Optimal network structure to induce the maximal small-world effect*

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In this paper, the general efficiency, which is the average of the global efficiency and the local efficiency, is defined to measure the communication efficiency of a network. The increasing ratio of the general efficiency of a small-world network relative to that of the corresponding regular network is used to measure the small-world effect quantitatively. The more considerable the small-world effect, the higher the general efficiency of a network with a certain cost is. It is shown that the small-world effect increases monotonically with the increase of the vertex number. The optimal rewiring probability to induce the best small-world effect is approximately 0.02 and the optimal average connection probability decreases monotonically with the increase of the vertex number. Therefore, the optimal network structure to induce the maximal small-world effect is the structure with the large vertex number (> 500), the small rewiring probability (≈ 0.02) and the small average connection probability (< 0.1). Many previous research results support our results.

Keywords: small-world network, communication efficiency, optimal network structure

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1. Introduction

In 1998, Watts and Strogatz proposed a novel graph model,[1] which is the so-called small-world (SW) network, it has since attracted much attention.[2–6] By rewiring a small fraction of edges, a regular network can be switched to the SW one.[1] The SW network possesses the advantages of the regular network and the random one simultaneously; that is, the short average distance between vertices and the high clustering coefficient.[1] Many biological, neural, technological, and social networks possess the main characteristics of the SW networks.[7–10] For example, many experimental results show that, although there are many kinds of neural networks in brains, the main pattern of neuronal connections in brains is of the SW type.[8,9] Some experimental results show that disruption or abnormality of the SW networks in brains can induce some psychiatric disorders, such as Alzheimer's disease,[11] schizophrenia,[12] attention-deficit/hyperactivity disorder,[13] etc. In addition to research on the SW neural networks, other research on the SW networks has also carried out extensively. Scaling in the SW networks has been studied, and it was found that there was one non-trivial length scale in the model, which was analogous to the correlation length in other systems.[14] Site percolation on the SW networks, which was proposed as a simple model of disease propagation, was also studied, and an approximate expression for the percolation probability was derived.[14] Some simple models of disease transmission on the SW networks were studied and it was shown that the resulting models displayed epidemic behavior when the transmission or infection probability rose above the threshold.[15] The study on synchronization of the SW networks revealed that the addition of random shortcuts could improve network synchronizability.[16] Spreading in the SW networks was also studied, and it was found that the slowing down of spreading is mainly caused by weight-topology correlations and the bursting activity patterns of individuals.[17] The dynamics of the SW networks is another interesting issue and was studied by Lyapunov stability approach and numerical simulation, it was shown that small-world connection may be efficient switch to control the dynamics of a system.[18] Robustness of the SW networks under intentional attacks is also an interesting issue. It was shown that with the increase of the rewiring probability the SW networks became more and more robust to intentional attacks.[19] In addition to the simulation results, some theoretical analyses on the SW networks were also carried out, and the theoretical expressions of the clustering coefficients,[20] the average distances,[21,22] and the degree distributions[20] of the SW networks were presented. Furthermore, a renormalization group approach was used to study small-world to fractal transition in the SW networks, and some interesting results were obtained.[23] During the last decade, extensive researches on the SW networks have also been carried out by Chinese scholars and many interesting results have been obtained.[24–31] Complete and phase synchronization in a heterogeneous SW neuronal network were studied, and it was found that the coupling strength necessary for complete or phase synchroniza-

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tion decreases with the increase of the neuron number and the node degree.[26] Phase synchronization on the SW networks with community structure was also studied, and it was shown that the network synchronizability can be enhanced markedly by increasing the fuzziness of community structure.[27] The problem of prisoner dilemma game (PDG) has aroused extensive interest for a long time, and the problem in the Newman–Watts SW networks has also been studied.[29] The effect of incubation period on epidemic spreading in the SW networks was also investigated by analytical method and numerical simulation, and it is shown that the epidemic threshold is independent of incubation period in the SW networks.[30]

Because the average distance and the clustering coefficient are two different concepts to measure the network characteristics, Latora and Marchiori[32,33] proposed the concept of communication efficiency to measure the two different parameters above. The global efficiency and the local efficiency are the counterparts of the average distance and the clustering coefficient, respectively. An SW network has both the high global efficiency and local efficiency.[32,33] In addition to the main advantage in information communication, the SW networks also have some other advantages in information processing, such as fast response, high coherent oscillation, and strong synchronization.[16,34] Owing to the above-mentioned advantages in information processing, the SW network is an ideal candidate for communication network. For example, many neural networks[35] and traffic networks[36] have both high global efficiency and high local efficiency, together with low cost. We can describe the SW characteristics by the network topology[1] or the network efficiency,[32,33] but they are just qualitative descriptions. In this paper, we construct a general parameter to measure the network efficiency quantitatively, the general efficiency, which is the average of the global and the local efficiencies. Because the average distance and the clustering coefficient are two equal parameters to measure the network characteristics and their counterparts,[1] the global efficiency and the local efficiency are also equally important to measure information communication of the networks.[32,33] Our construction of the general efficiency is reasonable. It is really found that the general efficiency of the SW network is highest.

In the aspect of communication efficiency, one of the most important issues is how to construct an ideal network which possesses the highest efficiency with the lowest cost. For example, how to construct a neural network possessing the highest efficiency with the lowest biological cost is a very important physiological issue because such a kind of neural network can process neural information more quickly and its consumption of metabolic energy is less. In another example, how to construct a traffic network possessing the highest efficiency with the lowest economical cost is also a very important issue because such a kind of traffic network can transport more vehicles and human beings in unit time and traffic jam has become an important social problem and has been the main constraint on economical development day by day. In this paper, the SW effect is proposed to measure quantitatively how a network is constructed to possess the highest efficiency with the lowest cost. The SW effect is measured quantitatively by increasing the percentage of the general efficiency of the SW network relative to that of the corresponding regular network. The larger the increasing percentage, the more considerable the SW effect is and the higher the communication efficiency of a network with a certain cost is. Our research results show that the SW effect increases monotonically with the increase of the vertex number. If the vertex number is 2×10^3, the increasing percentage of the general efficiency reaches 20%. It is also found that the optimal rewiring probability to induce the maximal SW effect is approximately 0.02. It is further demonstrated quantitatively that the network should be sparse for the maximal SW effect and the optimal average connection probability decrease monotonically as the vertex number increases. Therefore, in order to induce the maximal SW effect, the specific network should have the large vertex number, the small rewiring probability and the small average connection probability. In other words, the optimal structure for a network to possess the highest efficiency with the lowest cost is that the network size should be large, the network should be sparse and the rewiring probability should be small. Our results are confirmed by many previous results.[1,32,33,37–40]

2. Model

In 2001, Latora and Marchiori[32,33] proposed the concept of communication efficiency to measure the network characteristics. Now we restate concisely the concept of communication efficiency here. A generic graph G with N vertices and K edges is considered. The graph G is assumed to be unweighted and sparse; that is, each edge is equal and $K \ll N(N-1)/2$. The graph G is therefore represented by the connection matrix; that is, the $N\times N$ matrix whose entry a_{ij} is 1 if there is an edge connecting vertices i and j, and is 0 otherwise. Another important parameter of the graph G is the degree of vertex i; that is, the number k_i of edges connecting to vertex i. The average value of k_i is the average degree k of the graph G, which is defined as $k = 2K/N$. For each pair of vertices i and j, the shortest path with the minimal edges to connect the two vertices is found by the Dijkstra's algorithm, and the minimal edge number is denoted as d_{ij}. Because the information propagation time between vertices i and j is inversely proportional to d_{ij}, the communication efficiency ε_{ij} between the two vertices can be defined as $\varepsilon_{ij} = 1/d_{ij}$. When there is no path in the graph G between vertices i and j, $d_{ij} = +\infty$; that is, $\varepsilon_{ij} = 0$.

The global efficiency of G is defined as

$$E_{\text{glob}}(G) = \frac{\sum_{i\neq j\in G}\varepsilon_{ij}}{N(N-1)} = \frac{1}{N(N-1)}\sum_{i\neq j\in G}\frac{1}{d_{ij}}. \quad (1)$$

It is obvious that the global efficiency measures the communication efficiency in the global scale, which is the counterpart of the average distance.[32,33] On the other hand, the efficiency of any subgraph of the graph G can also be evaluated and, therefore, the local properties of the graph G can be characterized sequentially. In this paper, since the efficiency is defined also for a disconnected graph, we can characterize the local properties of the graph G by evaluating the efficiency of each subgraph G_i, where i is the arbitrary vertex in the graph G and G_i is the subgraph of the neighbors of vertex i. Therefore, the local efficiency E_{loc} is defined as

$$E_{\text{loc}} = \frac{1}{N}\sum_{i\in G}E_{\text{glob}}(G_i). \quad (2)$$

Obviously, the local efficiency E_{loc} is the counterpart of the clustering coefficient and measures the information communication on the local scale. Since $i \notin G_i$, the local efficiency E_{loc} shows the fault tolerant of the system and how efficiently the commutation is carried out between the first neighbors of vertex i when vertex i is removed.

In their milestone papers, Watts and Strogatz[1] measured the SW effect as short average distance and high clustering coefficient; however, their work is only qualitative. In their classical paper, Latora and Marchiori[32] measured the SW effect at relatively high global efficiency and high local efficiency;[32] however, this work is still qualitative. In the present paper, we try to measure the SW effect quantitatively. In the aspect of information communication, an ideal network should possess the highest communication efficiency with the lowest cost, which we have called the SW effect. In order to measure the SW effect quantitatively, we should use a general parameter to measure the network efficiency, which is called the general efficiency. Because information can propagate in the whole network or the subnets of the network, in order to obtain the complete description of communication characteristics of a specific network we should consider the information efficiency, both on the global scale and on the local scale, simultaneously. Therefore, the general efficiency should include the global efficiency and the local efficiency simultaneously. Furthermore, the global efficiency and the local efficiency are of the same importance for network communication and their counterparts, and the average distance and the clustering coefficient are also of the same importance for network topology. Therefore, the general efficiency E_{gen} is defined as the average of the global efficiency and the local efficiency; that is, $E_{\text{gen}} = (E_{\text{glob}} + E_{\text{loc}})/2$. Consequently, the SW effect is measured by the property that a specific network possesses the highest general efficiency with the lowest cost. Before the quantitative measurement of the SW effect, a basis should first be chosen. Because E_{gen} of the regular network is much larger than that of the corresponding random network for a sparse network, E_{gen} of the regular network, which is denoted as E_0, is chosen as the measuring basis. Therefore, we can use the increasing percentage of the global efficiency to measure the SW effect, which is defined as

$$\gamma_{\text{SW}} = \frac{E_{\text{gen}} - E_0}{E_0}. \quad (3)$$

The larger the value of γ_{SW}, the more considerable the SW effect is and the higher the network efficiency is. If γ_{SW} is maximal, it means that the network possesses the highest efficiency with the lowest cost. Because the studied networks in this paper are all unweighted, their cost can be measured quantitatively by the average connection probability.

3. Results

Firstly, a regular network with 2000 vertices and 60 edges for each vertex is constructed. Secondly, according to the Watts–Strogatz model, the SW network is reconstructed by rewiring the network with probability p_{rewire}.[1] The curves of E_{glob}, E_{loc}, and E_{gen} of the constructed SW network for different values of p_{rewire} are plotted in Fig. 1. According to the well-known results, E_{glob} increases monotonically and E_{loc} decreases monotonically with the increase of p_{rewire}.[32,33] As shown in Fig. 1, E_{gen} first increases and then decreases after reaching the maximum with the increase of p_{rewire}, which is the so-called SW effect. It is also found from Fig. 1 that E_{gen} of the regular network ($p_{\text{rewire}} = 0$) is larger than that of the corresponding random network ($p_{\text{rewire}} = 1$). Therefore, it is reasonable to choose E_{gen} of the regular network as the basis to measure the SW effect quantitatively. According to Eq. (3), the quantitative measurements of the SW effect (i.e. γ_{SW}) for different values of p_{rewire} are obtained and shown in Fig. 2. Figure 2 shows clearly the SW effect that γ_{SW} first increases and then decreases after reaching the maximum with the increase of p_{rewire}. The optimal p_{rewire} to induce the maximal efficiency is approximately 0.02, and the corresponding γ_{SW} is as large as 19%.

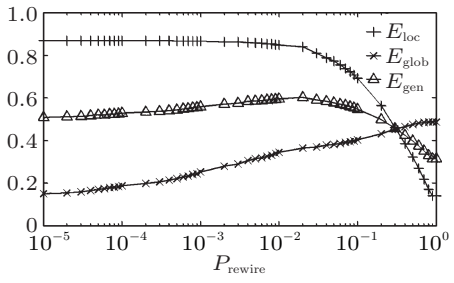


Fig. 1. Curves of E_{glob}, E_{loc}, and E_{gen} of the constructed SW networks versus p_{rewire}. The regular network with 2000 vertices and 60 edges for each vertex is rewired with probability p_{rewire}.

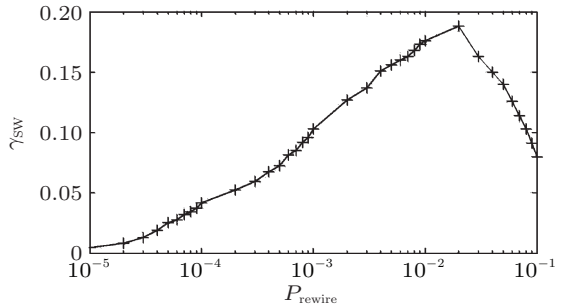


Fig. 2. Curve of γ_{SW} of the constructed SW networks versus p_{rewire}. The SW networks are the same ones as those in Fig. 1.

For an SW network, there are three parameters to measure the network, the vertex number N, the average edge number for each vertex (k), and the rewiring probability p_{rewire}. In addition to the parameter k, the average connection probability p_{ave}, which is approximately equal to k/N, can be used to measure the network connectivity. For a fixed vertex number N, the effects of p_{ave} and p_{rewire} on γ_{SW} are studied. The vertex number N is also set to be 2000, which is similar to those in Figs. 1 and 2. For $N = 2000$ and a certain value of p_{ave}, γ_{SW} reaches a maximum value when the rewiring probability p_{rewire} is approximately 0.02. For $N = 2000$ and $p_{rewire} = 0.02$, the curve of γ_{SW} versus p_{ave} is shown in Fig. 3. As the increase of p_{ave}, γ_{SW} first increases and then decreases when p_{ave} is over the optimum, 0.032. When p_{ave} is 0.032, γ_{SW} reaches a maximum value that is as large as 20%.

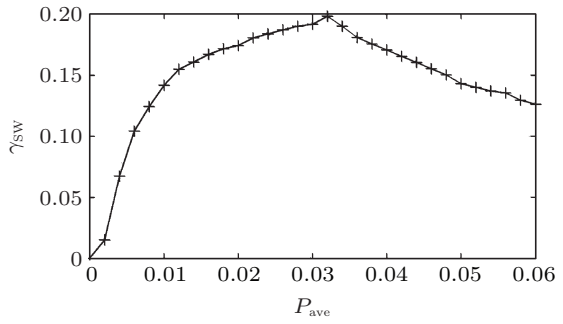


Fig. 3. Curve of γ_{SW} versus p_{ave} for $N = 2000$ and $p_{rewire} = 0.02$.

Following the above figure, we try to find the optimal values of p_{ave} and p_{rewire} to induce the maximal γ_{SW} for different values of vertex number N. The optimal p_{rewire} is almost a constant, which is approximately 0.02. The curve of the optimal p_{ave} versus N is plotted in Fig. 4. With the increase of N, the optimal p_{ave} decreases monotonically. If N is equal to 50, the optimal p_{ave} is as large as 0.24. If N is increased to 3×10^3, the optimal p_{ave} is reduced to 0.025. With the optimal values of p_{ave} and p_{rewire}, the curve of the maximal γ_{SW} versus N is plotted in Fig. 5, showing that the maximal γ_{SW} increases monotonically with the increase of N. When $N = 30$, the maximal γ_{SW} is equal to zero and there is not any SW effect. When $N = 50$, the maximal γ_{SW} is as small as 0.01 and the SW effect is very weak. For $N = 10^2$, 10^3, 2×10^3, and 3×10^3, the maximal γ_{SW} are 0.033, 0.16, 0.20, and 0.23, correspondingly.

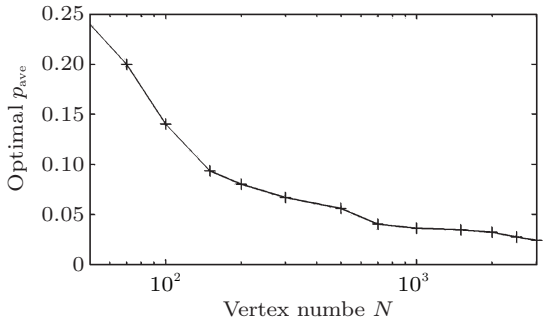


Fig. 4. Curve of optimal p_{ave} to induce the maximal SW effect versus vertex number N.

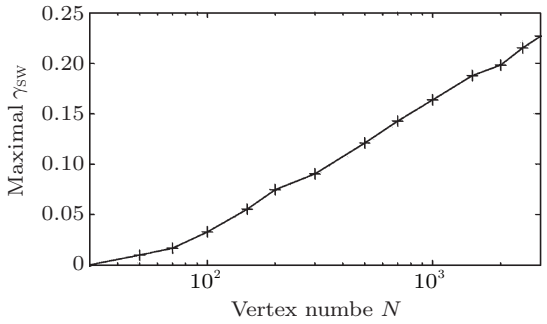


Fig. 5. Curve of the maximal γ_{SW} versus vertex number N.

4. Conclusions

In 2001, the concept of communication efficiency was proposed by Latora and Marchiori[32,33] to measure the network characteristics. The global efficiency corresponds to the average distance, and the local efficiency corresponds to the clustering coefficient.[32,33] In this paper, the general efficiency, which is the average of the global and the local efficiencies, is defined to measure the network efficiency. The increasing ratio of the general efficiency of a specific SW network relative to that of the corresponding regular network is used to measure the SW effect quantitatively. The more considerable the SW effect, the higher the efficiency of a network with a certain cost is. Our results show that the SW effect increases monotonically with vertex number. When the vertex number is 30, there is no SW effect. When the vertex number is 3×10^3, the increasing ratio of the general efficiency reaches 0.23. The optimal rewiring probability p_{rewire} to induce the best SW effect is almost a constant, 0.02, and the optimal average connection probability p_{ave} decreases monotonically with the increase of vertex number. When the vertex number is 3×10^3, the optimal p_{ave} is as small as 0.025. Therefore, in order to induce the best SW effect, the vertex number should be large (> 500), p_{rewire} should be small (≈ 0.02) and the network should be sparse ($p_{ave} < 0.1$). The SW networks in many previous papers possess the three characteristics mentioned above,[1,32,33,37–40] and, therefore, our results in this paper are supported by the previous works.

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