Maximal entanglement from photon-added nonlinear coherent states via unitary beam splitters

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In this paper, we construct photon-added f-deformed coherent states (PAf-DCSs) for nonlinear bosonic fields by discussing Klauder's minimal set of conditions required to obtain coherent states. Using this set of nonlinear states, we propose a very useful scheme for generating the maximal amount of entanglement via unitary beam splitters for different strength regimes of the input field α, deformation q and excitation number m. Therefore, the possibility to create highly entangled states and to control the entanglement is proposed. Moreover, the condition for a maximal and separable output beam state is obtained. Finally, we examine the statistical properties of the PAf-DCSs through the Mandel parameter and exploit a connection between this quantity and the behavior variation of the output state entanglement. Our result may open new perspectives in different tasks of quantum information processing.

Keywords: photon-added nonlinear coherent states, beam splitter, linear entropy, entanglement, statistical properties

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1. Introduction

In quantum mechanics, it is possible to characterize the maximal knowledge of a composite physical system and still not be able to assign a definite state to its constituent elements without reference to their relation to each other.[1,2] Such quantum system states are known as entangled states. Entanglement is a fascinating property of quantum systems, that has various important consequences for modern physics applications. It is a nonlocal correlation between distinct quantum systems, which has been referred to by Schrödinger in his 1935 discussion of the issues raised by the paradox of Einstein, Podolsky and Rosen[1,2] as the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought.[2] Entanglement lies at the heart of many problems in quantum mechanics and has attracted increasing attention in recent years, which has led to the development of different branches of physics.[3–10] Actually, it has emerged as a major resource for realizing several quantum protocol in several quantum information processing tasks, such as teleportation and certain quantum cryptographic protocols. It also plays an important role in quantum computing, making it possible that quantum computers can outperform their classical counterparts for several problems, such as prime factoring or searching. Furthermore, the generation of quantum entanglement naturally arises as goals in nowadays quantum control experiments when studying the nonclassical phenomena in quantum mechanics. In an experiment when entanglement is created, it is important to detect it. Thus, in many quantum physics experiments, the creation of an entangled state is followed by measurements. Based on the results of these measurements, the experimenters conclude that the produced state was entangled. However, in many-particle experiments, the possibilities for quantum control are very limited. In particular, the particles cannot be individually addressed. In such systems, the entanglement can be created and detected with collective operations.

Recently, various devices have been proposed and realized experimentally to generate quantum entanglement, such as a beam splitter,[11,12] cavity QED,[13] NMR systems,[14] etc. The entanglement generated by a beam splitter has been studied in considerable detail by Kim *et al.*[15] They showed that, in order to obtain entangled output states of a beam splitter, a necessary condition is that at least one of the input fields should be nonclassical. It was recently shown by Ivan *et al.* that if the input state to a beam splitter is a product state of the vacuum in one mode and a state with a sub-Poissonian distribution, then the output state will be entangled.[16] More recently, this result has been generalized to the states with super-Poissonian and sub-Poissonian statistics, by investigating the statistical properties of generalized spin coherent states for different kinds of deformations.[17]

Another concept widely used and applied in quantum information theory is the notion of coherent or quasiclassical states. These states make a very useful tool for the investigation of various problems in physics. Coherent states were first introduced by Schrödinger[18] in the context of the harmonic

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oscillator, who was interesting in finding quantum states which provide a close connection between quantum and classical formulations of a given physical system. Later, the notion of coherence has became very important in quantum optics due to Glauber,[19] as eigenstates of the annihilation operator $\hat{a}$ of the harmonic oscillator, while he demonstrated that these states have the interesting property of minimizing the Heisenberg uncertainty relation. On the other hand, the quantum groups were introduced as a mathematical description of deformed Lie algebra that gave the possibility to construct deformed coherent states. They were introduced as a natural extension of the notion of coherent states.[20,21] Recently, these states have attracted a lot of attention due to their possible applications in various branches of physics.[22–25] Such states exhibit some nonclassical properties, such as photon antibunching,[26] sub-Poissonian photon statistics[27] and squeezing,[28,29] etc. (for a review see Ref. [21]). Moreover, it has been experimentally observed that real laser, bunched and antibunched light possess a photon number statistics that can be super-Poissonian or sub-Poissonian.[30,31]

On the other hand, the search for physically inspired Hamiltonians, which may display definite features about q-deformation effects, is still open. Concrete applications of the formalism have been explored more recently[32–35] and these complement previous mathematical efforts, such as studies of generalized q-deformed oscillators.[36,37] In a series of papers, Wess *et al.*[38–40] and Zhang *et al.*[41–43] studied different nonstandard q-deformation schemes and obtained the corresponding Hamiltonians. Previously, these ideas were applied to study fermion–boson couplings[44] and the possible correspondence between q-deformations and boundary conditions,[32] by comparing the spectrum of a q-deformed harmonic oscillator with that of a finite-range potential. Interestingly, the link between the q-deformation and the finite range structure of the resulting Hamiltonian is found, where the existence of such a correspondence is shown, for the case of a one-dimensional harmonic oscillator. There, the effect of the q-deformation materializes in a certain region of the spectrum, which looks like the one belonging to a one-dimensional finite range potential.[32] In this manner, we aim at fixing realistic values of q, which is a non-observable quantity, from observables, such as the depth of the potential and its range. The optimum values of the q-parameter have achieved an overall improved fit to the experimental energies related to the Hamiltonian, by exploiting the physical nature of q deformation when applied to the bosonic fields. Moreover, it is shown that the q-deformation plays a significant role in understanding higher-order effects in the many-body interaction.[45] Then, the notion of deformed coherent states was straightforwardly introduced and the generation of such nonlinear coherent states became real possibilities of trapped systems.

Some years ago, Katriel and Solomon[46] described applications to the characterization of the photon statistics of nonideal lasers, nonclassical light and deformed photon states using f-deformed coherent states. Moreover, Mancini and Man'ko[47] studied the dissipative dynamics of f-deformed coherent states superposition. They found that such a kind of superposition can be more robust against decoherence than the usual Schrödinger cat states. Here, we are going to build the PAf-DCSs, exploiting their properties in the creation of nonlocal correlation using the effect of a beam splitter and show that these states exhibit a richer structure for generating maximal entanglement by a proper choice of the different parameters involved in nonlinear coherent states. Furthermore, we study the statistical properties of the input optical field across Mandel's parameter and giving the physical interpretation on the behavior variation of the output state entanglement. In particular, we shall show that the Mandel parameter may be used as an indicator of the entanglement behavior in this scheme.

This paper is organized as follows. In Section 2, we describe the construction of PAf-DCSs by analyzing the minimal set of conditions to obtain Klauder's coherent states. In Section 3, we investigate the entanglement generation from a beam splitter when an input state is comprised of a PAf-DCS in one input arm and a vacuum state in the other arm, in terms of different parameters involved in the state and giving the condition for the separability and maximal entanglement of the output state. In Section 4, we examine the statistical properties of the PAf-DCSs and provide the connection between the nonlinearity of the input field and the output state entanglement. In Section 5, we summarize the main results.

2. Photon-added nonlinear bosonic coherent states

In this work, we are interested to explain f-deformed Heisenberg–Weyl algebra defined by the algebra generated by the operator $\{1, A, A^\dagger, N\}$. The f-deformed Heisenberg–Weyl operators are defined as follows:

$$A = af(N) = f(N+1)a, \quad A^\dagger = f(N)a^\dagger = a^\dagger f(N+1), \tag{1}$$

where $N = a^\dagger a$ is the usual number operator and $a, a^\dagger$ are the annihilation and creation boson operators of the ordinary harmonic oscillator algebra respectively. They obey the following commutation relations

$$\begin{aligned} &[N, A] = -A, \quad [N, A^\dagger] = A^\dagger, \\ &[A, A^\dagger] = (N+1)f^2(N+1) - Nf^2(N). \end{aligned} \tag{2}$$

The function f, which is a characteristic for the deformation, has a dependance on a deformation parameter q where in the $q \to 1$ limit ($f(q=1)$), the deformation disappears and the usual quantum algebra is recovered.

Let us present the form of the photon-added f-deformed bosonic coherent states (PAf-DBCS) and discuss the minimal set of conditions required to obtain Klauder's minimal coherent states. We begin with no-photon excitation case, where the f-deformed coherent state can be constructed as eigenstates of the annihilation operator A

$$A|\alpha,f\rangle = \alpha|\alpha,f\rangle \tag{3}$$

and it is defined as the superposition of the complete orthogonal basis $|n\rangle$ of the nonlinear field Hilbert space[46,47]

$$|\alpha,f\rangle = \mathcal{N}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}(f(n)!}|n\rangle, \quad \mathcal{N} = \left[\exp_f\left[\alpha^2\right]\right]^{-1/2}, \tag{4}$$

where we have considered $\alpha \in \mathbb{R}$ and we have introduced

$$\exp_f[x] = \sum_{n=0}^{\infty}\frac{x^n}{n!(f(n)!)^2}, \quad f(n)! = f(n)f(n-1)\cdots f(0). \tag{5}$$

The function $\exp_f$ is a deformation version of the usual exponential function. They become coincident when f is the identity. Notice that $\exp_f[x]\exp_f[y] \neq \exp_f[x+y]$ and $[\exp_f[x]]^a \neq \exp_f[ax]$, i.e., we have a non-extensive exponential that can be found in many physical problems.[48,49]

The purpose of this paper is to construct an important special family of states obtained by adding photons to a conventional f-deformed bosonic coherent state $|\alpha,f,m\rangle$, m-photon excitations of the mode in the f-deformed coherent state. These excited states can be obtained by repeated application of the creation operator $A^\dagger$ to the conventional f-deformed coherent states $|\alpha,f\rangle$

$$||\alpha,f,m\rangle = \left(A^\dagger\right)^m|\alpha,f\rangle. \tag{6}$$

By making use of the following expression

$$\left(A^\dagger\right)^m|n\rangle = \sqrt{\frac{(m+n)!}{n!}}\frac{f(n+m)!}{f(n)!}|n+m\rangle, \tag{7}$$

then, the excited f-deformed coherent states are given by

$$|\alpha,f,m\rangle = N_m\left(|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\sqrt{(n+m)!}f(n+m)!}{n!\left(f(n)!\right)^2}\alpha^n|n+m\rangle, \tag{8}$$

where N_m is the normalization constant

$$N_m\left(|\alpha|^2\right) = \left(\sum_{n=0}^{\infty}\frac{(n+m)!\left(f(n+m)!\right)^2}{(n!)^2\left(f(n)!\right)^4}|\alpha|^{2n}\right)^{-1/2}, \tag{9}$$

and m is a positive integer being the number of added quanta (or added photons).

According to Klauder,[50] the minimal set of requirements to be imposed on a state $|z\rangle$ to be a coherent state is

(i) Normalizability

$$\langle z|z\rangle = 1; \tag{10}$$

(ii) Continuity in the label

$$\left|z-z'\right| \to 0 \quad \Rightarrow \quad \left||z\rangle - |z'\rangle\right| \to 0; \tag{11}$$

(iii) Resolution of unity

$$\iint_{\mathcal{C}} \mathrm{d}^2z|z\rangle\langle z|W(|z|^2) = \mathbf{1}. \tag{12}$$

In general, the first two conditions are easily satisfied. This is not the case for the third. In fact, this condition restricts considerably the choice of the states to be considered.

Now, we are going to analyze the above minimal set of conditions to obtain Klauder's coherent states for PAf-DCSs. To this end, we assume the existence of a positive Weight function $W_m(|\alpha|^2)$ so that an integral over the complex plane exists and gives the result

$$\iint_{\mathcal{C}} \mathrm{d}^2\alpha|\alpha,f,m\rangle\langle\alpha,f,m|W_m(|\alpha|^2) = \sum_n |n+m\rangle\langle n+m| = \mathbf{1}. \tag{13}$$

Thus, in order to resolve the identity operator as in Eq. (13), one should find the adequate Weight function $W_m\left(|\alpha|^2\right)$. By means of a change of the complex variables in terms of polar coordinates ($\alpha = r\mathrm{e}^{\mathrm{i}\theta}$), using the completeness of the states $|n+m\rangle$ and integrate on the angular variable θ, the resolution of the identity operator (13) can be brought to solving the following integral equation

$$\int_0^{\infty} x^n N_m^2(x)W_m(x)\mathrm{d}x = \frac{1}{\pi}\frac{(n!)^2\left(f(n)!\right)^4}{(n+m)!\left(f(n+m)!\right)^2}, \tag{14}$$

where $x = r^2$. Instead of solving Eq. (13) for $W_m(x)$, we investigate the existence of solutions for the following integrable equation

$$\int_0^{\infty} x^n\hat{W}_m(x)\mathrm{d}x = \rho_m(n), \tag{15}$$

where

$$\hat{W}_m(x) = N_m^2(x)W_m(x), \quad \rho_m(n) = \frac{1}{\pi}\frac{(n!)^2\left(f(n)!\right)^4}{(n+m)!\left(f(n+m)!\right)^2}. \tag{16}$$

Relation (15) is the well-known Stieltjes moment problem; this integral equation does not admit a general solution. In fact, whether a solution exists or not, the form of this solution in the positive case depends on the form of the box function. So at this stage, the only thing that can be affirmed is that equation (8) defines PAf-DCS if and only if equation (15) admits a solution.

For solving this integral equation, we use the Fourier transforms method: multiplying Eq. (15) by $((\mathrm{i}y)^n/n!)$ and summing over n yields

$$\int_0^{\infty}\mathrm{e}^{\mathrm{i}xy}\hat{W}_m(x)\mathrm{d}x = \sum_{n=0}^{\infty}\left(\frac{(\mathrm{i}y)^n}{n!}\right)\rho_m(n) := \bar{W}_m(y). \tag{17}$$

In the case where the series defining the function $\bar{W}_m(y)$ above converges, the inverse Fourier transform is given by

$$\hat{W}_m(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}xy}\bar{W}_m(y)\mathrm{d}y. \tag{18}$$

Finally, the Weight function $W_m(x)$ allowing for a resolution of the identity operator, in the form given in Eq. (13), is written

as

$$W_m(x) = \frac{1}{2\pi \mathcal{N}_m^2(x)} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}xy} \bar{W}_m(y)\,\mathrm{d}y. \tag{19}$$

In the $f \to 1$ and $m \to 0$ limits, i.e., the undeformed situation with no-photon excitation, the PAf-DBCSs reduce to the expression of the Glauber coherent states with normalization function $N_0(|\alpha|^2) = \exp\left(-|\alpha|^2/2\right)$ and the Weight function $1/\pi$.

3. Effect of a beam splitter on photon-added f-deformed bosonic coherent states

In this section, we discuss how a beam splitter acts on an input state comprised of a state, $|\alpha, f, m\rangle$ to be studied, in one input port and a vacuum state $|0\rangle$ in the other port, given the condition for the separability and entanglement of the output state.

A beam splitter is a linear optical device used to generate quantum entanglement between two modes. Its effect, as a lossless four-port device, is described by a unitary transformation connecting the input fields and the output fields. Quantum mechanically, the action of a beam splitter can be described by a unitary operator $\widehat{U}_{\mathrm{bs}}$ that relates the input state to the output state by

$$|\mathrm{out}\rangle = \widehat{U}_{\mathrm{bs}}|\mathrm{int}\rangle, \tag{20}$$

where the beam splitter operator is[15]

$$\widehat{U}_{\mathrm{bs}} = \exp\left[\frac{\theta}{2}\left(a^{\dagger} b\, \mathrm{e}^{\mathrm{i}\phi} - a b^{\dagger}\, \mathrm{e}^{-\mathrm{i}\phi}\right)\right]. \tag{21}$$

Here, a and b are the annihilation operators of the two input fields and $a^{\dagger}$ and $b^{\dagger}$ are their Hermitian conjugates; ϕ denotes the phase difference between the reflected and transmitted fields.

In order to obtain the beam splitter transformation when an excited nonlinear bosonic field is incident on one input port, we first introduce the effect of a beam splitter on an input state comprised of a Fock state in the input beam, that is $|n\rangle$, and a ground Fock state in the other

$$\widehat{U}_{\mathrm{bs}}|n\rangle|0\rangle = \sum_{p=0}^{n} \binom{n}{p}^{1/2} T^p R^{(n-p)} |p\rangle|n-p\rangle. \tag{22}$$

The quantities T and R are the transmissivity and the reflectivity of the beam splitter, respectively, obeying the normalization condition $|T|^2 + |R|^2 = 1$. Through the balance of this paper, we shall assume that our beam splitter is 50:50 and that the reflected beam receives a phase shift of $\pi/2$. Thus we set $T = 1/\sqrt{2}$ and $R = \mathrm{i}/\sqrt{2}$, such that we have

$$\widehat{U}_{\mathrm{bs}}|n\rangle|0\rangle = \sum_{p=0}^{n} \binom{n}{p}^{1/2} \left(\frac{1}{\sqrt{2}}\right)^{p} \left(\frac{\mathrm{i}}{\sqrt{2}}\right)^{(n-p)} |p\rangle|n-p\rangle. \tag{23}$$

It is well known that if a Glauber state defined as $|\alpha\rangle = \exp\left(-|\alpha|^2/2\right) \sum_{n=0}^{\infty} \alpha^n/\sqrt{n!}|n\rangle$ is incident on one input port and a vacuum state on the other, then the output state is separable with zero entanglement

$$|\mathrm{out}\rangle = \widehat{U}_{\mathrm{bs}}|\alpha\rangle|0\rangle = \left|\frac{\alpha}{\sqrt{2}}\right\rangle \otimes \left|\frac{\mathrm{i}\alpha}{\sqrt{2}}\right\rangle. \tag{24}$$

The Glauber coherent states are the only states that, when passed through one input port of a beam splitter, result in product states at the output. Any other states at the input result in entangled states at the output.[51]

Let us now examine the entanglement generated via a 50:50 beam splitter when an excited field is injected into one input port and a vacuum state is injected into the other. In this case, the output state is obtained by using both Eqs. (20) and (22)

$$\begin{aligned} |\mathrm{out}\rangle &= \widehat{U}_{\mathrm{bs}}|\alpha, f, m\rangle|0\rangle \\ &= \frac{N_m\left(|\alpha|^2\right)}{\alpha^m} \sum_{n=0}^{\infty} \sum_{p=0}^{n+m} \left(\frac{(n+m)!f(n+m)!}{n!\left(f(n)!\right)^2}\right) \frac{\alpha^p}{\sqrt{p!}} T^p \\ &\quad \times \frac{\alpha^{(n+m-p)}}{\sqrt{(n+m-p)!}} R^{(n+m-p)} |p\rangle|n+m-p\rangle. \end{aligned} \tag{25}$$

From the expression of the output state, we can exhibit the condition that allows us to determine whether the outcome of the beam splitter is entangled. Recently, Hillery and Zubairy[52] developed two inequalities for the detection of entanglement. Consider a composite bipartite system ab (i.e., a system consisting of two subsystems denoted by a and b respectively). The Hilbert space for the composite system is $H = H_a \otimes H_b$ where H_a is the Hilbert space for subsystem a and H_b is the Hilbert space for subsystem b. Let A and B be two operators on H_a and H_b respectively. The state of the composite system is entangled if either

$$\begin{aligned} &|\langle AB^{+}\rangle|^2 > \langle A^{+}AB^{+}B\rangle, \\ &|\langle AB\rangle|^2 > \langle A^{+}A\rangle\langle B^{+}B\rangle. \end{aligned} \tag{26}$$

These inequalities exhibit sufficient conditions for the entanglement, but if they are not satisfied, we cannot say whether the total state is entangled or not. Some of the implications of these conditions have been explored for the case that the systems are field modes and, in the case that one is a field mode and one is a collection of atoms,[53] also in the case for the states of two-spin systems in the context of $U_q(su(2))$ quantum algebra.[54] Here, we would like to explore their implications for the output fields defined by Eq. (25). Let a_1 be the annihilation operator for the field mode 1 and b_2 be the annihilation operator of field mode 2. The corresponding creation operators are $a_1^{\dagger}$ and $b_2^{\dagger}$ respectively. Our entanglement conditions for the two output modes of the beam splitter become

$$\begin{aligned} &|\langle a_1 b_2^{\dagger}\rangle|^2 > \langle a_1^{\dagger}a_1 b_2^{\dagger}b_2\rangle, \\ &|\langle a_1 b_2\rangle|^2 > \langle a_1^{\dagger}a_1\rangle\langle b_2^{\dagger}b_2\rangle. \end{aligned} \tag{27}$$

Using Eq. (25), we find that

$$\langle \text{out}|a_1 b_2|\text{out}\rangle = \sum_{n=0}^{\infty}\sum_{p=0}^{n+m} C^{*n-2,\alpha}_{p-1,m} C^{n,\alpha}_{p,m}\left[p(n+m-p)\right]^{1/2}, \quad (28)$$

$$\langle \text{out}|a_1^{\dagger} a_1|\text{out}\rangle = \sum_{n=0}^{\infty}\sum_{p=0}^{n+m} |C^{n,\alpha}_{p,m}|^2 p, \quad (29)$$

and

$$\langle \text{out}|b_2^{\dagger} b_2|\text{out}\rangle = \sum_{n=0}^{\infty}\sum_{p=0}^{n+m} |C^{n,\alpha}_{p,m}|^2 (n+m-p), \quad (30)$$

where

$$C^{n,\alpha}_{p,m} = \frac{\mathcal{N}_m\left(|\alpha|^2\right)}{\alpha^m}\left(\frac{(n+m)!f(n+m)!}{n!\left(f(n)!\right)^2}\right) \times \frac{\alpha^p}{\sqrt{p!}} t^p \frac{\alpha^{(n+m-p)}}{\sqrt{(n+m-p)!}} r^{(n+m-p)}.$$

Therefore, the condition entanglement becomes

$$\left|\sum_{n=0}^{\infty}\sum_{p=0}^{n+m} C^{*n-2,\alpha}_{p-1,m} C^{n,\alpha}_{p,m}\left[p(n+m-p)\right]^{1/2}\right|^2 > \sum_{n=0}^{\infty}\sum_{p=0}^{n+m} |C^{n,\alpha}_{p,m}|^2 p \times \sum_{n=0}^{\infty}\sum_{p=0}^{n+m} |C^{n,\alpha}_{p,m}|^2 (n+m-p). \quad (31)$$

We use the linear entropy as a measure of entanglement to investigate the degree of purity of the beam splitter output state. For a bipartite pure state ρ_{AB}, it is defined as

$$S = 1 - \text{Tr}\left(\rho_A\right)^2, \quad (32)$$

where $\rho_A = \text{Tr}_B\left(\rho_{AB}\right)$ is the reduced density operator of system A. Here we are interested in the degree of purity of the output mode A whose state ρ_A is obtained from $\rho = |\text{out}\rangle\langle\text{out}|$ by tracing over the other output, mode B. From Eq. (25), we obtain

$$\rho_A = \frac{N_m^2\left(|\alpha|^2\right)}{|\alpha|^{2m}} \sum_{p,p'=m}^{\infty}\sum_{h=0}^{\infty}\left(\frac{(p+h)!(p'+h)!f(p+h)!f(p'+h)!}{(p+h-m)!(p'+h-m)!\left(f(p+h-m)!\right)^2\left(f(p'+h-m)\right)^2}\right)\frac{|\alpha|^{2h}}{h!}|T|^{2h}\frac{\alpha^p\overline{\alpha^{p'}}}{\sqrt{p!p'!}}R^p\overline{R^{p'}}|p\rangle\langle p'|. \quad (33)$$

After calculation, we find for the linear entropy

$$S = 1 - \frac{N_m^4\left(|\alpha|^2\right)}{|\alpha|^{4m}} \sum_{p,p'=m}^{\infty}\sum_{h,h'=0}^{\infty} g\left(p,p',h,h'\right)\frac{|\alpha|^{2(h+h'+p+p')}}{h!h'!p!p'!}|T|^{2(h+h')}|R|^{2(p+p')}, \quad (34)$$

where

$$g\left(p,p',h,h'\right) = \frac{(p+h)!(p'+h)!(p+h')!(p'+h')!f(p+h)!f(p'+h)!f(p+h')!f(p'+h')!}{(p+h-m)!(p'+h-m)!(p+h'-m)!(p'+h'-m)!\left(f(p+h-m)!f(p'+h-m)!f(p+h'-m)!f(p'+h'-m)!\right)^2}. \quad (35)$$

From Eqs. (33) and (35), it is clear that the linear entropy depends on the outcome of the measurement of the observable α, furthermore S, depending on the specific form of f.

Let us consider the standard deformation defined by[55]

$$f_{\text{M}}(n) = \sqrt{\frac{1}{n}\frac{q^n - q^{-n}}{q - q^{-1}}}, \quad q \in \mathbb{R}. \quad (36)$$

This deformation can be used for the characterization of the photon statistics of laser outputs reasonably close to threshold, single-atom resonance fluorescence, the micromaser field and absorption by two-level atoms. Specifying certain values of the deformation parameter, we get some new algebras with interesting properties. The quantum algebras with these deformation functions have been considered to be a mere mathematical curiosity that may be useful for developing exactly solvable toy models, especially for statistical and many-body physics, where exactly solvable models are rare. In fact, a wide literature concerning deformed algebras with these deformations has been developed in the last few decades in different fields of physics. Statistical mechanics of deformed systems are considered in various contexts too; examples include the statistical properties of bosons in different dimensions, statistical distribution of a gas of bosons and specific heat of a quantum deformed lattice.[56]

To explore the influence of different parameters on the entanglement behavior of the beam output state in our scheme, we exhibit numerically the variation of the linear entropy S for different values of strength regimes $|\alpha|$, nonlinearity q and the number of added quanta m of the input field. Figure 1 displays the function S for $|\text{out}\rangle$ in terms of q and $|\alpha|$ for various values of the parameter m. The dashed–dotted red line presents the variation of S for $m = 8$, the dashed green line is for $m = 5$, the solid blue line is for $m = 3$ and the solid black line is for $m = 1$. Firstly, it is important to mention the separation condition for the output state in our scheme, which gives a strong point for this work. In this context, we can observe that in the case of no-photon excitation and undeformed limit (i.e., $m = 0$ and $q = 1$), the PAf-DCSs tend to Glauber states, and thus they become separable (product of coherent states) after passing by the beam splitter with zero entanglement. Whereas for the case $m \neq 0$ or $q \neq 1$, the output state is then always entangled for any choices of the parameters that characterize the PAf-DCSs. In Fig. 1(a), the linear entropy is plotted versus the parameter q for various values of m for $|\alpha| = 2$. In general, the amount of entanglement is highly dependent on the values of the excitation number and the deformation parameter, where the plots clearly show that the linear entropy has a different order in terms of these parameters. For each value of q, the linear entropy increases with increasing m. Furthermore, we find that the value of linear entropy becomes large

as the deformation parameter q gets farther from 1 (from left or right). Interestingly, the output state tends to stabilize at its maximal entanglement as m and q becomes significantly large, exhibiting EPR state. From these results, one may find that m and q can augment the linear entropy amount. That is to say, the increasing of the nonlinearity and excitation of the input field will enhance the entanglement of the output state. In Fig. 1(b), the linear entropy is plotted in terms of the amplitude $|\alpha|$ for different values of m in the undeformed $q \to 1$ limit. This figure shows that the amplitude has a destructive effect on the degree of the output state entanglement for different values of the excitation and deformation. From another side, the dependance on the excitation number shows that the increase of the input field photon number is always accompanied by the increase of quantum entanglement for different strength regimes of the input field. Interestingly, the output state attains its maximal entanglement for small values of the amplitude, which corresponds to a weak regime of the field. In this sense, the output state entanglement is very sensitive to the effect of the kind of input field and its different strength regimes. In this case, one may find that α may restrain the amount of entanglement of the output state. From the results shown above, we can generate a high amount of entanglement even for larger values of the amplitude (strong regime of the input field) through controlling the linearity and the enhancement of the photon excitation in the input field of the beam splitter. These outcomes show that the PAf-DCSs are very useful for generating a high amount of entanglement in this scheme by a proper choice of the input parameters.

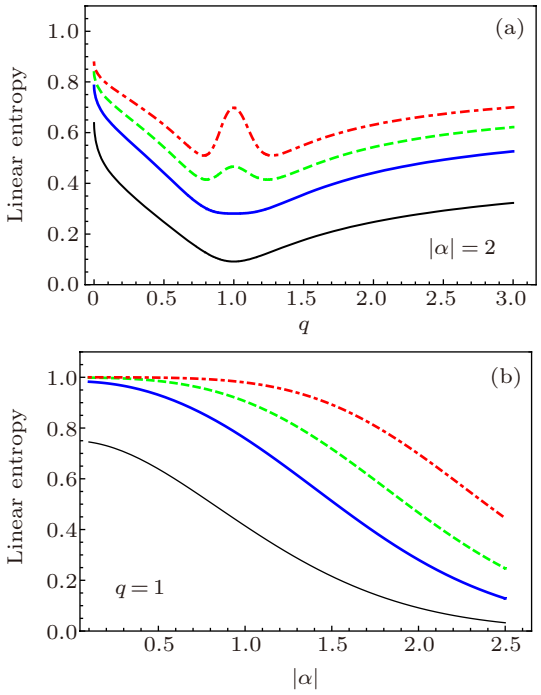


Fig. 1. (color online) The linear entropy of the output state as a function of q (a) and amplitude $|\alpha|$ (b). Red (dashed–dotted line): linear entropy for $m = 8$; green (dashed line): linear entropy for $m = 5$; Blue (solid line): linear entropy for $m = 3$; Black (solid line): linear entropy for $m = 1$. The excitation photon number and deformation may enhance the amount of entanglement of the beam output state.

4. Mandel's parameter

Let us investigate the statistical properties of the input optical field and explore a link between this quantity and the behavior variation of the entanglement in the beam output state. To this end, we study Mandel's Q-parameter defined by[59]

$$Q = \frac{\langle(\Delta\hat{n})^2\rangle - \langle\hat{n}\rangle}{\langle\hat{n}\rangle}, \tag{37}$$

where $\langle N\rangle$ is the average number of particles in the state and $\langle(\Delta N)^2\rangle$ is the variance of the distribution. This parameter is a good indication to determine whether a field state has a sub-Poissonian or is amplitude squeezed; this latter case being manifestly nonclassical (if $-1 \le Q < 0$), Poissonian (if $Q = 0$) or super-Poissonian photon number distribution (if $Q > 0$). For an ordinary coherent state, one would require $Q = 0$ and that the fluctuations of the field quadratures be the same as for the vacuum.

For the case of the input state $|\alpha, f, m\rangle$, we find that

$$Q = \frac{N_m^2\left(|\alpha|^2\right)}{|\alpha|^{2m}} \sum_{n=m}^{\infty} \frac{nn!\left(f(n)!\right)^2}{\left((n-m)!\right)^2\left(f(n-m)!\right)^4}|\alpha|^{2m}, \tag{38}$$

which is clearly dependent on different parameters involved in the coherent state.

Now, we are going to explain our results by analyzing the physical properties of the PAf-CSs and compare them with ordinary ones, i.e., Glauber coherent states, in this considered case. This study leads us to understand and exploit the link between the statistical properties of the nonlinearity of the input field and the entanglement generated in the beam output state. Mandel's parameter is a useful tool to investigate different physical properties of quantum systems that contain all the information of the optical field, being completely equivalent to the density operator and it can be interpreted as an information measure for such a joint measurement. Mandel's parameter clearly distinguishes coherent states and it can be used as a measure of the statistical properties of the optical fields. It conjectured that the zero value is obtained for ordinary coherent states that exhibit the Heisenberg uncertainty. Here, we trace the PAf-DCS from what we think as the most classical, $q = 1$ and $m = 0$, to the least classical, $q \neq 1$ and $m \neq 0$, and exploit its correlation to the degree of output state entanglement.

In order to explore the influence of different parameters on the statistical properties of the PAf-DCSs, we have displayed the variation of Mandel's parameter Q in terms of the different parameters in Fig. 2. The dashed–dotted red line presents the variation of Q for $m = 12$, the dashed green line is for $m = 10$, the solid blue line is for $m = 8$ and the solid black line is for $m = 5$. In Fig. 2(a), we show Mandel's parameter of the PAf-DCS in terms of the parameter q for various values of the number of added quanta m with $|\alpha| = 2$. Mandel's parameter satisfies the inequality $-1 \le Q \le 0$, indicating

sub-Poissonian statistics, apart from the trivial case of the vacuum field. Interestingly, Q approaches zero as q is close to one and reaches zero in the limit $q \to 1$ and $m = 0$, showing that the PAf-DCSs approach and equal Glauber states in these limits. We notice that the plots of Mandel's paremter have similar behavior as the linear entropy for the different parameter effects. Indeed, the parameter Q decreases as the deformation parameter moves away from one. This means that the PAf-DCSs become more quantum mechanical accompanied with an augmentation of the output state entanglement. Furthermore, Mandel's parameter decreases with increasing excitation number m and stabilizes at the minimal values $Q = -1$ for high values of m, exhibiting the maximal entanglement of the output state. Figure 2(b) shows the Mandel parameter of the input field in terms of $|\alpha|$ for different values of m with $q = 1$. From the figure, we can see that the parameter also satisfies the inequality $-1 \leq Q \leq 0$ for all ranges of $|\alpha|$ indicating a sub-Poissonian distribution. Interestingly, the Mandel parameter increases with increasing $|\alpha|$ showing that the field becomes more classical; such a limit of the PAf-DCS tends to factorize after beam splitting, indicating the destruction of the degree of the entanglement in the output state. From the above results, the Mandel parameter of the input field is an inverse monotonic function of the linear entropy of the output beam without a direct relation between them. Thus, for our propose, one may expect the behavior of the output state entanglement from the variation of the nonlinearity of the input fields.

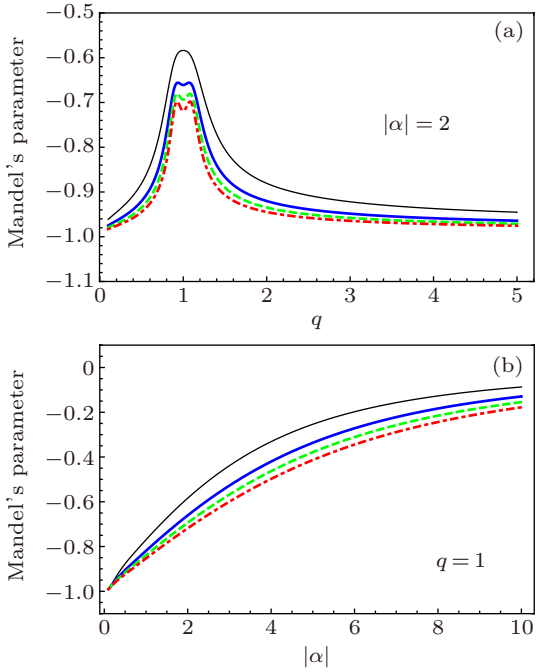


Fig. 2. (color online) The Mandel parameter of the input state as a function of the amplitude $|\alpha|$ (a) and deformation parameter q (b). Red (dashed–dotted line): linear entropy for $m = 12$; green (dashed line): linear entropy for $m = 10$; blue (solid line): linear entropy for $m = 5$; black (solid line): linear entropy for $m = 3$.

5. Conclusion

To summarize, we have constructed the photon-added f-deformed coherent states (PAf-DCSs) by discussing Klauder's minimal set of conditions required to obtain coherent states. Using this set of nonlinear states, we have described quantitatively the entanglement generated via a 50 : 50 beam splitter when a PAf-DBCS is injected on one input beam and the vacuum is injected in the other one, considering a large class of nonlinear input fields and taking into account the photon excitation and deformation effect. Our results present a very useful scheme for generating the maximal amount of entanglement via unitary beam splitters for different strength regimes of the input optical field α, deformation q and excitation number m. The condition for a maximal and separable output beam state is also discussed. Furthermore, our scheme provides an advantage for maximal entanglement, even for strong regimes of the input optical fields. Finally, we have examined the statistical properties of the PAf-DCSs through Mandel's parameter and exploit a connection between this quantity and the behavior variation of the output state entanglement. Then, it is found that the Mandel parameter may be used as an indicator of the entanglement behavior in this scheme. Our result may open new perspectives in different tasks of quantum information processing.

The PAf-DCSs provide a much richer structure for developing different tasks of quantum information processing. They are useful to generate and measure the entanglement and their use is not only of theoretical purpose but also of some practical importance due to their experimental accessibility. In the future, it will be important to study the entanglement generated via a beam splitter when the input state is defined in a mixed state, which makes a useful contribution for further understanding the correlations process related to the beam splitter device. Another interesting line is to study the entanglement dynamics of the beam output states considering the environment effects.

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