# Phase transition of Bose–Einstein condensate under decoherence*

Zheng Qiang(郑 强)[a], Yi Shan-Feng(易善峰)[a], and Hu Chang-Gang(胡长刚)[b]†

[a] School of Mathematics and Computer Science, Guizhou Normal University, Guiyang 550001, China
[b] School of Chemistry and Material Science, Guizhou Normal University, Guiyang 550001, China

(Received 28 April 2013; revised manuscript received 16 July 2013; published online 10 December 2013)

The effect of decoherence on the phase transition of a Bose–Einstein condensate in a symmetric double-well potential is determined by the mean atom number difference. It still has two phases, the tunneling phase and the self-trapping phase, even under decoherence. The density matrix and the operator fidelity also show very different behaviors in the two phases. This suggests that operator fidelity can be used to characterize the phase transition of this Bose–Einstein condensate model, even under decoherence.

**Keywords:** Bose–Einstein condensate, phase transition, operator fidelity

**PACS:** 64.60.–i, 03.75.Mn, 03.75.Kk **DOI:** 10.1088/1674-1056/23/2/026401

## 1. Introduction

Since the Bose–Einstein condensate (BEC) has been realized in experiment, a new epoch for studying its dynamic properties has been launched. An exciting field of study is BEC in optical lattices.[1] Generally, optical lattices provide advantages in that a vast number of potentials can be created and they are almost completely controlled by changing the laser field. This feature is certainly not available in any other solid-state experiment, thereby, making the BEC in optical lattices an ideal quantum simulator[2] to verify the predictions of condensed matter theories. For example, the quantum phase transition[3] between a superfluid phase and Mott insulator phase, which was originally developed from the Bose–Hubbard model,[4] has been experimentally observed.[5]

Recently, motivated by the work of Zanardi[6] and Quan,[7] further investigations have been carried out, showing that fidelity at the critical point of quantum phase transition decays much faster than in regions far from the critical point.[8,9] One merit of this approach to characterizing the quantum phase transition is that it does not require any a priori knowledge about the correlation functions or order parameters because fidelity is a purely geometrical quantity in the Hilbert space.

Decoherence[10] inevitably occurs as a quantum system interacts with its external environment. There are some sources related to the decoherence of BEC. The first source is continuous exchanging particles between condensate and noncondensate atoms. The second one comes from the presence of lasers trapping or detecting atoms, which will couple them to the vacuum modes of the electromagnetic field. The study of the effect of decoherence on phase transition is an interesting topic.[11] Recently, the effect of decoherence on the phase transition of BEC in a double-well potential has been investigated.[12] Actually, it is not easy to fully describe decoherence. In the present paper, we adopt a relatively simple Milburn's decoherence model[13] to study its effect on the quantum phase transition of BEC in a double-well potential, which extends the mean-field level study in Ref. [12].

## 2. Milburn's decoherence model

According to Milburn's decoherence model, the dynamics of a system is governed by the master equation[13,14]

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\mathrm{i}[H,\rho(t)] - \frac{\gamma}{2}[H,[H,\rho(t)]]. \tag{1}$$

Here, the Planck constant is set to be unity (i.e., $\hbar = 1$) and the intrinsic decoherence parameter $\gamma$ is a constant. Milburn's decoherence model gives a simple modification of standard quantum mechanics and the coherence is automatically destroyed when the system evolves. In the limit $\gamma \to 0$, Milburn's equation reduces to the standard von Neumann equation.

The formal solution of Eq. (1) can be written as[15–17]

$$\rho(t) = \sum_{k=0}^{\infty} \frac{(\gamma t)^k}{k!} M(t)^k \rho(0) M^{\dagger}(t)^k, \tag{2}$$

with

$$M(t)^k = H^k \mathrm{e}^{-\mathrm{i}Ht} \exp\left(-\frac{\gamma t}{2}H^2\right), \tag{3}$$

and $\rho(0)$ is the initial state of the system. On the basis of the eigenvectors $|\varphi_n\rangle$ of $H$ with eigenvalues $E_n$, the element of the density matrix at any time $t$ becomes

$$\langle\varphi_n|\rho(t)|\varphi_m\rangle = \exp\left[\frac{-\gamma t}{2}(E_m - E_n)^2 - \mathrm{i}(E_m - E_n)t\right] \times \langle\varphi_m|\rho(0)|\varphi_n\rangle. \tag{4}$$

*Project supported by the National Natural Science Foundation of China (Grant No. 11065005) and the Creative Talent Programme in University of Guizhou Province, China.

†Corresponding author. E-mail: hudmtop01@sina.com

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When the density matrix $\rho(t)$ is given, all of the properties of the system can be determined. In the following, we will adopt Milburn's decoherence model to study the effect of decoherence on the phase transition of BEC in a double-well potential.

## 3. Phase transiton of BEC under decoherence

### 3.1. Model

Consider an $N$-atom BEC in a double-well potential, with the following second quantized Hamiltonian:[18]

$$H_{\rm BEC} = \frac{\gamma}{2}(a^\dagger a - b^\dagger b) + \frac{c}{2N}(a^\dagger a^\dagger aa + b^\dagger b^\dagger bb) - \frac{v}{2}(a^\dagger b + b^\dagger a). \quad (5)$$

Here $\gamma = E_a - E_b$ is the level separation between the two wells,

$$E_i = \int [(1/2m)|\nabla\varphi_i|^2 + V(r)|\varphi_i|^2|]\,{\rm d}r,$$

$c = (4\pi aN/m)\int |\varphi_i|^4\,{\rm d}r$ is the effective interaction constant of atoms, determined by the s-wave scattering length between atoms,

$$v = \int [(1/2m)\nabla\varphi_a\nabla\varphi_b + V(r)\varphi_a\varphi_b]\,{\rm d}r$$

is the coupling strength between the two wells, $\varphi_i$ $(i = a, b)$ is the wave function for each well, and $a^\dagger$ and $b^\dagger$ $(a,\ b)$ are boson creation (annihilation) operators for the two wells, respectively.

In the following, we focus on the symmetric double-well potential $\gamma = 0$, set the coupling strength $v$ to be unity, and consider the variation of $c$ ($c > 0$, referring to the repulsive interaction between atoms). Experimentally, the atom–atom interactions can be adjusted via Feshbach resonances.[19] The eigenvalues and eigenvectors of $H_{\rm BEC}$ are obtained by adopting direct numerical diagonalization[20] in the Fock space $|n\rangle$, which denotes that there are $n$ atoms in the first well. By substituting these eigenvalues and eigenvectors into Eq. (4), the density matrix of the system can be obtained. Throughout this paper, the initial state is $|0\rangle$.

### 3.2. Two phases under decoherence

As is well known, when the evolution of the system is unitary, the BEC system has two phases: a tunneling phase and a self-trapping phase.[21] For a relatively weak nonlinear term (i.e., small value of $c$), if all the atoms are initially in one of the two wells, then they can travel to other well by tunneling. On the other hand, for a sufficiently large $c$, the tunneling effect can be suppressed, resulting in a self-trapping phase. These two phases have been observed experimentally[22] and explained in detail analytically.[23] When the initial state of the system is $|0\rangle$, the critical value is $c_{\rm cr} = 2$ in the mean-field framework.[24] It is an interesting to consider the effect of decoherence on the dynamic phase transition of this BEC model.

Firstly, we consider the atom number in the first well

$$N_1 = {\rm Tr}[n\rho(t)]/N \quad (6)$$

and the mean atom number difference $s$

$$s = \{N_1 - N_2\} \quad (7)$$

to display the effect of decoherence on the phase transition of this model. Here $n = a^\dagger a$, $N_2 = 1 - N_1$, and $\{\cdot\}$ denotes the average over time. As an example, in Fig. 1 we plot the variations of $N_1$ with time $t$. Here, figure 1(a) corresponds to the tunneling phase and figure 1(b) to the self-trapping phase. The evolutions of $N_1$ in the unitary and decoherence cases are very similar, except that the amplitude in the latter case damps gradually. In order to clearly display the two phases, figure 2 plots the variation of $s$ with coupling strength $c$. The $s = 0$ corresponds to the tunneling phase and $s > 0$ to the self-trapping phase. The variation of $s$ for the unitary case and the decoherence case are almost indistinguishable. The system still has the two phases, even if its evolution is non-unitary.

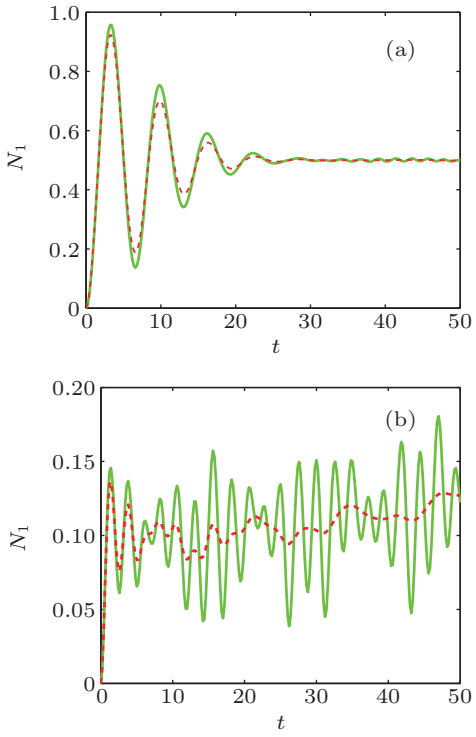



**Fig. 1.** (color online) Evolutions of mean atom number $N_1$ in the first well for (a) $c = 1$ and (b) $c = 3$. The solid and dash line correspond to the unitary $\gamma = 0$ and decoherence $\gamma = 0.1$, respectively.

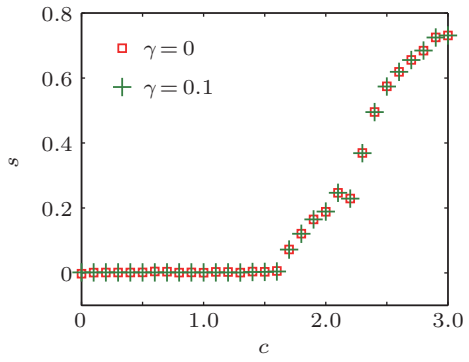



**Fig. 2.** (color online) Variation of mean atom number difference $s$ with coupling strength $c$. The square and the circle correspond to the unitary $\gamma = 0$ and decoherence $\gamma = 0.1$ cases, respectively. They are almost indistinguishable.

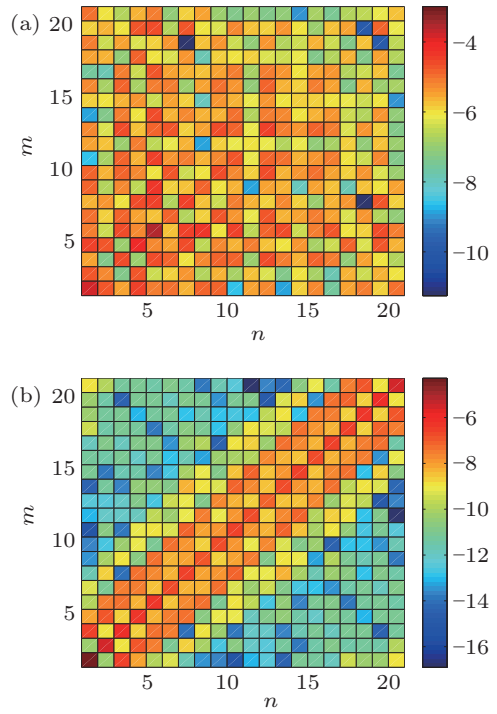



**Fig. 3.** (color online) Square of the amplitude of the density matrix $|\rho|^2$ in the Fock basis at time $t = 50$. The system is in the tunnelling phase with $c = 1$. Panels (a) and (b) correspond to $\gamma = 0$ and $\gamma = 0.1$, respectively. The $z$ axis is in the natural logarithmic scale.

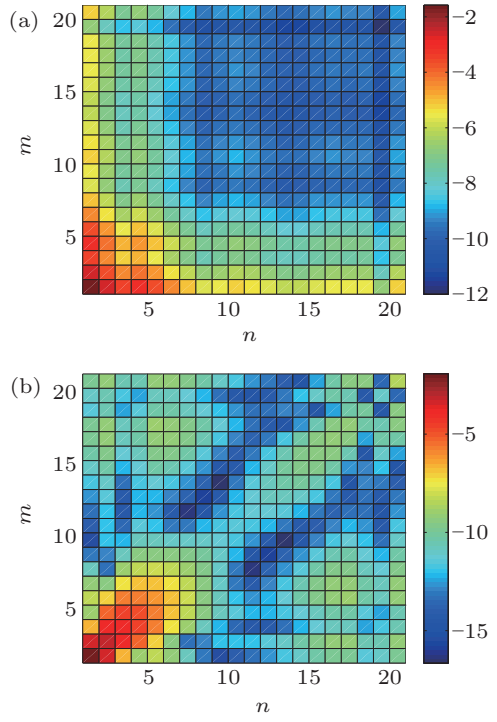



**Fig. 4.** (color online) The same as Fig. 3, except that the system is in the self-trapping phase with $c = 3$.

We also study the evolutions of $|\rho|^2$ under decoherence. Figures 3 and 4 correspond to the tunnelling phase and the self-trapping phase, respectively. In the tunnelling phase, $|\rho|^2$ is almost distributed in the whole Fock basis. The off-diagonal elements of $|\rho|^2$ imply the system with somewhat coherence. Under the effect of decoherence $\gamma = 0.1$, only the diagonal elements of density matrix remain. However, in the self-trapping phase, the matrix $|\rho|^2$ is localized in a limited Fock basis. The coherence in the self-trapping phase is also destroyed under decoherence, which is similar to that in the tunnelling phase.

For a closed BEC model, the state of the system will always be a pure state if the initial state is pure. Then, one can adopt a Loschmidt echo to characterize the dynamical phase transition.[25] When the state of the system becomes a mixed state under decoherence, the Loschmidt echo does not work. Fortunately, for a mixed state we can adopt operator fidelity, defined as[26] $f = \mathrm{Tr}[\rho_\varepsilon^\dagger \rho]$, with $\rho_\varepsilon$ corresponding to the density matrix of the perturbation system and $\dagger$ denoting the Hermite conjugate. To characterize the phase transition of this BEC system under decoherence, we adopt the operator fidelity

$$f = \mathrm{Tr}[\rho_{c'}^\dagger \rho_c]. \tag{8}$$

Here $c' = c + \delta c$.

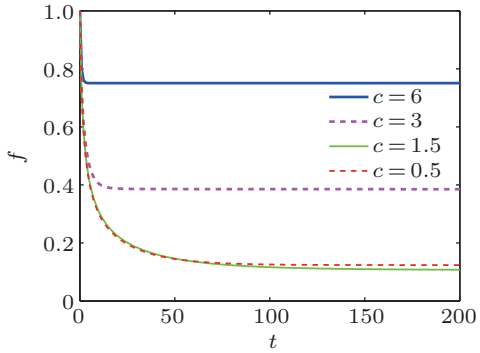



**Fig. 5.** (color online) Evolution of operator fidelity $f$ under the different coupling strength $c$, with perturbation strength $\delta c = 10^{-3}$ and decoherence strength $\gamma = 0.05$.

For example, figure 5 presents the time evolutions of $f$ with $c = 0.5, 1.5, 3, 6$. The evolutions of the operator fidelity $f$ show similar decays in the two phases: an initially fast decay and a following steady value $f_{\mathrm{steady}}$. However, the values of $f_{\mathrm{steady}}$ display different behaviors in the two phases, as shown in Fig. 6. In the tunnelling phase, the values of $f_{\mathrm{steady}}$ are approximately identical; while in the self-trapping phase, $f_{\mathrm{steady}}$ is approximately linearly dependent on $c$. Extensively numerical simulations indicate that the relationship shown in Fig. 6 is independent of $\gamma$ and $\delta c$ if the value of $\delta c$ is less than $10^{-1}$ (not shown here). The upper numerical results can be understood with the following arguments. From Eq. (4), it is easy to see that with a small $t$, the unitary term is dominant. For a large $t$, the main contribution comes from factor containing the term $\gamma t$, and the unitary property is completely destroyed and the density matrix of the system approaches to a steady state under decoherence. In this case, the operator fidelity becomes constant.

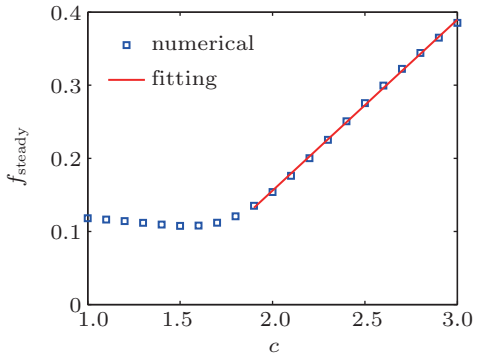



**Fig. 6.** (color online) Variation of the steady value of operator fidelity $f_{\text{steady}}$ with coupling strength $c$. The squares denote the numerical results, and the solid line represents the fitting one. The parameters are the same as those in Fig. 3.

The upper linear relationship can also be understood approximately in the following. Figure 2 shows approximately $s \propto c$. Note $s \propto N_1 \sim \text{Tr}[n\rho(t)]$. Taking into consideration the facts that (i) both $\rho_{c'}^{\dagger}$ and $n$ are operators, (ii) in the self-trapping phase, the state of the system is populated only in some limited Fock states with the increase of the interaction strength $c$, and (iii) for a sufficiently long time, the coherence of density matrix of BEC is completely destroyed by the decoherence factor, we can approximate $n$ as $\rho_{c'}$, $\rho_{c'} \sim n$. Then, the operator fidelity $f$ in Eq. (8) has a similar form to that of $N_1$. Therefore, $f_{\text{steady}}$ is linearly dependent on $c$.

## 4. Conclusion

In conclusion, the operator fidelity of BEC in a symmetric double-well potential under decoherence is studied by numerical simulations. This BEC system has two phases, even under decoherence; that is, the tunneling phase and the self-trapping phase, which is indicated by the mean atom number difference and the density matrix in Fock basis. Moreover, the operator fidelity of the steady state is an approximately constant in the tunneling phase, while it increases linearly with the increase of atom interaction strength in the self-trapping phase. The operator fidelity is one of candidates for characterizing the dynamical phase transition of the BEC model under decoherence.

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