Nonlinear resonance phenomenon of one-dimensional Bose–Einstein condensate under periodic modulation*

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We investigate the effect of an external periodic modulation on the one-dimensional (1D) Bose–Einstein condensate with harmonic trapping potential. By numerically solving the Gross–Pitaevskii equation with symplectic algorithm, the nonlinear resonance phenomenon is shown and the corresponding Fourier spectrum is given. The autoresonance phenomenon is also presented under almost periodic external modulation, and it shows that the condensate eventually evolves into quasi-periodic oscillation.

Keywords: Bose–Einstein condensate, resonance, periodic modulation, symplectic method

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1. Introduction

In recent years, much attention has been paid to the investigation of the dynamics of a Bose–Einstein condensate (BEC) under external modulation.[1–6] Chaos is studied in the time evolution of the one-dimensional (1D) condensate in harmonic potential under external modulations.[5] Multi-band excitation with zero quasimomentum transfer for interacting bosons loaded in a 1D optical lattice is reported.[6] One main interest is the dynamics of the condensate under periodic modulation.[7–10] This is due to the resonance phenomenon occurred when the driving frequency and the inherent natural frequency of the system are matched, which is manifested by greatly increased amplitude. It is well known that BEC can be described by the mean-field Gross–Pitaevskii (GP) equation, which incorporates the trapping potential and the interaction between the atoms in the condensate.[11–13] The latter gives a contribution to the nonlinear term.

The GP equation uses a mean-field approach and has a form similar with the nonlinear Schrödinger equation. It can be transformed into a Hamilton system, which is a nonlinear Hamilton system, and it has the symplecic structure that is preserved. The symplectic method can preserve the symplecic structure of the nonlinear system and the normalization of the wavefuction is naturally preserved in computation, which is superior to some other methods in long-time and many-step computation when applied to nonlinear differential equations.[14–16]

A periodic driving force is an efficient and effective tool for studying the behavior of condensate. For BEC in a harmonic trapping potential, the effect of resonance can either be studied by modulating the trapping frequency periodically, or by modulating the interaction between the atoms periodically. In practice, this is done by a temporally variation of the scattering length a_s, which characterizes the two-body atomic interactions. Such a variation can be achieved experimentally by Feshbach resonance technique, where a_s is varied as a function of external magnetic field. The control of scattering length using Feshbach resonance has been reported experimentally and theoretically.[17–21] The scattering length can cover the continuum of positive and negative values. When $a_s > 0$, the interaction is repulsive. When $a_s < 0$, the interaction is attractive. The sign of the scattering length can be changed experimentally, which means the interaction of the atoms in the trap can change from repulsive to attractive and *vice versa*.[17]

The resonance of BEC in spherical trap and axial trap has been studied extensively.[9] An almost periodic driving force will bring peculiar dynamics of BEC, and the subharmonic autoresonance phenomenon was reported in the spherical case.[10]

In this paper, we shall perform a numerical study of the resonance of a condensate with harmonic potential in the 1D case. In addition, the nonlinear resonance is studied by varying the scattering length of the condensate. Both periodic driving force and almost periodic driving force cases are reported and comparisons are made. In particular, we find that the condensate eventually evolves into quasi-periodic oscillation under the modulation of an almost periodic nonlinearity characterized by thicker closed trajectories in the phase space.

2. Numerical model

In laboratory, the condensate is essentially 1D when its lateral motion is confined or negligible. The dynamics of the

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condensate can be well studied by the 1D time-dependent GP equation[12,13]

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2}m\omega_0^2x^2+U_0N|\psi(x,t)|^2\right]\psi(x,t) = \mathrm{i}\hbar\frac{\partial\psi(x,t)}{\partial t}. \quad (1)$$

Here, $U_0=4\pi\hbar^2a_s/m$ is the interaction between atoms with a_s presenting the atomic scattering length and m the mass of a single atom. N is the number of atoms in the condensate, and ω_0 is the oscillation frequency of the trap. $\psi(x,t)$ is the wavefunction of the condensate and is normalized to unity. Equation (1) can be written in dimensionless form as

$$\left[-\frac{\partial^2}{\partial\xi^2}+\frac{\xi^2}{4}+\alpha|\Phi(\xi,\tau)|^2\right]\Phi(\xi,\tau)=\mathrm{i}\frac{\partial\Phi(\xi,\tau)}{\partial\tau}, \quad (2)$$

by $\xi=x/a$, $\tau=\omega_0t$, and $\Phi(\xi,\tau)=\psi(x,t)\sqrt{a}$, where $a=\sqrt{\hbar/2m\omega_0}$ is the size of the harmonic trapping potential. Then, the nonlinear coefficient reads $\alpha=8\pi Na_s/a$.

By $\Phi(\xi,\tau)=u(\xi,\tau)+\mathrm{i}v(\xi,\tau)$, equation (2) can be written into Hamiltonian equations. We then apply a symplectic algorithm to it, and the normalization of the wavefunction is naturally preserved.[14–16] In this paper, the Euler-centered scheme is applied,[16] the space step is $h=0.1$, and the boundary is taken to be $|\xi|_{\max}=20$. The time step is $\tau=0.001$, the computation time is from 0 to 5.0×10^3, and the total computation step is 5.0×10^6. In our computation, the norm of the wavefunction is preserved to be 10^{-11}.

In studying the dynamics of condensate, the root mean square (RMS) radius of the condensate is always used. The dimensionless form of RMS radius is $R=\sqrt{\langle\xi^2\rangle}$, and

$$\langle\xi^2\rangle=\int_{-\infty}^{\infty}\xi^2|\Phi(\xi,\tau)|^2\,\mathrm{d}\xi. \quad (3)$$

3. Nonlinear resonance phenomenon under periodic modulation

Resonance will happen when the external driving frequency matches the natural frequency of the system or its integral multiple. When the nonlinearity $\alpha=0$, the natural frequency of the system is exactly twice the oscillation frequency of the trap, $2\omega_0$, and this value will vary with the increasing of α.[9] This is a kind of frequency shift,[7] and the frequency shift will increase as the nonlinearity increases. We only consider the small nonlinearity case and the natural frequency of the oscillation is approximately $2\omega_0$. Experimentally, it is feasible to control the scattering length a_s through Feshbach resonance. Since a_s is involved in the nonlinear term, for convenience, we take the modulated nonlinear coefficient to be $\alpha=c\cos[\int\omega(\tau)\mathrm{d}\tau]$, where $\omega(\tau)=\Omega-\mu\tau$ is the frequency of the driving with Ω and μ constants, c is a constant controlling the strength of the driving. In physics, this describes the evolution of the interaction of atoms with time under external modulation. This definition is convenient for us to study both the case of periodically varying the scattering length[10] and the case of almost periodically varying the scattering length. When $\mu=0$, the nonlinear coefficient will become $\alpha=c\cos[\Omega\tau]$, which is a periodic function of τ corresponding to the exact periodic modulation. Here, Ω is the frequency of variation of the nonlinear coefficient. When μ is taken to be a small value, the nonlinear coefficient will become $\alpha=c\cos[\Omega\tau-\mu\tau^2/2]$. It is nearly periodic, corresponding to an almost periodic modulation; that is, the variation of the scattering length a_s is almost periodic. The constant c has to be less than a critical value in order to avoid collapse for attractive interaction when the nonlinearity is negative.[8,9] We choose a smaller value of c since the natural frequency will be almost equal to $2\omega_0$ under this condition. For convenience, the dimensionless form of the trap frequency $\omega_0=1$ is used.

Firstly, we consider the case of $\mu=0$, the driving frequency is constant $\omega=\Omega$, and the driving force is exactly periodic with time, $\alpha=c\cos[\Omega\tau]$. Resonance oscillation is expected when $\Omega=2n, n=\pm1,\pm2,\ldots$. In Fig. 1 we present the RMS radius versus time. In Figs. 1(a) and 1(b) we plot the results of $c=0.01$, $c=0.02$, and for $\Omega=2$. The RMS radius oscillates with large amplitude, which corresponds to the lowest harmonic oscillation, $n=1$. The amplitude of the oscillations of the RMS radius increases and subsequently decreases which forms a cycle, such a cycle repeats several times singles nonlinear resonance. With the increase of c, the strength of the modulation becomes stronger and the period of the cycle decreases and the RMS radius grows to a larger value. In Fig. 1(c) we plot the result for $c=0.01$, $\Omega=4$. From the repeated cycle of the amplitude of oscillation we can identify that it is a nonlinear resonance. This oscillation form is different from that of Fig. 1(a) and Fig. 1(b) because $\Omega=4$ is two times the natural frequency, that is $n=2$, which corresponds to the first excited state of the harmonic oscillation. Through comparison, we can see that the period of the cycle of nonlinear oscillation decreases with the increasing of the driving frequency. In Fig. 1(d) we plot the result for off-resonance oscillation case $c=0.01$, $\Omega=2.2$ where the large amplitude oscillation breaks down and a small oscillation repeats rapidly throughout our computation. In Fig. 1(c) and Fig. 1(d) the amplitudes of the oscillation are small, which may be hard to observe in experiment. However, in the off-resonance case [Fig. 1(d)], we have small amplitude oscillations around an equilibrium value. In the resonance case [Fig. 1(c)], the driving frequency increases, the nonlinearity becomes larger, and then the frequency shift increases, so the resonance phenomenon is not very obvious and the amplitude of the oscillation is small. Taken as an example, we employ the trap frequency in the radial direction to be $\omega_0=640\times2\pi$ rad/s, and the trap frequency in the axial direction to be zero.[18] The unit

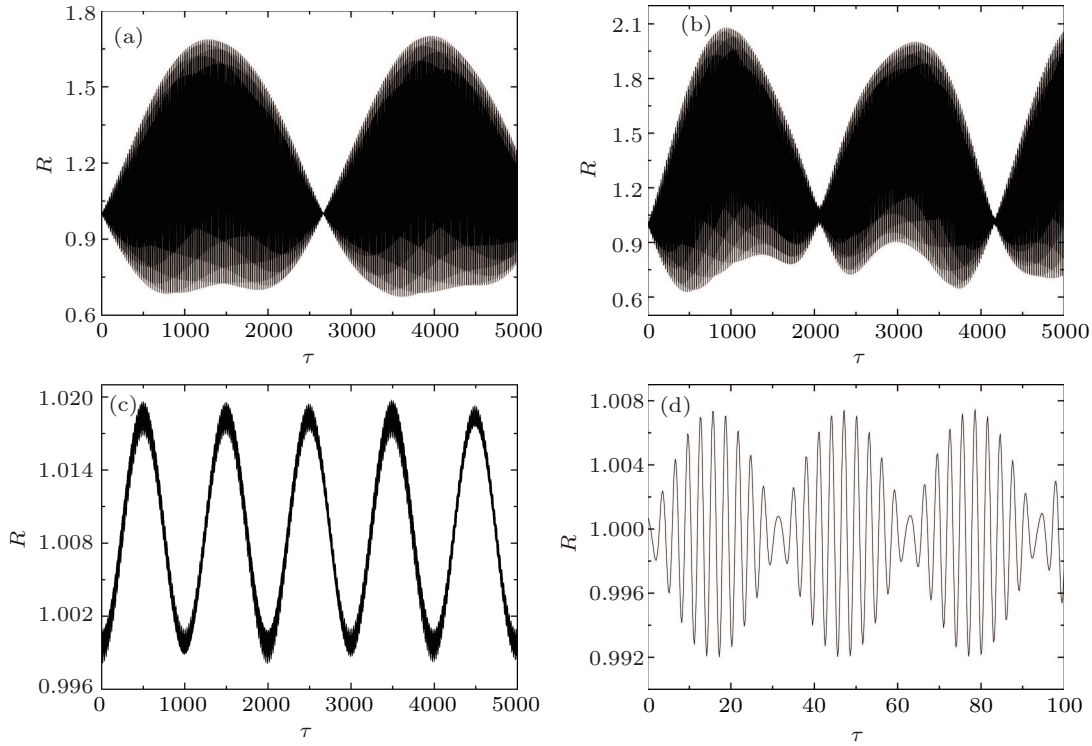


Fig. 1. The RMS radius R versus time τ for BEC in a harmonic trapping potential subject to a periodic variation of nonlinearity $\alpha = c\cos[\Omega\tau]$ for (a) $c = 0.01$, $\Omega = 2$, (b) $c = 0.02$, $\Omega = 2$, (c) $c = 0.01$, $\Omega = 4$, and (d) $c = 0.01$, $\Omega = 2.2$. Time in unit of ω_0^{-1}, length in unit of a.

of length is $a = 1.15$ μm, the unit of time is $\omega_0^{-1} = 0.25$ ms. In Fig. 1(b) the period of the growth and decay cycle is about 2000 units of dimensionless time corresponds to about 0.5 s, which can be measured precisely.

We present the Fourier spectrum of the RMS radius for the nonlinear resonance $c = 0.01$, $\Omega = 2$ in Fig. 2(a), and the corresponding evolution of the probability density of the condensate in Fig. 2(b). Figure 2(a) shows that there is an isolated discrete distribution with equal frequency interval, the fundamental frequency of the periodic oscillation is shown to be $f_1 = 0.318$. Figure 2(b) shows that the condensate keeps its spatially localized structures and temporally periodic evolutions under nonlinear periodic modulations. The condensate undergoes periodic evolution in a style different from that kind of the breather mode oscillation, where the amplitude of oscillation remains constant.[22] At resonance, the amplitude of oscillation grows and decays, which forms a cycle.

Secondly, we consider the case of $\mu = 0.01$, the driving frequency slowly varies with time $\omega(\tau) = \Omega - \mu\tau$, and the driving force is almost periodic with time. It is adiabatic modulation. The driving frequency varies slowly and whenever it equals the integral multiples of the natural frequency of the system, $\omega(\tau) = 2n$, $n = \pm1, \pm2, \ldots$, a kind of subharmonic autoresonance featured by successive stepwise increase oscillation in RMS radius happens.[10] In Fig. 3(a) we plot the RMS radius for $c = 0.01$, $\Omega = 2$, $\mu = 0.01$, which shows that the RMS radius oscillates in a stepwise increase manor at time $\tau = 400$, 600, respectively. The amplitude of the increasing of RMS radius is small because the value $c = 0.01$ is

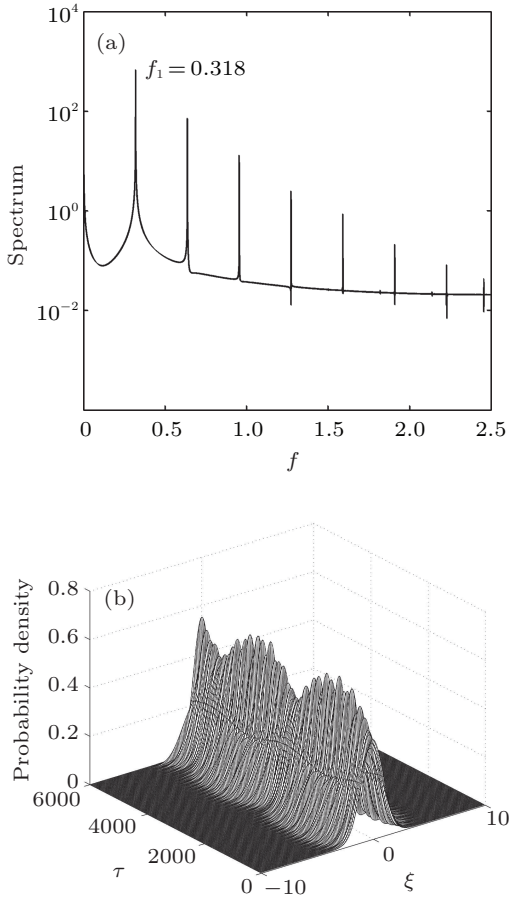


Fig. 2. (a) The Fourier spectrum for RMS radius and (b) the evolution of the probability density of the condensate with $c = 0.01$, $\Omega = 2$. Time in unit of ω_0^{-1}, length in unit of a, and density in unit of a^{-1}.

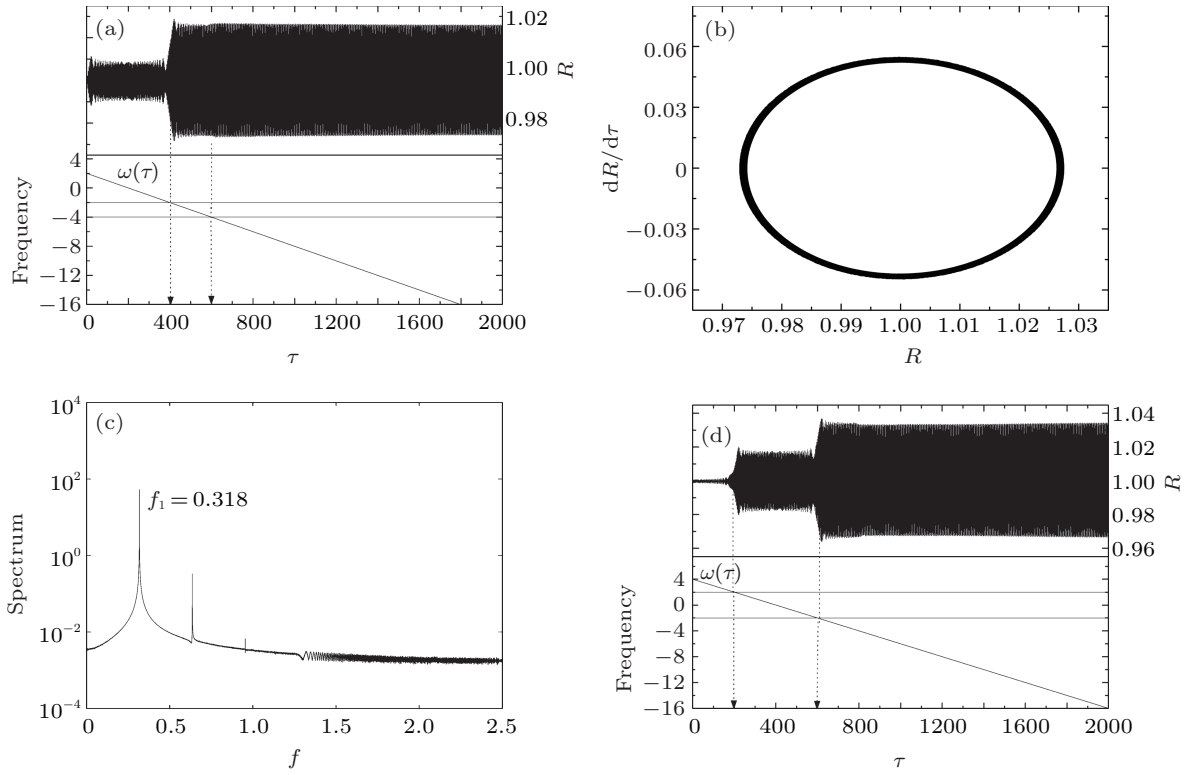


Fig. 3. Numerical results of (a) the RMS radius, (b) phase trajectory, and (c) Fourier spectrum of the condensate for $c = 0.01$, $\Omega = 2$, $\mu = 0.01$. (d) The RMS radius R versus time τ for $c = 0.01$, $\Omega = 4$, $\mu = 0.01$. The lower panel is the corresponding plot of $\omega(\tau) = \Omega - \mu\tau$. Time in unit of ω_0^{-1}, length in unit of a.

small. When the strength of the modulation is larger, this amplitude will increase and can more easily be detected. The lower panel is the plot of $\omega(\tau) = 2 - 0.01\tau$, we can work out that the subharmonic autoresonance happens at $\omega = -2, -4$. The stepwise increase at $\tau = 600$ is not very obvious and such a subharmonic autoresonance goes on but is not manifested clearly, which indicates that the condensates eventually evolves into steady periodic oscillation. We plot the corresponding phase trajectory and Fourier spectrum in Fig. 3(b) and Fig. 3(c), respectively. In order to avoid transient chaos induced by the nonlinear modulation, we omit the computation values of the first 10^6 steps. We can see that the trajectory in phase space is a single loop but is thicker, which corresponds to quasi-periodic recurrence. The Fourier spectrum shows three isolated discrete frequencies with equal interval, the fundamental frequency is $f_1 = 0.318$, which indicates that the condensate eventually turns into quasi-periodic oscillation. In this case $\mu = 0.01$, the driving force is quasi-periodic, and the modulation is adiabatic. Under the continuous action of the quasi-periodic driving force, the phase space is compressed into a quasi-periodic recurrence, the system is adapted to the driving force, and is then eventually stabilized.[23]

In Fig. 3(d) we show the RMS radius for $c = 0.01$, $\Omega = 4$, $\mu = 0.01$, and the lower panel is the plot of $\omega(\tau) = 4 - 0.01\tau$. The stepwise increase of the oscillation of RMS radius is more obvious, happens at time $\tau = 200, 600$, and is consistent with $\omega = 2, -2$. The amplitude of the oscillation is larger than that of Fig. 3(a). After the first several stepwise oscillations, the condensate eventually goes into quasi-periodic oscillation. In comparison with the periodic driving case, the condensate can be stabilized under the quasi-periodic driving force.

A comparison is given in Fig. 4 for a stronger modulation case $c = 0.05$, $\Omega = 2$, and $\mu = 0.01$, which shows that the amplitude of RMS radius is larger than that of Fig. 3(a) and the stepwise increase manor is more obvious at time $\tau = 400, 600$, respectively. We can see that the condensate goes into stabilization beyond time $\tau = 850$, which is a kind of quasi-periodic oscillation due to the continuous action of the quasi-periodic driving force.

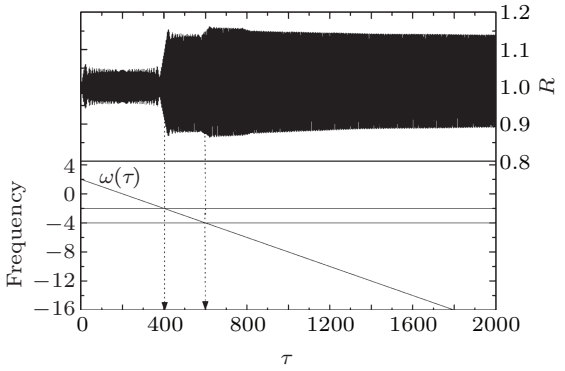


Fig. 4. Numerical results of the RMS radius for $c = 0.05$, $\Omega = 2$, $\mu = 0.01$. The lower panel is the corresponding plot of $\omega(\tau) = \Omega - \mu\tau$. Time in unit of ω_0^{-1}, length in unit of a.

4. Conclusion

In conclusion, the effects of periodic driving and almost periodic driving on the 1D BECs by modulating the scattering length have been studied by symplectic computation. The norm of the wavefunction is preserved well and the symplectic method is a proper method for investigating this kind of problem.

The modulation-induced dynamics strongly depend on the value of the driving frequency. It is shown that the nonlinear resonance happens when the driven frequency equals the integral multiples of the natural frequency of the condensate. The driving frequency contributes to the nonlinear term, so the nonlinear term influences the system greatly at certain discrete values. The subharmonic autoresonance is shown under almost periodic driving, and the successive stepwise increase oscillation of the condensate eventually goes into quasi-periodic oscillation. Such a study would provide theoretical prediction and present a method to control the dynamics of the condensate.

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