Heat transfer for boundary layers with cross flow

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An analysis is presented to study the dual nature of solutions for the forced convective boundary layer flow and heat transfer in a cross flow with viscous dissipation terms in the energy equation. The governing equations are transformed into a set of three self-similar ordinary differential equations by similarity transformations. These equations are solved numerically using the very efficient shooting method. This study reveals that the dual solutions of the transformed similarity equations for velocity and temperature distributions exist for certain values of the moving parameter, Prandtl number, and Eckert numbers. The reverse heat flux is observed for larger Eckert numbers; that is, heat absorption at the wall occurs.

Keywords: heat transfer, boundary layer, cross flow, viscous dissipation, dual solutions

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1. Introduction

The usual way in which a heat convection problem is modeled is, in general, to solve the differential equation governing the conservation of energy by assuming that the flow is driven either by a surface temperature or by a prescribed surface heat flux. A large number of calculations have been performed, especially for the large Reynolds number (or boundary layer) limit, for such surface conditions and for various geometrical configurations. Much of this work is well described in Bejan,[1] Schlichting and Gersten,[2] and Pop and Ingham.[3] A somewhat different driving mechanism for the forced convection heat transfer boundary layer flow was proposed by Kubitschek and Weidman,[4] and was also recently used by Aziz[5] and Magyari[6] to investigate the bottom surface of a plate that is heated by convection from a hot fluid at temperature T_f, which provides a heat transfer coefficient h_f.

On the other hand, the interesting cross flow problem over Blasius boundary layer was described by Sowerby[7] and Loos.[8] Klemp and Acrivos[9,10] demonstrated the boundary layer flow with a reverse flow region. Hussaini *et al.*[11] studied the boundary layer on a flat plate with a constant velocity opposite in the direction to that of uniform mainstream. Weidman[12] investigated the transverse flow over a flat plate moving with, or against, the uniform stream. The case of simultaneous effects of mass flux (suction) and tangential movement on the two-dimensional forced convection boundary layer flow (momentum boundary layer) passing through a semi-infinite flat plate beneath a uniform stream was also considered by Weidman *et al.*[13] Relating to the reverse flow, some other important investigations can be found in Refs. [14]–[37].

The main objective of the present investigation is to consider the forced convection heat transfer (thermal boundary layer) in the presence of viscous dissipation over a flat plate moving with a uniform velocity into and out of the origin beneath a uniform stream of a three-dimensional (streamwise and transverse) flow. The momentum (velocity) boundary layer of this problem has been considered by Weidman.[12] The governing partial differential equations (PDEs) are transformed into nonlinear self-similar ordinary differential equations (ODEs), which are then solved by the shooting method. Then, to explain the heat transfer characteristics, the computed numerical results are plotted in some figures.

2. Basic equations

Following Weidman,[12] we consider a three-dimensional uniform (streamwise and transverse) flow over a heated flat plate moving with the uniform velocity $-\lambda U$ into and out of the origin located at $x = 0$, where x is the coordinate measured along the flat plate, λ is a dimensionless constant (say, moving parameter), and U is the constant velocity of the outer flow. It is assumed that the constant temperature of the plate is T_w, while the uniform temperature of the free stream (inviscid fluid) is T_∞. It is also assumed that the cross-flow is of infinite extent in the spanwise direction and, thus, fully-developed. Hence, the velocity and temperature fields are independent of the coordinate z, so that the boundary layer equations with the viscous dissipation terms included into the energy equations can be written as (see Bejan[1] and Weidman[12])

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} = \nu\frac{\partial^2 w}{\partial y^2}, \tag{3}$$

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$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 w}{\partial y^2}+\frac{\nu}{c_p}\left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right], \quad (4)$$

along with the boundary conditions

$$\begin{cases} u=-\lambda U, \ v=0, \ w=0, \ T=T_{\rm w} \quad \text{at} \quad y=0, \\ u=U, \ w=W_0, \ T=T_\infty \quad \text{as} \quad y\to\infty. \end{cases} \quad (5)$$

Here, y is the normal coordinate in the plane of the flow, z is the spanwise coordinate in the direction of the transverse flow, u, v, and w are the velocity components along x, y, and z axes, respectively, W_0 is the transverse velocity at the origin (e.g. $w=W_0$ at $x=0$ for $y>0$), T is the fluid temperature, α is the thermal diffusivity of the fluid, ν is the kinematic viscosity of the fluid, and $c_{\rm p}$ is the specific heat at constant pressure. It is worth pointing out that for $\lambda<0$ the plate moves out of the origin $x=0$ and for $\lambda>0$ the plate moves into the origin.

Following Weidman,[12] we look for a similarity solution of Eqs. (1)–(4) with the boundary conditions (Eq. (5)) of the following form:

$$\begin{cases} \psi=\sqrt{2\nu U x}f(\eta), \ w=W_0 g(\eta), \\ \theta(\eta)=(T-T_\infty)/(T_{\rm w}-T_\infty), \ \eta=y\sqrt{U/2\nu x}, \end{cases} \quad (6)$$

where ψ is the stream function defined in the usual notation as $u=\partial\psi/\partial y$ and $v=-\partial\psi/\partial x$, and η is the similarity variable.

Substituting Eq. (6) into Eqs. (2)–(4), we obtain the following ODEs:

$$f'''+ff''=0, \quad (7)$$

$$g''+fg'=0, \quad (8)$$

$$\frac{1}{Pr}\theta''+f\theta'+Ec_1 f''^2+Ec_2 g'^2=0, \quad (9)$$

and the boundary conditions (Eq. (5)) become

$$\begin{cases} f(\eta)=0, \ f'(\eta)=-\lambda, \ g(\eta)=0, \ \theta(\eta)=1 \ \text{at} \ \eta=0, \\ f'(\eta)\to 1, \ g(\eta)\to 1, \ \theta(\eta)\to 0 \ \text{as} \ \eta\to\infty, \end{cases} \quad (10)$$

where primes denote differentiation with respect to η, $Pr=\nu/\alpha$ is the Prandtl number, and Ec_1 and Ec_2 are the Eckert numbers, which are defined as

$$Ec_1=\frac{U^2}{c_{\rm p}(T_{\rm w}-T_\infty)}, \ Ec_2=\frac{W_0^2}{c_{\rm p}(T_{\rm w}-T_\infty)}. \quad (11)$$

The physical quantities of interest are the skin friction coefficient for the streamwise flow $C_{\rm fx}$, the skin friction coefficient for the cross flow $C_{\rm fz}$, and the local Nusselt number Nu_x, which are defined as

$$C_{\rm fx}=\frac{\tau_{\rm wx}}{\rho U^2}, \quad C_{\rm fz}=\frac{\tau_{\rm wz}}{\rho W_0^2}, \quad Nu_x=\frac{xq_{\rm w}}{k(T_{\rm w}-T_\infty)}, \quad (12)$$

where ρ is the density, k is the thermal conductivity of the fluid, $\tau_{\rm wx}$ and $\tau_{\rm wz}$ are the wall shear stresses along the streamwise direction and along the transverse direction, respectively, and $q_{\rm w}$ is the heat flux from the flat plate, which are defined as

$$\tau_{\rm wx}=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}, \quad \tau_{\rm wz}=\mu\left(\frac{\partial w}{\partial y}\right)_{y=0},$$

$$q_{\rm w}=-k\left(\frac{\partial T}{\partial y}\right)_{y=0}. \quad (13)$$

By substituting Eq. (6) into Eq. (13) and using Eq. (12), we obtain

$$(2Re_x)^{1/2}C_{\rm fx}=f''(0), \quad (2Re_x)^{1/2}(W_0/U)C_{\rm fz}=g'(0),$$
$$(Re_x/2)^{1/2}Nu_x=-\theta'(0), \quad (14)$$

where $Re_x=Ux/\nu$ is the local Reynolds number.

3. Numerical solution and discussion

As we know, the decoupled ordinary differential equations (ODEs) (7)–(9) subject to the boundary conditions (7) form a two-point boundary value problem (BVP), which is solved numerically using the shooting method,[38–44] by converting it into an initial value problem (IVP). In this method we need to choose a suitable finite value of $\eta\to\eta_\infty$, say $\eta=\eta_\infty$. We set

$$f'=p, \ p'=q, \ q'=-fq, \quad (15)$$

$$g'=h, \ h'=-fh, \quad (16)$$

$$\theta'=z, \ z'=-Pr(fz+Ec_1 q^2+Ec_2 h^2), \quad (17)$$

with the boundary conditions

$$f(0)=0, \ p(0)=-\lambda, \ g(0)=0 \ \text{and} \ \theta(0)=1. \quad (18)$$

In order to integrate Eqs. (15)–(17) with Eq. (18) as an initial value problem the values for $q(0)$, that is, $f''(0)$; $h(0)$, that is, $g'(0)$, and $z(0)$, that is, $\theta'(0)$ are required but no such values are given in the boundary. The suitable guess values for $f''(0)$, $g'(0)$, and $\theta'(0)$ are chosen and then an integration is carried out. We compare the calculated values for $f'(\eta)$, $g(\eta)$, and $\theta(\eta)$ at $\eta=\eta_\infty(=20)$ with given boundary conditions $f'(\eta_\infty)=1$, $g(\eta_\infty)=1$, and $\theta(\eta_\infty)=0$ and adjust the estimated values, $f''(0)$, $g'(0)$, and $\theta'(0)$, to give a better approximation for the solution. We take the series of values for $f''(0)$, $g'(0)$, and $\theta'(0)$, and apply the fourth order classical Runge–Kutta method with step-size $\Delta\eta=0.01$. The above procedure is repeated until we obtain the converged results within a tolerance limit of 10^{-5}.

Using the above numerical scheme, solutions are obtained for some values of the moving parameter λ, Prandtl number Pr, and the Eckert numbers Ec_1 and Ec_2. The dual solutions are obtained by setting different initial guesses for the values of the reduced skin friction coefficients $f''(0)$, $g'(0)$, and the reduced local Nusselt number $-\theta'(0)$, where all profiles satisfy the far field boundary conditions (10) asymptotically but with different shapes. The results are given to carry out a parametric study showing influences of several of these

non-dimensional parameters. It is found that, for $\lambda < 0$, solutions of Eqs. (7)–(9) along with the boundary conditions (10) can be obtained while for $\lambda \geq 0$ these equations have solutions only in the range $0 \leq \lambda \leq \lambda_c$, where $\lambda_c(> 0)$ is a critical value of λ, so that for $\lambda > \lambda_c(> 0)$, equations (7)–(9) have no solutions. However, for λ in the range $0 \leq \lambda \leq \lambda_c$, the solution is not unique, there being two (dual) solutions, upper and lower branch solutions for a given value of λ. This can be seen from Figs. 1–4, where $f''(0)$, $g'(0)$, and $-\theta'(0)$ are plotted each as a function of λ. The values of λ_c for $f''(0)$ and $g'(0)$ are both 0.3541, that is, $\lambda_c = 0.3541$, which is in perfect agreement with the value found by Weidman.[12] Therefore, we are confident that the present numerical results are very accurate. Furthermore, it is seen from Figs. 2–4 that the value of λ_c for $-\theta'(0)$ is $\lambda_c = 0.3541$ for any value of Pr, Ec_1, and Ec_2; that is, the value of λ_c is the same as that for $f''(0)$ and $g'(0)$. Therefore, the range of the dual solution does not change for any change of the parameters in Eq. (9) and even in Eq. (8) (though there is no such parameter in Eq. (8)). The dual solution range for forced convective flow changes only when there are some other external forces imposed on the flow field-like magnetic field, suction/blowing, resistance due to porous medium, etc., for which some extra terms are added either in Eq. (7) or in the streamwise velocity boundary conditions. Furthermore, from Fig. 2, it is observed that in the case of dual solutions ($\lambda \geq 0$) the value of $-\theta'(0)$ decreases with the increase of Pr for both solution branches but for the unique solution case ($\lambda < 0$), $-\theta'(0)$ increases with the increase of Pr. In addition, for the increase of the Eckert numbers Ec_1 and Ec_2, the value of $-\theta'(0)$ decreases for upper branch solution as well as for lower branch solution. So, due to the increase of Eckert number the rate of heat transfer reduces; that is, the viscous dissipation causes the reduction in the rate of heat transfer. From Fig. 3, it is important to note that for $\lambda = -1$ the Eckert number Ec_1 does not affect the heat transfer. Actually, for $\lambda = -1$, equation (7) has an exact solution in the form $f(\eta) = \eta$ and equation (9) becomes independent of Ec_1.

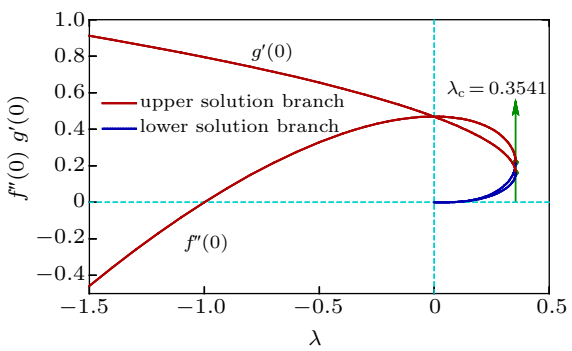


Fig. 1. (color online) Variations of $f''(0)$ and $g'(0)$ with λ.

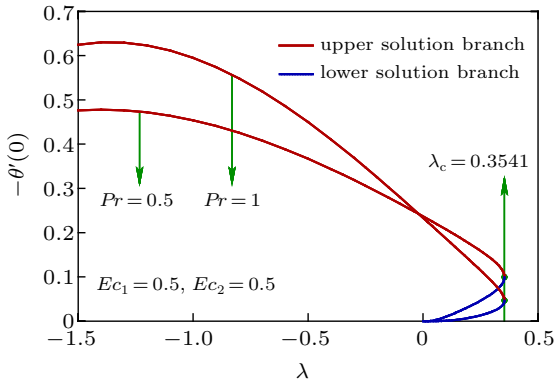


Fig. 2. (color online) Variations of $-\theta'(0)$ with λ for different values of Pr.

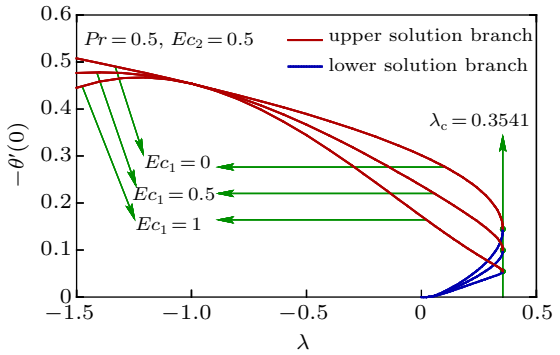


Fig. 3. (color online) Variations of $-\theta'(0)$ with λ for different values of Ec_1.

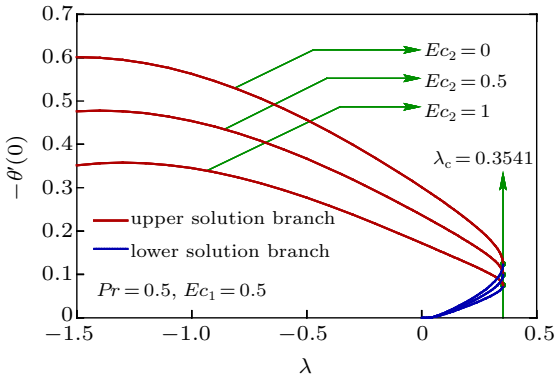


Fig. 4. (color online) Variations of $-\theta'(0)$ with λ for different values of Ec_2.

We expect that the upper branch solution is stable, while the lower branch solution is not. It is, however, worth mentioning that according to the studies by Weidman *et al.*,[13,45] Merrill and Weidman,[46] and Postelnicu and Pop,[47] the upper branch solution is stable and physically realizable because the upper branch solution is the only solution when $\lambda > 0$, while the lower branch solution is not stable. Although the lower branch solution is less important physically for this problem, from a mathematical point of view this lower branch solution is very important because a similar equation may appear in some other physical problems where the lower branch solution acquires proper physical meaning.

The dimensionless streamwise and cross velocity profiles $f'(\eta)$ and $g(\eta)$ are plotted for several values of the moving parameter $\lambda(> 0)$ in Figs. 5 and 6. On the other hand, figures 8–11(b) show the temperature distributions for several values of the moving parameter $\lambda(> 0)$, Prandtl number Pr,

and the Eckert numbers Ec_1 and Ec_2. It is seen from Fig. 5 that the momentum boundary layer thickness increases with the increase of the moving parameter λ for the upper branch solutions, while the momentum boundary layer thickness decreases with the increase of $\lambda(>0)$ for the lower branch solutions. It should be noted that the momentum boundary layer thickness for lower branch solution is larger than the thickness for upper branch solution. The dimensionless reduced stream function $f(\eta)$ (for $\lambda > 0$) in Fig. 7 initially decreases to some negative values and then it increases from negative to positive; that is, the profiles exhibit a reverse flow region. This happens because of the opposite directions of the free stream velocity and the wall velocity. The dual temperature profiles $\theta(\eta)$ for some of the values of the parameters $\lambda(>0)$, Prandtl number Pr, and the Eckert numbers Ec_1 are illustrated in Figs. 8–11(b). It is noticed that, for fixed values of Pr, Ec_1, and Ec_2 the behavior of the temperature profile $\theta(\eta)$ shown in Fig. 8 is similar to that of the velocity profile $f'(\eta)$ shown in Fig. 5; that is, the thermal boundary layer thickness increases with the increase of the moving parameter $\lambda(>0)$ for the upper branch solution, while the thermal boundary layer thickness decreases with the increase of $\lambda(>0)$ for the lower branch solution. It should also be noticed that the thermal boundary layer thickness for lower branch solution is larger than the thickness for upper branch solution. However, for fixed values of λ, Ec_1, and Ec_2 the temperature at a point decreases monotonically with the increase of the Prandtl number Pr for the upper branch solution (except very near the wall), while it first increases and then decreases for the lower branch solution, as can be seen from Fig. 9. What should also be noticed is that the thermal boundary layer thickness decreases with the increase of Pr. Finally, figures 10(a)–11(b) demonstrate that for fixed values of λ and Pr the thermal boundary layer thickness increases with the increase of the parameters Ec_1 and Ec_2. From Figs. 10(b) and 11(b) it follows that for higher values of Ec_1 and Ec_2, the thermal overshoot occurs in both solution branches. So, for large values of Ec_1 and Ec_2, the temperature in the ambient fluid is higher than the wall temperature, consequently heat absorption at the surface occurs; that is, the direction of heat flux is changed.

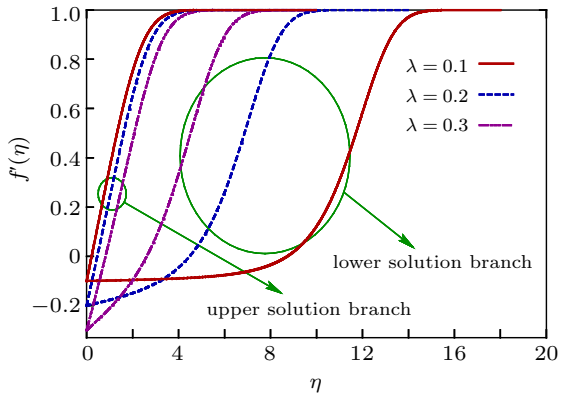


Fig. 5. (color online) Plots of streamwise velocity profile $f'(\eta)$ for different values of λ (dual solutions).

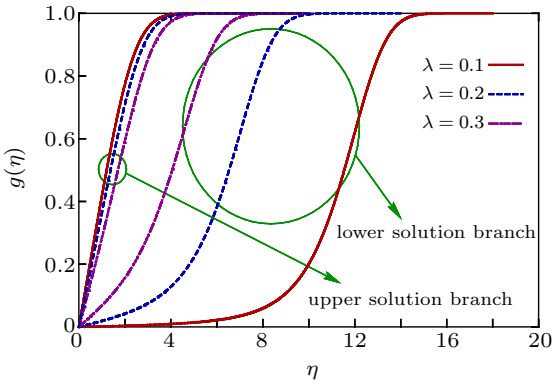


Fig. 6. (color online) Plots of cross-flow velocity profile $g(\eta)$ for different values of λ (dual solutions).

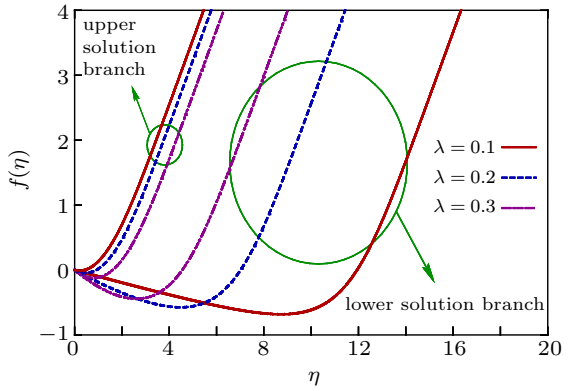


Fig. 7. (color online) Plots of reduced stream function $f(\eta)$ for different values of λ (dual solutions).

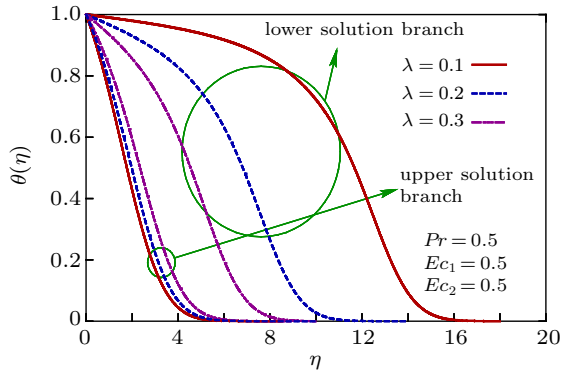


Fig. 8. (color online) Plots of temperature profile $\theta(\eta)$ for different values of λ (dual solutions).

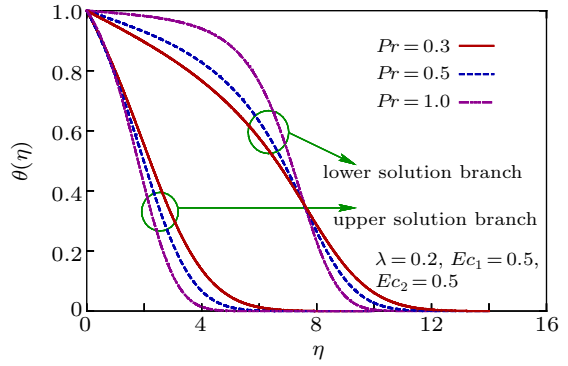


Fig. 9. (color online) Plots of temperature profile $\theta(\eta)$ for different values of Pr (dual solutions).

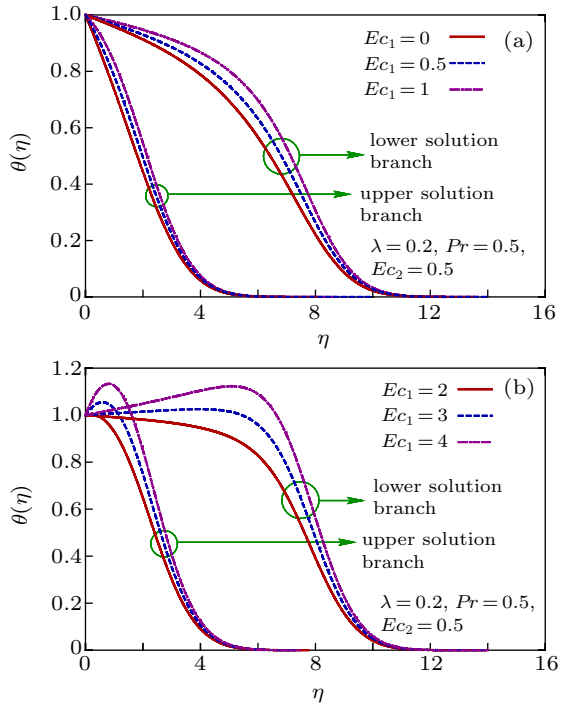


Fig. 10. (color online) Plots of temperature profile $\theta(\eta)$ for (a) $Ec_1 = 0$, 0.5, and 1, and (b) $Ec_1 = 2$, 3, and 4 (dual solutions).

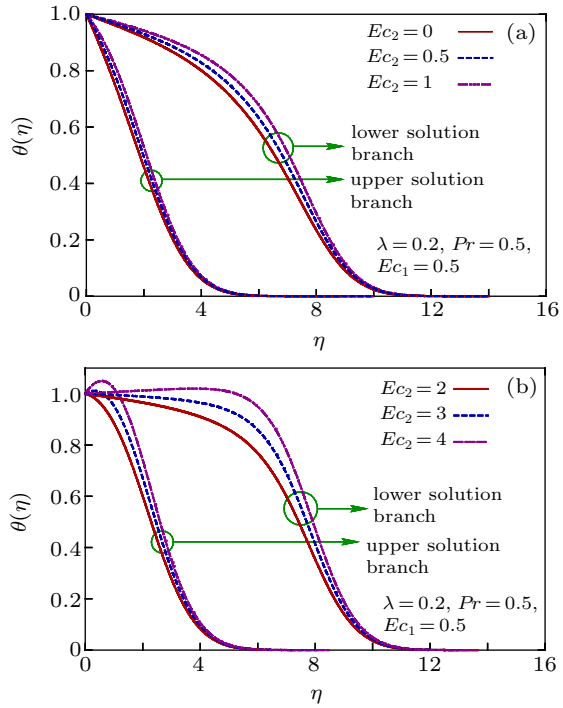


Fig. 11. (color online) Plots of temperature profile $\theta(\eta)$ for (a) $Ec_2 = 0$, 0.5, and 1, and (b) $Ec_2 = 2$, 3, and 4 (dual solutions).

4. Conclusions

The heat transfer in a cross flow with viscous dissipation effect is investigated. The similarity transformations are used to convert the governing equations. Then, the transformed ODEs are solved numerically using the shooting method. The dual solutions (depending on the moving parameter) of the temperature distribution are obtained in addition to the velocity field. It is also observed that, due to viscous dissipation, the rate of the heat transfer decreases. In the presence of viscous dissipation, the increase of Prandtl number results in a decrease in the rate of heat transfer for the dual solution case. For larger values of Eckert numbers, heat absorption at the surface occurs.

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References

[1] Bejan A 2004 *Convective Heat Transfer*, 3rd edn. (New York: Wiley)
[2] Schlichting H and Gersten K 2000 *Boundary-Layer Theory* (New York: Springer)
[3] Pop I and Ingham D B 2001 *Convective Heat Transfer: Mathematical and Computational Modelling of Viscous Fluids and Porous Media* (Oxford: Pergamon)
[4] Kubitschek J P and Weidman P D 2003 *Int. J. Heat Mass Transfer* **46** 3697
[5] Aziz A 2009 *Commun. Nonlinear Sci. Numer. Simul.* **14** 1064
[6] Magyari E 2011 *Commun. Nonlinear Sci. Numer. Simul.* **16** 599
[7] Sowerby L 1954 *Rep. Aero. Res. Coun. Lond.*, No. 16832
[8] Loos H G 1955 *J. Acro. Sci.* **22** 35
[9] Klemp J B and Acrivos A A 1972 *J. Fluid Mech.* **53** 177
[10] Klemp J B and Acrivos A A 1976 *J. Fluid Mech.* **76** 363
[11] Hussaini M Y, Lakin W D and Nachman A 1987 *SIAM J. Appl. Math.* **47** 699
[12] Weidman P D 1997 *J. Appl. Math. Phys. (ZAMP)* **48** 341
[13] Weidman P D, Kubitschek D G and Davis A M J 2006 *Int. J. Eng. Sci.* **44** 730
[14] Fang T 2003 *Acta Mech.* **163** 161
[15] Fang T 2003 *Acta Mech.* **163** 183
[16] Pop I, Ishak A and Nazar R 2007 *Chin. Phys. Lett.* **24** 2274
[17] Pop I, Nazar R and Ishak A 2007 *Chin. Phys. Lett.* **24** 2895
[18] Fang T and Lee C F 2009 *Acta Mech.* **204** 235
[19] Ishak A 2009 *Chin. Phys. Lett.* **26** 034701
[20] Zhang J, Fang T and Yao S S 2009 *Chin. Phys. Lett.* **26** 014703
[21] Fang T, Zhang J and Yao S S 2010 *Chin. Phys. Lett.* **27** 124702
[22] Ishak A, Lok Y Y and Pop I 2010 *Chem. Eng. Commun.* **197** 1417
[23] Bhattacharyya K and Layek G C 2011 *Int. J. Heat Mass Transfer* **54** 302
[24] Bhattacharyya K, Mukhopadhyay S and Layek G C 2011 *Int. J. Heat Mass Transfer* **54** 308
[25] Lok Y Y, Ishak A and Pop I 2011 *Int. J. Numer. Meth. Heat Fluid Flow* **21** 61
[26] Yacob N A, Ishak A and Pop I 2011 *Comput. Fluids* **47** 16
[27] Mahapatra T R, Nandy S K and Gupta A S 2011 *ASME J. Appl. Mech.* **78** 021015
[28] Bhattacharyya K 2011 *Chin. Phys. Lett.* **28** 084702
[29] Bhattacharyya K 2011 *Int. Commun. Heat Mass Transfer* **38** 917
[30] Rosali H, Ishak A and Pop I 2011 *Int. Commun. Heat Mass Transfer* **38** 1029
[31] Bhattacharyya K and Vajravelu K 2012 *Commun. Nonlinear Sci. Numer. Simul.* **17** 2728
[32] Mukhopadhyay S, Bhattacharyya K and Layek G C 2011 *Int. J. Heat Mass Transfer* **54** 2751
[33] Bhattacharyya K 2012 *Int. J. Heat Mass Transfer* **55** 3482
[34] Bhattacharyya K, Layek G C and Gorla R S R 2012 *Int. J. Fluid Mech. Res.* **39** 438
[35] Bhattacharyya K 2013 *Ain Shams Eng. J.* **4** 259
[36] Bhattacharyya K 2013 *Chin. Phys. B* **22** 074705

[37] Bhattacharyya K, Hayat T and Alsaedi A 2013 *Chin. Phys. B* **22** 024702
[38] Bhattacharyya K, Mukhopadhyay S and Layek G C 2011 *Chin. Phys. Lett.* **28** 024701
[39] Bhattacharyya K 2011 *Chin. Phys. Lett.* **28** 074701
[40] Bhattacharyya K and Layek G C 2011 *Chem. Eng. Commun.* **198** 1354
[41] Bhattacharyya K and Layek G C 2011 *Chin. Phys. Lett.* **28** 084705
[42] Bhattacharyya K, Mukhopadhyay S and Layek G C 2011 *Chin. Phys. Lett.* **28** 094702
[43] Bhattacharyya K and Pop I 2011 *Magnetohydrodynamics* **47** 337
[44] Bhattacharyya K, Mukhopadhyay S, Layek G C and Pop I 2012 *Int. J. Heat Mass Transfer* **55** 2945
[45] Weidman P D, Davis A M J and Kubitschek D G 2008 *J. Appl. Math. Phys.* (ZAMP) **59** 313
[46] Paullet J and Weidman P 2007 *Int. J. Nonlinear Mech.* **42** 1084
[47] Postelnicu A and Pop I 2011 *Appl. Math. Comput.* **217** 4359