 050201

[10] Khalique C M and Adem K R 2011 *Math. Comput. Model.* **54** 184

[11] Johnpillai A G, Kara A H and Biswas A 2013 *Appl. Math. Lett.* **26** 376

[12] Kaur L and Gupta R K 2013 *Math. Meth. Appl. Sci.* **36** 584

[13] Gupta R K and Bansal A 2013 *Appl. Math. Comput.* **219** 5290

[14] Wang M L, Li X and Zhang 2008 *J. Phys. Lett. A* **372** 417

[15] Zhang S, Dong L, Ba J M and Sun Y N 2009 *Phys. Lett. A* **373** 905

[16] Guo S and Zhou Y 2010 *Appl. Math. Comput.* **215** 3214

[17] Li B Q and Ma Y L 2011 *Commun. Nonlinear Sci. Numer. Simulat.* **16** 144

[18] Alexander F J and Lebowitz J L 1990 *J. Phys. A: Math. Gen.* **23** 375

[19] Alexander F J and Lebowitz J L 1994 *J. Phys. A: Math. Gen.* **27** 683

[20] Saied E A and Abd El-Rahman R G 1998 *J. Stat. Phys.* **94** 639

[21] Elwakil S A, Zahran M A and Sabry R 2005 *ZAMP* **56** 986