Applications of quantum Fourier transform in photon-added coherent state*

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Quantum Fourier transform is realized by the Hadamard gate in a quantum computer, which can also be considered as a Hadamard transform. We introduce the Hadamard transformed photon-added coherent state (HTPACS), which is obtained by letting the photon-added coherent state (PACS) across the quantum Hadamard gate, from this result. It is found that the HTPACS can be considered as a coordinate–momentum mutual exchanging followed by a squeezing transform of the PACS. In addition, the non-classical statistical properties of HTPACS, such as squeezing coefficient, Mandel parameter, etc., are also discussed.

Keywords: quantum Fourier transform, Hadamard gate, non-classical state

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1. Introduction

Quantum computers can efficiently perform some complex calculations that are not feasible on a classical computer.[1,2] The quantum computer's algorithms based upon quantum Fourier transform are the most important quantum algorithms, which is also widely used in classical algorithms. For example, the Deutsch–Jozsa algorithm is one of this type of quantum algorithm. The discrete Fourier transform is defined by

$$y_k \equiv \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} jk/N} x_j, \tag{1}$$

which is usually used to describe the transforming $x_0, ..., x_{N-1}$ of N complex numbers into a set of complex numbers $y_0, ..., y_{N-1}$. Many of the important quantum algorithms involve some type of Fourier transforms. The Hadamard transform which was used in the Deutsch–Jozsa algorithm is also an example of this generalized class of Fourier transforms. Shor's fast algorithm for factoring and discrete logarithm, which is the most well known important quantum algorithm, is also based upon the Fourier transform defined in Eq. (1) of Ref. [3]. The fast Fourier transform on a classical computer takes about $N \log(N) = n2^n$ steps to Fourier transform $N = 2^n$. However, the Fourier transform can be finished using roughly $\log^2(N) = n^2$ steps. From the above discussion, we can see that quantum Fourier transform has an exponential saving.

The point of realizing quantum Fourier transform lies in finding a physical implementation of the quantum gates between quantum bits. The language of quantum computation can describe the changes occurring when a quantum state passes through a quantum gate. The Hadamard transform is expressed as

$$|j\rangle = \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} \mathrm{e}^{2\pi \mathrm{i} jk/2^n} |k\rangle, \tag{2}$$

which is a linear transformation on n quits by its action on computational basis states $|j\rangle$ and $0 \leq j \leq 2^n - 1$. The continuous Hadamard transform is defined as[4]

$$H|x\rangle = \frac{1}{\sqrt{\pi}\sigma} \int_{-\infty}^{\infty} \mathrm{e}^{2\mathrm{i}xy/\sigma^2} |y\rangle \, \mathrm{d}y, \tag{3}$$

where σ is the scale length.

The study of non-classical quantum states has been another important topic in quantum information and quantum computing for the last few years.[5–7] All non-classical states show either sub-Poissonian or super-Poissonian distribution of photons, which are different from the Poisson distribution of the classical states.[8–10] Several different experimental schemes of generating non-classical states also have been proposed recently.[11,12] Among them, Parigi *et al.* have been successfully demonstrated that subtracting photons from and/or adding photons to quantum states may exhibit an abundance of nonclassical properties.[13] Photon subtraction or addition can improve entanglement between Gaussian states,[14,15] loophole-free tests of Bell's inequality,[16] and quantum computing.[17]

In this paper, we introduce the HTPACS as a new non-classical quantum state, which is obtained by letting the PACS pass through a quantum Hadamard gate. The HTPACS exhibits the function of a quantum Hadamard gate. The paper is organized as follows. In Section 2, we shall study the properties of the Hadamard operator and the normalization factor of

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the HTPACS is also derived. In Section 3, squeezing effects of the HTPACS are discussed. The Mandel Q-factor and the second-order correlation function of the HTPACS are given in Sections 4 and 5. To see the function of a quantum Hadamard gate more clearly, we analyze the distribution function of the HTPACS in Section 6. The main results of our research are summarized in the last section.

2. Normalization of HTPACS

We define the HTPACS $|\alpha\rangle_H$ as

$$|\alpha\rangle_H = N_m^{-1/2} H a^\dagger |\alpha\rangle, \tag{4}$$

where $N_m^{-1/2}$ is the normalization factor to be determined, H is the Hadamard operator, $a^\dagger$ is the creation operator and $|\alpha\rangle$ is the coherent state. From Eq. (2), the Hadamard operator (HO) is introduced in the integration form as[18]

$$H = \frac{1}{\sqrt{\pi}\sigma} \iint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y \mathrm{e}^{2\mathrm{i}xy/\sigma^2} |y\rangle \langle x|, \tag{5}$$

where $|y\rangle$ and $|x\rangle$ are the coordinate eigenstate. The coordinate eigenstate is expressed as

$$|x\rangle = \pi^{-1/4} \exp\left(-\frac{x^2}{2} + \sqrt{2}xa^\dagger - \frac{a^{\dagger 2}}{2}\right)|0\rangle, \tag{6}$$

where $[a, a^\dagger] = 1$, a is the annihilation operator,

With the help of the technique of integration within an ordered product (IWOP) of operators,[19,20] we have the normal ordering of the Hadamard operator as

$$H = \frac{2\sigma}{\sqrt{\sigma^2+4}} : \exp\left(Aa^{\dagger 2} + Ba^\dagger a + a^2\right) :, \tag{7}$$

where

$$A = \frac{\sigma^4 - 4}{2(\sigma^4+4)}, \quad B = \frac{4i\sigma^2}{\sigma^4+4} - 1.$$

From Eq. (5), we have

$$HH^\dagger = H^\dagger H = 1. \tag{8}$$

It can be seen from Eq. (8) that H is a unitary operator. The transform under HO is

$$HaH^{-1} = Ca - Da^\dagger, \quad Ha^\dagger H^{-1} = Da - Ca^\dagger, \tag{9}$$

where

$$C = \frac{1}{4\mathrm{i}\sigma^2}\left(\sigma^4+4\right), \quad D = \frac{1}{4\mathrm{i}\sigma^2}\left(\sigma^4-4\right).$$

To fully describe a quantum state, its normalization is usually important. Using Eqs. (4) and (8), we achieve our goal as

$$N_m = \langle\alpha| aH^\dagger H a^\dagger |\alpha\rangle = \langle\alpha|\left(1 + a^\dagger a\right)|\alpha\rangle = 1 + |\alpha|^2. \tag{10}$$

3. Squeezing effects of HTPACS

In this section, we explore why the squeezing quantitatively changes when the HO is acted on the PACS. The optimized nonclassical depth turns out to be[21]

$$S_{\mathrm{opt}} = -2\left|\langle a^{\dagger 2}\rangle - \langle a^\dagger\rangle^2\right| + 2\langle a^\dagger a\rangle - 2\left|\langle a^\dagger\rangle\right|^2. \tag{11}$$

The negative value of S_{opt} in the range $[-1, 0)$ indicates squeezing or nonclassical case.

For the HTPACS, by using Eq. (9), we derive

$$\begin{aligned}\langle a^\dagger\rangle_H &= N_m^{-1}\langle\alpha| aH^\dagger a^\dagger H a^\dagger |\alpha\rangle \\ &= N_m^{-1}\langle\alpha| a\left(Ca^\dagger - Da\right)a^\dagger |\alpha\rangle \\ &= N_m^{-1}\langle\alpha|\left(Caa^{\dagger 2} - Da^2 a^\dagger\right)|\alpha\rangle. \end{aligned} \tag{12}$$

Using the operator's formula[19]

$$a^n a^{\dagger m} = (-\mathrm{i})^{m+n} : \mathrm{H}_{m,n}\left(\mathrm{i}a^\dagger, \mathrm{i}a\right) :, \tag{13}$$

equation (12) can be expressed as

$$\langle a^\dagger\rangle_H = N_m^{-1}\mathrm{i}\left[C\mathrm{H}_{1,2}\left(\mathrm{i}\alpha^*, \mathrm{i}\alpha\right) - D\mathrm{H}_{2,1}\left(\mathrm{i}\alpha^*, \mathrm{i}\alpha\right)\right], \tag{14}$$

where $\mathrm{H}_{m,n}\left(\mathrm{i}\alpha^*, \mathrm{i}\alpha\right)$ is two-variable Hermite polynomials

$$\mathrm{H}_{m,n} = \sum_{l=0}^{\min(m,n)} \frac{m!n!(-1)^l}{l!(m-l)!(n-l)!}(\mathrm{i}\alpha)^{m-l}(\mathrm{i}\alpha^*)^{n-l}, \tag{15}$$

here and hereafter, the notations $\mathrm{H}_{m,n}$ denote $\mathrm{H}_{m,n}\left(\mathrm{i}\alpha^*, \mathrm{i}\alpha\right)$ for writing convenience.

Similarly, one can get

$$\langle a^{\dagger 2}\rangle = N_m^{-1}\left(C^2\mathrm{H}_{1,3} + D^2\mathrm{H}_{3,1} - 2CD\mathrm{H}_{2,2} - CD\mathrm{H}_{1,1}\right), \tag{16}$$

and

$$\begin{aligned}\langle a^\dagger a\rangle &= N_m^{-1}[CD\left(\mathrm{H}_{1,3} + \mathrm{H}_{3,1}\right) \\ &\quad - \left(C^2 + D^2\right)\mathrm{H}_{2,2} - C^2\mathrm{H}_{1,1}]. \end{aligned} \tag{17}$$

Substituting Eqs. (14), (16), and (17) into Eq. (11), one can get the S_{opt} of HTPACS. To see clearly the variation of the S_{opt} for the HTPACS with the parameters α and σ, we give the plots of the S_{opt} in Fig. 1. For different parameters of the coherent state, the variation trend is similar. With the increase of the strength of the coherent state, the positive values of S_{opt} in the HTPACS gradually vanish, which exhibits the non-classical effects.

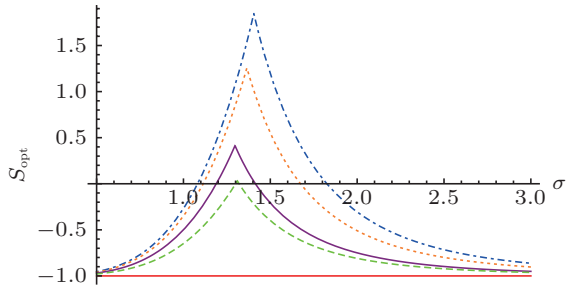


Fig. 1. (color online) Degree of quadrature squeezing S_{opt} for the HTPACS as a function of scale length σ with different parameters $\alpha = 0.2$ (dot–dashed curve), 0.5 (dotted curve), 1 (solid curve), 2 (dashed curve). The bottom line is $S_{\mathrm{opt}} = -1$.

4. Mandel Q–factor for HTPACS

Mandel introduced the so-called Mandel Q-factor to describe the violation from the photo number distribution of the state as Poissonian statistics. To see clearly the photon statistical properties of HTPACS, we examine Mandel's Q-parameter, which is defined as[22,23]

$$Q = \frac{\langle a^{\dagger 2}a^2\rangle - \langle a^\dagger a\rangle^2}{\langle a^\dagger a\rangle}. \tag{18}$$

The states for $Q = 0$, $Q < 0$, and $Q > 0$, respectively, correspond to the cases of Poissonian, sub-Poissonian and super-Poissonian statistics.

In order to get the Mandel Q-factor, we calculate the average value of $a^{\dagger 2}a^2$ as follows:

$$\begin{aligned}\langle a^{\dagger 2}a^2\rangle_H &= N_m^{-1}\langle\alpha|a(Ca^\dagger - Da)^2(Da^\dagger - Ca)^2a^\dagger|\alpha\rangle\\ &= N_m^{-1}\langle\alpha|\big[CDa(CDa^{\dagger 5} + C^2a^{\dagger 3} + D^2a^{\dagger 3} + CDa^\dagger)\\ &\quad - 2CD^2a^2(Da^{\dagger 4} + 2Ca^{\dagger 2})\\ &\quad + Da^3(D^3a^{\dagger 3} + CD^2a^\dagger + C^3a^\dagger)\\ &\quad - CD^3a^4a^{\dagger 2} + C^2D^2a^5a^\dagger + C^4aa^{\dagger 2}a^2a^\dagger\\ &\quad - 2C^3Daa^{\dagger 2}aa^{\dagger 2} - 2C^3Da^2a^\dagger aa^\dagger\\ &\quad + 4C^2D^2a^2a^\dagger aa^{\dagger 2}\big]|\alpha\rangle,\end{aligned} \tag{19}$$

where the Bogoliubov transform in Eq. (9) is used.

Using the operator's formula in Ref. [17],

$$\begin{aligned}&a^n a^{\dagger k}a^j a^{\dagger m}\\ &= \sum_{l=0}^{\min[k,j]}\frac{(-\mathrm{i})^{k+n+j+m}}{l!\,(k-l)!\,(j-l)!}:\mathrm{H}_{m+k-l,n+j-l}\left(\mathrm{i}a^\dagger,\mathrm{i}a\right):,\end{aligned} \tag{20}$$

and Eq. (13), we have

$$\begin{aligned}&\langle a^{\dagger 2}a^2\rangle_H\\ &= N_m^{-1}\big[(-CD\mathrm{H}_{1,5} + C^2\mathrm{H}_{1,3}\\ &\quad + D^2\mathrm{H}_{1,3} - CD\mathrm{H}_{1,1})CD - 2(2CH_{2,2} - DH_{2,4})CD^2\\ &\quad + D(-D^3H_{3,3} + CD^2H_{3,1} + C^3H_{3,1})\\ &\quad + CD^3H_{4,2} - C^2D^2H_{5,1} + \mu + \nu,\end{aligned} \tag{21}$$

where

$$\begin{aligned}\mu &= -C^4\sum_{l=0}^{2}\frac{1}{l!\,((2-l)!)^2}\mathrm{H}_{3-l,3-l},\\ \nu &= 2C^2D\sum_{l=0}^{1}\frac{1}{(1-l)!l!}\left[\frac{C}{(2-l)!}\mathrm{H}_{4-l,2-l}\right.\\ &\quad \left. + \frac{C}{(2-l)!}\mathrm{H}_{2-l,4-l} - \frac{2D}{(1-l)!}\mathrm{H}_{3-l,3-l}\right].\end{aligned} \tag{22}$$

With the help of Eqs. (17), (18), and (21), we show the plot of Q-parameter for the HTPACS as a function of the parameter σ with different coherent states in Fig. 2. From Fig. 2, one can see clearly that the Q-parameter increases along with the increasing of the parameter. It changes from positive value to negative value at the point about $\sigma = 1.67$.

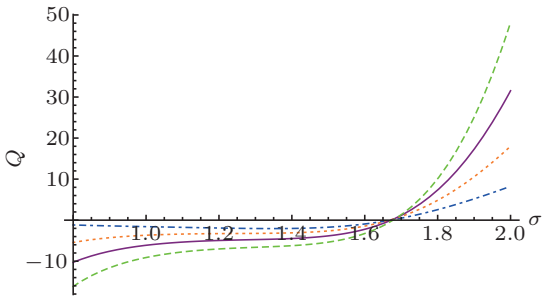


Fig. 2. (color online) The Mandel Q-parameter as a function of scale length σ of quantum Hadamard for the HTPACS with different parameters $\alpha = 1$ (dot–dashed curve), 1.5 (dotted curve), 2 (solid curve), 2.5 (dashed curve).

5. Second-order correlation function

Although in some special cases, there are quantum states in which super-/sub-Poissonian statistical behavior appears with a bunching/antibunching effect via the Mandel Q-factor, this is not absolutely true. To investigate bunching or antibunching effects, the second-order correlation function is widely used. The second-order correlation is defined as[24]

$$g^2 = \frac{\langle a^{\dagger 2}a^2\rangle}{\langle a^\dagger a\rangle^2}. \tag{23}$$

If $g^2 = 1$, the optical field is Poissonian, while for $g^2 < 1$ the field is sub-Poissonian, and for $g^2 > 1$ the field is super-Poissonian. Substituting Eqs. (17) and (21) into Eq. (23), we plot the second-order correlation function of the HTPACS as a function of σ with different parameters α in Fig. 3.

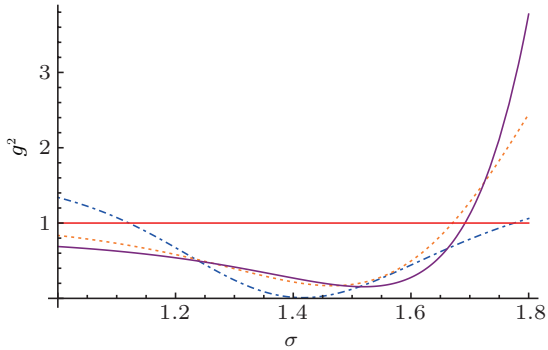


Fig. 3. (color online) The second-order correlation function of the HTPACS as a function of the parameter σ for different $\alpha = 0.1$ (dot–dashed curve), 1 (dotted curve), 5 (solid curve). The horizontal line is $g^2 = 1$.

From Fig. 3, we find g^2 of the HTPACS has a rather complicated behavior depending on the scale length σ and the complex number α. In general, the distribution function changes from sub-Poissonian to super-Poissonian as the increasing of the scale length σ.

6. Distribution function of the quadrature in HTPACS

The distribution function of the quadrature in the HTPACS is defined as

$$P = |\langle q\,|\alpha\rangle_H|^2, \tag{24}$$

where $\langle q|$ is the coordinate representation, i.e.,

$$\langle q| = \langle 0|\,\pi^{-1/4}\exp\left(-\frac{q^2}{2}+\sqrt{2}qa-\frac{a^2}{2}\right). \tag{25}$$

Using the normal ordering of HO in Eq. (7), we can write the explicit form of the wave function of state HTPACS as[25–27]

$$\begin{aligned}\langle q|\alpha\rangle_H &= N_m^{-1/2}\frac{2\sigma}{\sqrt{\sigma^2+4}}\exp\left[(1+A-B)\,\alpha^2\right.\\ &\left.+\sqrt{2}q\alpha\,(B-2A)+2Aq^2\right]\langle q|\alpha\rangle,\end{aligned} \tag{26}$$

where $\langle q|\alpha\rangle$ is the coherent in coordinate representation,

$$\langle q|\alpha\rangle = \pi^{-1/4}\exp\left(-\frac{q^2}{2}+\sqrt{2}\alpha q-\frac{\alpha^2}{2}-\frac{|\alpha|^2}{2}\right). \tag{27}$$

Substituting Eq. (27) into (26) and using Eq. (24), we get

$$P = \pi^{-1/2}N_m^{-1}\frac{4\sigma^2}{\sigma^2+4}\,|\omega|^2, \tag{28}$$

where

$$\begin{aligned}\omega &= \exp\left[\left(A-B+\frac{1}{2}\right)\alpha^2+\sqrt{2}q\alpha\,(B-2A+1)\right.\\ &\left.+\frac{4A-1}{2}q^2-\frac{|\alpha|^2}{2}\right].\end{aligned} \tag{29}$$

To see the function of a quantum Hadamard gate more clearly, we first give the distribution function's plot of PACS by

$$\begin{aligned}P' &= \pi^{-1/2}N_m^{-1}\left(\sqrt{2}q-\alpha\right)^2\\ &\times\left[\exp\left(-\frac{q^2}{2}+\sqrt{2}\alpha q-\frac{\alpha^2}{2}-\frac{|\alpha|^2}{2}\right)\right]^2,\end{aligned} \tag{30}$$

in Fig. 4. Using Eq. (28), the distribution function of the HTPACS is plotted as a function of q with different scale length σ in Fig. 5. Comparing Figs. 4 and 5, it is interesting to find that the quantum Hadamard gate not only shows squeezing properties, but also has the function of a parity operator. We also see that the maximal distribution function P of the HTPACS becomes larger as σ increases.

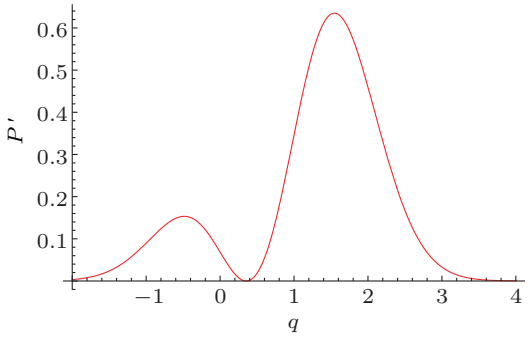


Fig. 4. (color online) The distribution P' of PACS for $\alpha = 0.5$.

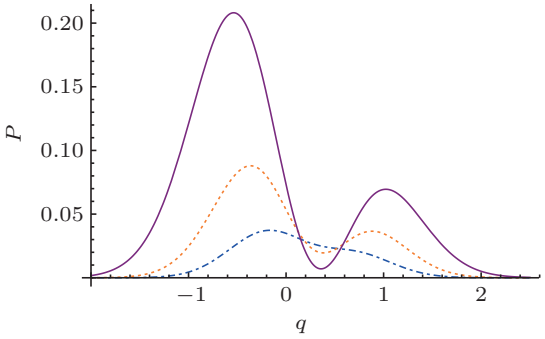


Fig. 5. (color online) The distribution P of the HTPACS for $\alpha = 0.5$ and $\sigma = 0.8$ (dot–dashed curve), 1 (dotted curve), 1.2 (solid curve).

7. Conclusion

Let us summarize the main results of this paper. We have introduced a new non-classical state that is generated by the action of a quantum Hadamard gate on a PACS. Also we have found several new analytical expressions for the functions characterizing the HTPACS, such as squeezing coefficient, Mandel parameter and the second-order correlation function. It should be pointed out that the HTPACS does not always exhibit squeezing effects. So the HTPACS cannot be simply considered as a squeezed state. We have demonstrated that the quantum Hadamard gate shows squeezing properties and the function of a parity operator in some special cases.

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