A novel (G'/G)-expansion method and its application to the Boussinesq equation

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In this article, a novel (G'/G)-expansion method is proposed to search for the traveling wave solutions of nonlinear evolution equations. We construct abundant traveling wave solutions involving parameters to the Boussinesq equation by means of the suggested method. The performance of the method is reliable and useful, and gives more general exact solutions than the existing methods. The new (G'/G)-expansion method provides not only more general forms of solutions but also cuspon, peakon, soliton, and periodic waves.

Keywords: (G'/G)-expansion method, Boussinesq equation, solitary wave solutions, auxiliary nonlinear ordinary differential equation

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1. Introduction

As is well known, nonlinear evolution equations (NLEEs) are widely used to describe complex phenomena in various fields of science, especially in physics, plasma physics, fluid physics, quantum field theory, biophysics, chemical kinematics, geochemistry, electricity, propagation of shallow water waves, high-energy physics, condensed matter physics, quantum mechanics, optical fibers, elastic media, etc. The analytical solutions of such equations are of fundamental importance to reveal the inner mechanisms of the phenomena. Among the possible solutions to NLEEs, certain special form solutions may depend only on a single combination of variables, such as solitons. In mathematics and physics, a soliton is a self-reinforcing solitary wave, a wave packet or pulse that upholds its profile while it travels at constant speed. Due to the availability of symbolic computation software, direct methods to search for exact solutions have attracted much attention. In past years, many powerful and direct methods have been developed to find special solutions, such as: wave of translation,[1] the inverse scattering transform,[2] the Hirota's bilinear method,[3] the Darboux transformation method,[4] the Backlund transformation method,[5] the tanh method,[6] the Jacobi elliptic function method,[7] the symmetry method,[8] the Painleve expansion method,[9] the Exp-function method,[10,11] the Adomian decomposition method,[12] the homogeneous balance method,[13] etc.

Lately, Wang *et al.*[14] introduced an expansion technique called the (G'/G)-expansion method and demonstrated that it is a powerful technique for seeking for the analytic solutions of NLEEs. Making use of this method, some useful equations are studied in Refs. [15]–[19].

In order to establish the efficiency and persistence of the (G'/G)-expansion method, and to extend the range of applicability, further research has been carried out. For example, Zhang *et al.*[20] generalized the (G'/G)-expansion method for the evolution equations with variable coefficients. Zhang *et al.*[21] also presented an improved (G'/G)-expansion method to seek for general traveling wave solutions. Zayed[22] presented a new approach to making the (G'/G)-expansion method more reliable, where $G(\xi)$ satisfies the Jacobi elliptical equation $[G'(\xi)]^2 = e_2G^4(\xi) + e_1G^2(\xi) + e_0$. Zayed[23] presented a further alternative approach of this method, in which $G(\xi)$ satisfies the Riccati equation $G'(\xi) = A + BG^2(\xi)$, where A and B are arbitrary constants. Akbar *et al.*[16] presented a generalized and improved (G'/G)-expansion method that provides further solutions than the improved (G'/G)-expansion method.[21]

In this article, we offer a novel (G'/G)-expansion method to solve the NLEEs in mathematical physics and engineering. To illustrate the originality, consistency, and advantages of the proposed method, the Boussinesq equation is solved and abundant new families of exact solutions are found.

2. Description of the method

Suppose that the nonlinear evolution equation is of the form

$$P(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots) = 0, \tag{1}$$

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where P is a polynomial in $u(x,t)$ and its partial derivatives wherein the highest order partial derivatives and the nonlinear terms are concerned. The main steps of the method are as follows.

Step 1 Combining the real variables x and t by a compound variable ξ, we suppose that

$$u(x,t) = u(\xi), \quad \xi = x \pm Vt, \tag{2}$$

where V is the speed of the traveling wave. Equation (2) transforms Eq. (1) into an ordinary differential equation (ODE) for $u = u(\xi)$ as follows:

$$Q(u, u', u'', u''', \ldots) = 0, \tag{3}$$

where Q is a function of $u(\xi)$ and its derivatives in which prime refers to derivative with respect to ξ.

Step 2 Assume that the solution of Eq. (3) can be expressed in powers of $\psi(\xi)$ as follows:

$$u(\xi) = \sum_{j=-N}^{N} \alpha_j (\psi(\xi))^j, \tag{4}$$

where

$$\psi(\xi) = (d + \Phi(\xi)) \tag{5}$$

with $\Phi(\xi) = G'(\xi)/G(\xi)$.

Herein, α_{-N} or α_N may be zero, but both of them could not be zero simultaneously. $\alpha_j\,(j = 0, \pm1, \pm2, \ldots, \pm N)$ and d are constants to be determined later and $G = G(\xi)$ satisfies the second order nonlinear ODE:

$$GG'' = \lambda GG' + \mu G^2 + \nu (G')^2, \tag{6}$$

where the prime denotes the derivative with respect to ξ; λ, μ, and ν are real parameters.

The Cole–Hopf transformation $\Phi(\xi) = \ln(G(\xi))_\xi = G'(\xi)/G(\xi)$ reduces Eq. (6) into the Riccati equation:

$$\Phi'(\xi) = \mu + \lambda \Phi(\xi) + (\nu - 1)\Phi^2(\xi). \tag{7}$$

Equation (7) has twenty five individual solutions (see Ref. [25] by Zhu for details).

Step 3 The value of the positive integer N can be determined by balancing the highest order linear terms with the nonlinear terms of the highest order coming out of Eq. (3). If the degree of $u(\xi)$ is $D[u(\xi)] = n$, then the degree of the other expressions will be as follows:

$$D\left[\frac{\mathrm{d}^p u(\xi)}{\mathrm{d}\xi^p}\right] = n + p, \quad D\left[u^p \left(\frac{\mathrm{d}^q u(\xi)}{\mathrm{d}\xi^q}\right)^s\right] = np + s(n+q).$$

Step 4 Substitute Eq. (4) including Eqs. (5) and (6) into Eq. (3), then we will obtain polynomials in $[d + G'(\xi)/G(\xi)]^j$ and $[d + G'(\xi)/G(\xi)]^{-j}$, $(j = 0, 1, 2, \ldots, N)$. Collect each coefficient of the resulting polynomials and let it be zero; then, an over-determined set of algebraic equations for $\alpha_j\,(j = 0, \pm1, \pm2, \ldots, \pm N)$, d and V is obtained.

Step 5 Suppose that the value of the constants can be obtained by solving the algebraic equations obtained in Step 4. The values of the constants together with the solutions of Eq. (6) yield abundant exact traveling wave solutions of nonlinear evolution equation (1).

Discussion 1 It is significant to observe that if we replace λ by $-\lambda$ and μ by $-\mu$ and set $\nu = 0$ in Eq. (6), then the proposed new (G'/G)-expansion method coincides with Akbar *et al.*'s generalized and improved (G'/G)-expansion method.[16] On the other hand, if we set $d = 0$ in Eq. (5) and $\nu = 0$ in Eq. (6) then the proposed method is identical to the improved (G'/G)-expansion method presented by Zhang *et al.*[21] Again, if we set $d = 0$, $\nu = 0$, and negative the exponents of (G'/G) are zero in Eq. (4), then the proposed method turns into the basic (G'/G)-expansion method introduced by Wang *et al.*[14] Finally, if we let $\nu = 0$ in Eq. (6) and α_j $(j = 1, 2, 3, \ldots, N)$ are functions of x and t instead of constants then the proposed method is transformed into the generalized the (G'/G)-expansion method developed by Zhang *et al.*[20] Thus, the methods presented in Refs. [14], [16], [20], and [21] are only special cases of the proposed new (G'/G)-expansion method.

3. Application

In this section, we will employ the proposed method to obtain some new and more general exact traveling wave solutions of the celebrated Boussinesq equation.

Consider the Boussinesq equation

$$u_{tt} - u_{xx} - u_{xxxx} - 3(u^2)_{xx} = 0, \tag{8}$$

we will study the traveling wave solutions to Eq. (8). Substituting $u = u(\xi)$, $\xi = x - Vt$ into Eq. (8) and integrating twice, we obtain

$$(V^2 - 1)u - u'' - 3u^2 + C = 0. \tag{9}$$

where C is an integration constant. Using Eqs. (4) and (6) and considering the homogeneous balance between u'' and u^2 in Eq. (9), we obtain $2N = N + 2$, i.e, $N = 2$. Therefore, solution formula (4) becomes

$$\begin{aligned} u(\xi) = {} & \alpha_{-2}(\psi(\xi))^{-2} + \alpha_{-1}(\psi(\xi))^{-1} + \alpha_0 \\ & + \alpha_1(\psi(\xi)) + \alpha_2(\psi(\xi))^2. \end{aligned} \tag{10}$$

Substituting Eq. (10) into Eq. (9), the terms on the left-hand side of the resulting equation are transformed into polynomials in $[d + G'(\xi)/G(\xi)]^j$, $(j = 0, 1, 2, \ldots, N)$ and $[d + G'(\xi)/G(\xi)]^{-j}$, $(j = 0, 1, 2, \ldots, N)$. Equating the coefficients of the same power of these polynomials to zero, we

obtain an over-determined set of algebraic equations (for simplicity, we leave out the displaying of the equations) for α_0, α_1, α_2, α_{-1}, α_{-2}, d, C, and V. Solving the over-determined set of algebraic equations by using the symbolic computation software, such as Maple 13, we obtain

Set 1

$$\begin{aligned}
&\alpha_2 = 4\upsilon - 2\upsilon^2 - 2, \quad \alpha_1 = 4\upsilon^2 d - 2\lambda\upsilon - 8\upsilon d + 2\lambda + 4d,\\
&\alpha_0 = -2\upsilon^2 d^2 + 4\upsilon d^2 + 2\lambda d\upsilon - \frac{4}{3}\mu\nu - \frac{1}{6} + \frac{4}{3}\mu + \frac{1}{6}V^2\\
&\quad - 2d^2 - \frac{1}{6}\lambda^2 - 2d\lambda,\\
&V = V, \quad \alpha_{-1} = 0, \quad \alpha_{-2} = 0, \quad d = d,\\
&C = -\frac{2}{3}\mu\upsilon\lambda^2 - \frac{1}{12} + \frac{4}{3}\mu^2 + \frac{1}{6}V^2 + \frac{2}{3}\mu\lambda^2 - \frac{8}{3}\mu^2\upsilon\\
&\quad + \frac{4}{3}\mu^2\upsilon^2 - \frac{1}{12}V^4 + \frac{1}{12}\lambda^4,
\end{aligned} \tag{11}$$

where d, λ, μ, and ν are all arbitrary constants.

Set 2

$$\begin{aligned}
&\alpha_0 = -2\upsilon^2 d^2 + 4d^2\upsilon + 2d\lambda\upsilon - \frac{4}{3}\mu\upsilon - \frac{1}{6} + \frac{4}{3}\mu + \frac{1}{6}V^2\\
&\quad - 2d^2 - \frac{1}{6}\lambda^2 - 2\lambda d,\\
&\alpha_{-1} = 4d^3 - 2\mu\lambda + 2\lambda^2 d - 4\mu d + 6\lambda d^2 + 4\upsilon^2 d^3\\
&\quad - 8\upsilon d^3 - 6\lambda d^2\upsilon + 4\mu\upsilon d,\\
&\alpha_{-2} = 4\upsilon d^4 - 4\mu\upsilon d^2 - 2\mu^2 + 4\lambda d^3\upsilon - 4\lambda d^3 - 2\lambda^2 d^2\\
&\quad - 2\upsilon^2 d^4 + 4\mu d^2 - 2d^4 + 4\mu\lambda d,\\
&V = V, \quad d = d, \quad \alpha_2 = 0, \quad \alpha_1 = 0,\\
&C = -\frac{2}{3}\mu\upsilon\lambda^2 - \frac{1}{12} + \frac{4}{3}\mu^2 + \frac{1}{6}V^2 + \frac{2}{3}\mu\lambda^2 - \frac{8}{3}\mu^2\upsilon\\
&\quad + \frac{4}{3}\mu^2\upsilon^2 - \frac{1}{12}V^4 + \frac{1}{12}\lambda^4,
\end{aligned} \tag{12}$$

where d, λ, μ, and ν are all arbitrary constants.

Set 3

$$\begin{aligned}
&\alpha_2 = 4\upsilon - 2\upsilon^2 - 2, \ \ \alpha_0 = -\frac{4}{3}\mu\upsilon + \frac{1}{3}\lambda^2 + \frac{4}{3}\mu - \frac{1}{6} + \frac{1}{6}V^2,\\
&\alpha_{-2} = -\frac{1}{8}\Big(\frac{16\mu^2\upsilon^2 - 8\mu\upsilon\lambda^2 - 32\mu^2\upsilon + 16\mu^2 + 8\mu\lambda^2 + \lambda^4}{(\upsilon-1)^2}\Big),\\
&V = V, \quad d = \frac{\lambda}{2(\nu-1)}, \quad \alpha_1 = 0, \quad \alpha_{-1} = 0,\\
&C = \frac{64}{3}\mu^2\upsilon^2 - \frac{32}{3}\mu\upsilon\lambda^2 - \frac{128}{3}\mu^2\upsilon - \frac{1}{12} - \frac{1}{12}V^4 + \frac{32}{3}\mu\lambda^2\\
&\quad + \frac{64}{3}\mu^2 + \frac{1}{6}V^2 + \frac{4}{3}\lambda^4,
\end{aligned} \tag{13}$$

where λ, μ, and ν are all arbitrary constants.

Substituting Eqs. (11)–(13) into solution formula (10), we obtain

$$\begin{aligned}
u_1(\xi) = &\Big(-2\upsilon^2 d^2 + 4\upsilon d^2 + 2\lambda d\upsilon - \frac{4}{3}\mu\upsilon - \frac{1}{6} + \frac{4}{3}\mu\\
&+ \frac{1}{6}V^2 - 2d^2 - \frac{1}{6}\lambda^2 - 2d\lambda\Big)\\
&+ (4\upsilon^2 d - 2\lambda\upsilon - 8\upsilon d + 2\lambda + 4d)(d + (G'/G))\\
&+ (4\upsilon - 2\upsilon^2 - 2) \times (d + (G'/G))^2,
\end{aligned} \tag{14}$$

$$\begin{aligned}
u_2(\xi) = &\Big(-2\upsilon^2 d^2 + 4\upsilon d^2 + 2\lambda d\upsilon - \frac{4}{3}\mu\upsilon - \frac{1}{6} + \frac{4}{3}\mu\\
&+ \frac{1}{6}V^2 - 2d^2 - \frac{1}{6}\lambda^2 - 2d\lambda\Big)\\
&+ (4d^3 - 2\mu\lambda + 2d\lambda^2 - 4d\mu + 6d^2\lambda + 4d^3\upsilon^2\\
&- 8d^3\upsilon - 6d^2\lambda\upsilon + 4d\mu\upsilon) \times (d + (G'/G))^{-1}\\
&+ (4d^4\upsilon - 4d^2\mu\upsilon - 2\mu^2 + 4\lambda\upsilon d^3 - 4\lambda d^3\\
&- 2\lambda^2 d^2 - 2\upsilon^2 d^4 + 4\mu d^2 - 2d^4 + 4\mu\lambda d)\\
&\times (d + (G'/G))^{-2},
\end{aligned} \tag{15}$$

$$\begin{aligned}
u_3(\xi) = &\Big(-\frac{4}{3}\mu\upsilon + \frac{1}{3}\lambda^2 + \frac{4}{3}\mu - \frac{1}{6} + \frac{1}{6}V^2\Big) + (4\upsilon - 2\upsilon^2 - 2)\\
&\times \Big(\frac{\lambda}{2(\upsilon-1)} + (G'/G)\Big)^2 + \Big\{-\frac{1}{8}\frac{1}{(\upsilon-1)^2}\\
&\times \big(16\mu^2\upsilon^2 - 8\mu\upsilon\lambda^2 - 32\mu^2\upsilon + 16\mu^2 + 8\mu\lambda^2 + \lambda^4\big)\Big\}\\
&\times \Big(\frac{\lambda}{2(\upsilon-1)} + (G'/G)\Big)^{-2},
\end{aligned} \tag{16}$$

where $\xi = x - Vt$, d, λ, μ, and υ are all arbitrary constants.

Substituting the solutions $G(\xi)$ of Eq. (6) into Eq. (14) and simplifying the resulting equation, we obtain the following solutions.

When $\Omega = \lambda^2 - 4\mu\nu + 4\mu > 0$ and $\lambda(\nu - 1) \neq 0$ (or $\mu(\nu - 1) \neq 0$), we have

$$\begin{aligned}
u_{1_1}(\xi) = &(4\upsilon - 2\upsilon^2 - 2)\Big\{d - \frac{1}{2(\nu-1)}\\
&\times \Big(\lambda + \sqrt{\Omega}\tanh\Big(\frac{1}{2}\sqrt{\Omega}\xi\Big)\Big)\Big\}^2\\
&+ (4\upsilon^2 d - 2\lambda\upsilon - 8\upsilon d + 2\lambda + 4d)\\
&\times \Big\{d - \frac{1}{2(\nu-1)}\Big(\lambda + \sqrt{\Omega}\tanh\Big(\frac{1}{2}\sqrt{\Omega}\xi\Big)\Big)\Big\}\\
&+ \Big(-2\upsilon^2 d^2 + 4\upsilon d^2 + 2\lambda d\upsilon - \frac{4}{3}\mu\nu - \frac{1}{6}\\
&+ \frac{4}{3}\mu + \frac{1}{6}V^2 - 2d^2 - \frac{1}{6}\lambda^2 - 2d\lambda\Big),
\end{aligned} \tag{17}$$

$$\begin{aligned}
u_{1_2}(\xi) = &(4\upsilon - 2\upsilon^2 - 2)\Big\{d - \frac{1}{2(\nu-1)}\\
&\times \Big(\lambda + \sqrt{\Omega}\coth\Big(\frac{1}{2}\sqrt{\Omega}\xi\Big)\Big)\Big\}^2\\
&+ (4\upsilon^2 d - 2\lambda\upsilon - 8\upsilon d + 2\lambda + 4d)\\
&\times \Big\{d - \frac{1}{2(\nu-1)}\Big(\lambda + \sqrt{\Omega}\coth\Big(\frac{1}{2}\sqrt{\Omega}\xi\Big)\Big)\Big\}\\
&+ \Big(-2\upsilon^2 d^2 + 4\upsilon d^2 + 2\lambda d\upsilon - \frac{4}{3}\mu\nu - \frac{1}{6}\\
&+ \frac{4}{3}\mu + \frac{1}{6}V^2 - 2d^2 - \frac{1}{6}\lambda^2 - 2d\lambda\Big),
\end{aligned} \tag{18}$$

$$\begin{aligned}
u_{1_3}(\xi) = &(4\upsilon - 2\upsilon^2 - 2)\Big[d - \frac{1}{2(\nu-1)}\\
&\times \big\{\lambda + \sqrt{\Omega}\big(\tanh\big(\sqrt{\Omega}\xi\big) \pm \mathrm{i}\,\mathrm{sech}\big(\sqrt{\Omega}\xi\big)\big)\big\}\Big]^2\\
&+ (4\upsilon^2 d - 2\lambda\upsilon - 8\upsilon d + 2\lambda + 4d)\Big[d - \frac{1}{2(\nu-1)}
\end{aligned}$$

$$\times\{\lambda+\sqrt{\Omega}(\tanh(\sqrt{\Omega}\xi)\pm \mathrm{i}\,\mathrm{sech}(\sqrt{\Omega}\xi))\}\Big]$$
$$+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}$$
$$+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \tag{19}$$

$$u_{1_4}(\xi)=(4\upsilon-2\upsilon^2-2)\Big[d-\frac{1}{2(\nu-1)}$$
$$\times\{\lambda+\sqrt{\Omega}(\coth(\sqrt{\Omega}\xi)\pm\mathrm{csch}(\sqrt{\Omega}\xi))\}\Big]^2$$
$$+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)\times\Big[d-\frac{1}{2(\nu-1)}$$
$$\times\{\lambda+\sqrt{\Omega}(\coth(\sqrt{\Omega}\xi)\pm\mathrm{csch}(\sqrt{\Omega}\xi))\}\Big]$$
$$+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}$$
$$+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \tag{20}$$

$$u_{1_5}(\xi)=(4\upsilon-2\upsilon^2-2)\Big[d-\frac{1}{4(\nu-1)}$$
$$\times\Big\{2\lambda+\sqrt{\Omega}\Big(\tanh\Big(\frac{1}{4}\sqrt{\Omega}\xi\Big)+\coth\Big(\frac{1}{4}\sqrt{\Omega}\xi\Big)\Big)\Big\}\Big]^2$$
$$+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)\times\Big[d-\frac{1}{4(\nu-1)}$$
$$\times\Big\{2\lambda+\sqrt{\Omega}\Big(\tanh\Big(\frac{1}{4}\sqrt{\Omega}\xi\Big)+\coth\Big(\frac{1}{4}\sqrt{\Omega}\xi\Big)\Big)\Big\}\Big]$$
$$+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}$$
$$+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \tag{21}$$

$$u_{1_6}(\xi)=(4\upsilon-2\upsilon^2-2)\Big[d+\frac{1}{2(\nu-1)}$$
$$\times\Big\{-\lambda+\frac{\pm\sqrt{\Omega(A^2+B^2)}-A\sqrt{\Omega}\cosh(\sqrt{\Omega}\xi)}{A\sinh(\sqrt{\Omega}\xi)+B}\Big\}\Big]^2$$
$$+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)\times\Big[d+\frac{1}{2(\nu-1)}$$
$$\times\Big\{-\lambda+\frac{\pm\sqrt{\Omega(A^2+B^2)}-A\sqrt{\Omega}\cosh(\sqrt{\Omega}\xi)}{A\sinh(\sqrt{\Omega}\xi)+B}\Big\}\Big]$$
$$+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}$$
$$+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \tag{22}$$

$$u_{1_7}(\xi)=(4\upsilon-2\upsilon^2-2)\Big[d+\frac{1}{2(\nu-1)}$$
$$\times\Big\{-\lambda+\frac{\pm\sqrt{\Omega(A^2+B^2)}+A\sqrt{\Omega}\cosh(\sqrt{\Omega}\xi)}{A\sinh(\sqrt{\Omega}\xi)+B}\Big\}\Big]^2$$
$$+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)\times\Big[d+\frac{1}{2(\nu-1)}$$
$$\times\Big\{-\lambda+\frac{\pm\sqrt{\Omega(A^2+B^2)}+A\sqrt{\Omega}\cosh(\sqrt{\Omega}\xi)}{A\sinh(\sqrt{\Omega}\xi)+B}\Big\}\Big]$$
$$+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}$$
$$+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \tag{23}$$

where A and B are both real constants;

$$u_{1_8}(\xi)=(4\upsilon-2\upsilon^2-2)$$
$$\times\Big\{d+\frac{2\mu\cosh\Big(\frac{1}{2}\sqrt{\Omega}\xi\Big)}{\sqrt{\Omega}\sinh\Big(\frac{1}{2}\sqrt{\Omega}\xi\Big)-\lambda\cosh\Big(\frac{1}{2}\sqrt{\Omega}\xi\Big)}\Big\}^2$$
$$+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)$$
$$\times\Big\{d+\frac{2\mu\cosh\Big(\frac{1}{2}\sqrt{\Omega}\xi\Big)}{\sqrt{\Omega}\sinh\Big(\frac{1}{2}\sqrt{\Omega}\xi\Big)-\lambda\cosh\Big(\frac{1}{2}\sqrt{\Omega}\xi\Big)}\Big\}$$
$$+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}$$
$$+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \tag{24}$$

$$u_{1_9}(\xi)=(4\upsilon-2\upsilon^2-2)$$
$$\times\Big\{d+\frac{2\mu\sinh\Big(\frac{1}{2}\sqrt{\Omega}\xi\Big)}{\sqrt{\Omega}\cosh\Big(\frac{1}{2}\sqrt{\Omega}\xi\Big)-\lambda\sinh\Big(\frac{1}{2}\sqrt{\Omega}\xi\Big)}\Big\}^2$$
$$+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)$$
$$\times\Big\{d+\frac{2\mu\sinh\Big(\frac{1}{2}\sqrt{\Omega}\xi\Big)}{\sqrt{\Omega}\cosh\Big(\frac{1}{2}\sqrt{\Omega}\xi\Big)-\lambda\sinh\Big(\frac{1}{2}\sqrt{\Omega}\xi\Big)}\Big\}$$
$$+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}$$
$$+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \tag{25}$$

$$u_{1_{10}}(\xi)=(4\upsilon-2\upsilon^2-2)$$
$$\times\Big\{d+\frac{2\mu\cosh(\sqrt{\Omega}\xi)}{\sqrt{\Omega}\sinh(\sqrt{\Omega}\xi)-\lambda\cosh(\sqrt{\Omega}\xi)\pm\mathrm{i}\sqrt{\Omega}}\Big\}^2$$
$$+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)$$
$$\times\Big\{d+\frac{2\mu\cosh(\sqrt{\Omega}\xi)}{\sqrt{\Omega}\sinh(\sqrt{\Omega}\xi)-\lambda\cosh(\sqrt{\Omega}\xi)\pm\mathrm{i}\sqrt{\Omega}}\Big\}$$
$$+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}$$
$$+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \tag{26}$$

$$u_{1_{11}}(\xi)=(4\upsilon-2\upsilon^2-2)$$
$$\times\Big\{d+\frac{2\mu\sinh(\sqrt{\Omega}\xi)}{\sqrt{\Omega}\cosh(\sqrt{\Omega}\xi)-\lambda\sinh(\sqrt{\Omega}\xi)\pm\sqrt{\Omega}}\Big\}^2$$
$$+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)$$
$$\times\Big\{d+\frac{2\mu\sinh(\sqrt{\Omega}\xi)}{\sqrt{\Omega}\cosh(\sqrt{\Omega}\xi)-\lambda\sinh(\sqrt{\Omega}\xi)\pm\sqrt{\Omega}}\Big\}$$
$$+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}$$
$$+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big). \tag{27}$$

When $\Omega=\lambda^2-4\mu\nu+4\mu<0$ and $\lambda(\nu-1)\neq 0$ (or $\mu(\nu-1)\neq 0$), we have

$$u_{1_{12}}(\xi)=(4\upsilon-2\upsilon^2-2)$$
$$\times\Big\{d+\frac{1}{2(\nu-1)}\Big(-\lambda+\sqrt{-\Omega}\tan\Big(\frac{1}{2}\sqrt{-\Omega}\xi\Big)\Big)\Big\}^2$$

$$+(4\upsilon^2 d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)\times\Big\{d+\frac{1}{2(\nu-1)}\Big(-\lambda+\sqrt{-\Omega}\tan\Big(\frac{1}{2}\sqrt{-\Omega}\xi\Big)\Big)\Big\}+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d-\frac{4}{3}\mu\nu-\frac{1}{6}\upsilon+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \quad (28)$$

$$u_{1_{13}}(\xi)=(4\upsilon-2\upsilon^2-2)\times\Big\{d-\frac{1}{2(\nu-1)}\Big(\lambda+\sqrt{-\Omega}\cot\Big(\frac{1}{2}\sqrt{-\Omega}\xi\Big)\Big)\Big\}^2+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)\times\Big\{d-\frac{1}{2(\nu-1)}\Big(\lambda+\sqrt{-\Omega}\cot\Big(\frac{1}{2}\sqrt{-\Omega}\xi\Big)\Big)\Big\}+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \quad (29)$$

$$u_{1_{14}}(\xi)=(4\upsilon-2\upsilon^2-2)\times\Big[d+\frac{1}{2(\nu-1)}\Big\{-\lambda+\sqrt{-\Omega}\big(\tan(\sqrt{-\Omega}\xi)\pm\sec(\sqrt{-\Omega}\xi)\big)\Big\}\Big]^2+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)\times\Big[d+\frac{1}{2(\nu-1)}\Big\{-\lambda+\sqrt{-\Omega}\big(\tan(\sqrt{-\Omega}\xi)\pm\sec(\sqrt{-\Omega}\xi)\big)\Big\}\Big]+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \quad (30)$$

$$u_{1_{15}}(\xi)=(4\upsilon-2\upsilon^2-2)\times\Big[d-\frac{1}{2(\nu-1)}\Big\{\lambda+\sqrt{-\Omega}\big(\cot(\sqrt{-\Omega}\xi)\pm\csc(\sqrt{-\Omega}\xi)\big)\Big\}\Big]^2+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)\times\Big[d-\frac{1}{2(\nu-1)}\Big\{\lambda+\sqrt{-\Omega}\big(\cot(\sqrt{-\Omega}\xi)\pm\csc(\sqrt{-\Omega}\xi)\big)\Big\}\Big]+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \quad (31)$$

$$u_{1_{16}}(\xi)=(4\upsilon-2\upsilon^2-2)\times\Big[d+\frac{1}{4(\nu-1)}\Big\{-2\lambda+\sqrt{-\Omega}\Big(\tan\Big(\frac{1}{4}\sqrt{-\Omega}\xi\Big)-\cot\Big(\frac{1}{4}\sqrt{-\Omega}\xi\Big)\Big)\Big\}\Big]^2+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)\times\Big[d+\frac{1}{4(\nu-1)}\Big\{-2\lambda+\sqrt{-\Omega}\Big(\tan\Big(\frac{1}{4}\sqrt{-\Omega}\xi\Big)-\cot\Big(\frac{1}{4}\sqrt{-\Omega}\xi\Big)\Big)\Big\}\Big]+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \quad (32)$$

$$u_{1_{17}}(\xi)=(4\upsilon-2\upsilon^2-2)\times\Big[d+\frac{1}{2(\nu-1)}\Big\{-\lambda+\frac{\pm\sqrt{-\Omega(A^2-B^2)}-A\sqrt{-\Omega}\cos(\sqrt{-\Omega}\xi)}{A\sin(\sqrt{-\Omega}\xi)+B}\Big\}\Big]^2+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)\times\Big[d+\frac{1}{2(\nu-1)}\Big\{-\lambda+\frac{\pm\sqrt{-\Omega(A^2-B^2)}-A\sqrt{-\Omega}\cos(\sqrt{-\Omega}\xi)}{A\sin(\sqrt{-\Omega}\xi)+B}\Big\}\Big]+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \quad (33)$$

$$u_{1_{18}}(\xi)=(4\upsilon-2\upsilon^2-2)\times\Big[d+\frac{1}{2(\nu-1)}\Big\{-\lambda+\frac{\pm\sqrt{-\Omega(A^2-B^2)}+A\sqrt{-\Omega}\cos(\sqrt{-\Omega}\xi)}{A\sin(\sqrt{-\Omega}\xi)+B}\Big\}\Big]^2+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)\times\Big[d+\frac{1}{2(\nu-1)}\Big\{-\lambda+\frac{\pm\sqrt{-\Omega(A^2-B^2)}+A\sqrt{-\Omega}\cos(\sqrt{-\Omega}\xi)}{A\sin(\sqrt{-\Omega}\xi)+B}\Big\}\Big]+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \quad (34)$$

where A and B are both arbitrary constants such that $A^2-B^2>0$;

$$u_{1_{19}}(\xi)=(4\upsilon-2\upsilon^2-2)\times\Big\{d-\frac{2\mu\cos(\sqrt{-\Omega}\xi/2)}{\sqrt{-\Omega}\sin(\sqrt{-\Omega}\xi/2)+\lambda\cos(\sqrt{-\Omega}\xi/2)}\Big\}^2+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)\times\Big\{d-\frac{2\mu\cos(\sqrt{-\Omega}\xi/2)}{\sqrt{-\Omega}\sin(\sqrt{-\Omega}\xi/2)+\lambda\cos(\sqrt{-\Omega}\xi/2)}\Big\}+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \quad (35)$$

$$u_{1_{20}}(\xi)=(4\upsilon-2\upsilon^2-2)\times\Big\{d+\frac{2\mu\sin(\sqrt{-\Omega}\xi/2)}{\sqrt{-\Omega}\cos(\sqrt{-\Omega}\xi/2)-\lambda\sin(\sqrt{-\Omega}\xi/2)}\Big\}^2+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)$$

$$\times\Big\{d+\frac{2\mu\sin(\sqrt{-\Omega}\xi/2)}{\sqrt{-\Omega}\cos(\sqrt{-\Omega}\xi/2)-\lambda\sin(\sqrt{-\Omega}\xi/2)}\Big\}$$
$$+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}$$
$$+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \quad (36)$$

$$u_{1_{21}}(\xi)=(4\upsilon-2\upsilon^2-2)$$
$$\times\Big\{d-\frac{2\mu\cos(\sqrt{-\Omega}\xi)}{\sqrt{-\Omega}\sin(\sqrt{-\Omega}\xi)+\lambda\cos(\sqrt{-\Omega}\xi)\pm\sqrt{-\Omega}}\Big\}^2$$
$$+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)$$
$$\times\Big\{d-\frac{2\mu\cos(\sqrt{-\Omega}\xi)}{\sqrt{-\Omega}\sin(\sqrt{-\Omega}\xi)+\lambda\cos(\sqrt{-\Omega}\xi)\pm\sqrt{-\Omega}}\Big\}$$
$$+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}$$
$$+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \quad (37)$$

$$u_{1_{22}}(\xi)=(4\upsilon-2\upsilon^2-2)\times\Big\{d$$
$$+\frac{2\mu\sin(\sqrt{-\Omega}\xi/2)}{\sqrt{-\Omega}\cos(\sqrt{-\Omega}\xi/2)-\lambda\sin(\sqrt{-\Omega}\xi/2)\pm\sqrt{-\Omega}}\Big\}^2$$
$$+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)\times\Big\{d$$
$$+\frac{2\mu\sin(\sqrt{-\Omega}\xi/2)}{\sqrt{-\Omega}\cos(\sqrt{-\Omega}\xi/2)-\lambda\sin(\sqrt{-\Omega}\xi/2)\pm\sqrt{-\Omega}}\Big\}$$
$$+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}$$
$$+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big). \quad (38)$$

When $\mu=0$ and $\lambda(\nu-1)\neq0$, we have

$$u_{1_{23}}(\xi)=(4\upsilon-2\upsilon^2-2)$$
$$\times\Big\{d-\frac{\lambda k}{(\nu-1)\{k+\cosh(\lambda\xi)-\sinh(\lambda\xi)\}}\Big\}^2$$
$$+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)$$
$$\times\Big\{d-\frac{\lambda k}{(\nu-1)\{k+\cosh(\lambda\xi)-\sinh(\lambda\xi)\}}\Big\}$$
$$+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}$$
$$+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \quad (39)$$

$$u_{1_{24}}(\xi)=(4\upsilon-2\upsilon^2-2)$$
$$\times\Big\{d-\frac{\lambda\{\cosh(\lambda\xi)+\sinh(\lambda\xi)\}}{(\nu-1)\{k+\cosh(\lambda\xi)+\sinh(\lambda\xi)\}}\Big\}^2$$
$$+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)$$
$$\times\Big\{d-\frac{\lambda\{\cosh(\lambda\xi)+\sinh(\lambda\xi)\}}{(\nu-1)\{k+\cosh(\lambda\xi)+\sinh(\lambda\xi)\}}\Big\}$$
$$+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}$$
$$+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \quad (40)$$

where k is an arbitrary constant.

When $(\nu-1)\neq0$ and $\lambda=\mu=0$, the solution of Eq. (8) is

$$u_{1_{25}}(\xi)=(4\upsilon-2\upsilon^2-2)\Big\{d-\frac{1}{(\nu-1)\xi+c_1}\Big\}^2$$
$$+(4\upsilon^2d-2\lambda\upsilon-8\upsilon d+2\lambda+4d)$$
$$\times\Big\{d-\frac{1}{(\nu-1)\xi+c_1}\Big\}$$
$$+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}$$
$$+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \quad (41)$$

where c_1 is an arbitrary constant.

By substituting the solutions $G(\xi)$ of Eq. (6) into Eq. (15), and simplifying the resulting equation, we obtain the following solutions.

When $\Omega=\lambda^2-4\mu\nu+4\mu>0$ and $\lambda(\nu-1)\neq0$ (or $\mu(\nu-1)\neq0$),

$$u_{2_1}(\xi)=\big(4\upsilon d^4-4\mu\upsilon d^2-2\mu^2+4\lambda d^3\upsilon-4\lambda d^3$$
$$-2\lambda^2d^2-2\upsilon^2d^4+4\mu d^2-2d^4+4\mu\lambda d\big)$$
$$\times\Big\{d-\frac{1}{2(\nu-1)}\Big(\lambda+\sqrt{\Omega}\tanh\Big(\frac{1}{2}\sqrt{\Omega}\xi\Big)\Big)\Big\}^{-2}$$
$$+\big(4d^3-2\mu\lambda+2\lambda^2d-4\mu d+6\lambda d^2+4\upsilon^2d^3$$
$$-8\upsilon d^3-6\lambda d^2\upsilon+4\mu\upsilon d\big)$$
$$\times\Big\{d-\frac{1}{2(\nu-1)}\Big(\lambda+\sqrt{\Omega}\tanh\Big(\frac{1}{2}\sqrt{\Omega}\xi\Big)\Big)\Big\}^{-1}$$
$$+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}$$
$$+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \quad (42)$$

where $\xi=x-Vt$, d, λ, μ, and ν are arbitrary constants;

$$u_{2_2}(\xi)=(4\upsilon d^4-4\mu\upsilon d^2-2\mu^2+4\lambda d^3\upsilon-4\lambda d^3$$
$$-2\lambda^2d^2-2\upsilon^2d^4+4\mu d^2-2d^4+4\mu\lambda d)$$
$$\times\Big\{d-\frac{1}{2(\nu-1)}(\lambda+\sqrt{\Omega}\coth(\sqrt{\Omega}\xi/2))\Big\}^{-2}$$
$$+(4d^3-2\mu\lambda+2\lambda^2d-4\mu d+6\lambda d^2+4\upsilon^2d^3$$
$$-8\upsilon d^3-6\lambda d^2\upsilon+4\mu\upsilon d)$$
$$\times\Big\{d-\frac{1}{2(\nu-1)}(\lambda+\sqrt{\Omega}\coth(\sqrt{\Omega}\xi/2))\Big\}^{-1}$$
$$+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}$$
$$+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big), \quad (43)$$

$$u_{2_3}(\xi)=(4\upsilon d^4-4\mu\upsilon d^2-2\mu^2+4\lambda d^3\upsilon-4\lambda d^3$$
$$-2\lambda^2d^2-2\upsilon^2d^4+4\mu d^2-2d^4+4\mu\lambda d)$$
$$\times\Big[d-\frac{1}{2(\nu-1)}\Big\{\lambda$$
$$+\sqrt{\Omega}(\tanh(\sqrt{\Omega}\xi)\pm\mathrm{i}\,\mathrm{sech}(\sqrt{\Omega}\xi))\Big\}\Big]^{-2}$$
$$+(4d^3-2\mu\lambda+2\lambda^2d-4\mu d+6\lambda d^2+4\upsilon^2d^3$$
$$-8\upsilon d^3-6\lambda d^2\upsilon+4\mu\upsilon d)$$

$$\times\Big[d-\frac{1}{2(\nu-1)}\Big\{\lambda+\sqrt{\Omega}(\tanh(\sqrt{\Omega}\xi)\pm \mathrm{i}\,\mathrm{sech}(\sqrt{\Omega}\xi))\Big\}\Big]^{-1}+\Big(-2\upsilon^2d^2+4\upsilon d^2+2\lambda d\upsilon-\frac{4}{3}\mu\nu-\frac{1}{6}+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2d\lambda\Big). \quad (44)$$

The other families of exact solutions of Eq. (8) are omitted for convenience.

When $\Omega=\lambda^2-4\mu\nu+4\mu<0$ and $\lambda(\nu-1)\neq 0$ (or $\mu(\nu-1)\neq 0$), we have

$$\begin{aligned}u_{2_{12}}(\xi)=&(4\upsilon d^4-4\mu\upsilon d^2-2\mu^2+4\lambda d^3\upsilon-4\lambda d^3-2\lambda^2d^2-2\upsilon^2d^4+4\mu d^2-2d^4+4\mu\lambda d)\times\Big\{d+\frac{1}{2(\nu-1)}\\&\times\big(-\lambda+\sqrt{-\Omega}\tan(\sqrt{-\Omega}\xi/2)\big)\Big\}^{-2}+(4d^3-2\mu\lambda+2\lambda^2d-4\mu d+6\lambda d^2+4\upsilon^2d^3-8\upsilon d^3-6\lambda d^2\upsilon+4\mu\upsilon d)\\&\times\Big\{d+\frac{1}{2(\nu-1)}(-\lambda+\sqrt{-\Omega}\tan(\sqrt{-\Omega}\xi/2))\Big\}^{-1}+\Big(-2\upsilon^2d^2+4d^2\upsilon+2d\lambda\upsilon-\frac{4}{3}\mu\upsilon-\frac{1}{6}+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2\lambda d\Big), \quad (45)\\u_{2_{13}}(\xi)=&(4\upsilon d^4-4\mu\upsilon d^2-2\mu^2+4\lambda d^3\upsilon-4\lambda d^3-2\lambda^2d^2-2\upsilon^2d^4+4\mu d^2-2d^4+4\mu\lambda d)\\&\times\Big\{d-\frac{1}{2(\nu-1)}\big(\lambda+\sqrt{-\Omega}\cot(\sqrt{-\Omega}\xi/2)\big)\Big\}^{-2}+(4d^3-2\mu\lambda+2\lambda^2d-4\mu d+6\lambda d^2+4\upsilon^2d^3-8\upsilon d^3-6\lambda d^2\upsilon+4\mu\upsilon d)\\&\times\Big\{d-\frac{1}{2(\nu-1)}\big(\lambda+\sqrt{-\Omega}\cot(\sqrt{-\Omega}\xi/2)\big)\Big\}^{-1}+\Big(-2\upsilon^2d^2+4d^2\upsilon+2d\lambda\upsilon-\frac{4}{3}\mu\upsilon-\frac{1}{6}+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2\lambda d\Big), \quad (46)\\u_{2_{14}}(\xi)=&(4\upsilon d^4-4\mu\upsilon d^2-2\mu^2+4\lambda d^3\upsilon-4\lambda d^3-2\lambda^2d^2-2\upsilon^2d^4+4\mu d^2-2d^4+4\mu\lambda d)\\&\times\Big\{d+\frac{1}{2(\nu-1)}\Big\{-\lambda+\sqrt{-\Omega}(\tan(\sqrt{-\Omega}\xi)\pm\sec(\sqrt{-\Omega}\xi))\Big\}\Big\}^{-2}+(4d^3-2\mu\lambda+2\lambda^2d-4\mu d+6\lambda d^2+4\upsilon^2d^3-8\upsilon d^3-6\lambda d^2\upsilon+4\mu\upsilon d)\\&\times\Big\{d+\frac{1}{2(\nu-1)}\Big\{-\lambda+\sqrt{-\Omega}(\tan(\sqrt{-\Omega}\xi)\pm\sec(\sqrt{-\Omega}\xi))\Big\}\Big\}^{-1}+\Big(-2\upsilon^2d^2+4d^2\upsilon+2d\lambda\upsilon-\frac{4}{3}\mu\upsilon-\frac{1}{6}+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2\lambda d\Big). \quad (47)\end{aligned}$$

When $(\nu-1)\neq 0$ and $\lambda=\mu=0$, the solution of Eq. (8) is

$$\begin{aligned}u_{2_{25}}(\xi)=&(4\upsilon d^4-4\mu\upsilon d^2-2\mu^2+4\lambda d^3\upsilon-4\lambda d^3-2\lambda^2d^2-2\upsilon^2d^4+4\mu d^2-2d^4+4\mu\lambda d)\Big\{d-\frac{1}{(\nu-1)\xi+c_1}\Big\}^{-2}+(4d^3-2\mu\lambda+2\lambda^2d-4\mu d+6\lambda d^2+4\upsilon^2d^3-8\upsilon d^3-6\lambda d^2\upsilon+4\mu\upsilon d)\Big\{d-\frac{1}{(\nu-1)\xi+c_1}\Big\}^{-1}+\Big(-2\upsilon^2d^2+4d^2\upsilon+2d\lambda\upsilon-\frac{4}{3}\mu\upsilon-\frac{1}{6}+\frac{4}{3}\mu+\frac{1}{6}V^2-2d^2-\frac{1}{6}\lambda^2-2\lambda d\Big), \quad (48)\end{aligned}$$

where c_1 is an arbitrary constant.

We can write down the other families of exact solutions of Eq. (8), which are omitted for practicality.

Finally, substituting the solutions $G(\xi)$ of Eq. (6) into Eq. (16) and simplifying the resulting equation, we obtain the following solutions.

When $\Omega=\lambda^2-4\mu\nu+4\mu>0$ and $\lambda(\nu-1)\neq 0$ (or $\mu(\nu-1)\neq 0$), we obtain

$$\begin{aligned}u_{3_1}(\xi)=&(4\upsilon-2\upsilon^2-2)\Big\{\frac{1}{2(\nu-1)}(\sqrt{\Omega}\tanh(\sqrt{\Omega}\xi/2))\Big\}^2+\Big\{-\frac{1}{8}\Big(\frac{16\mu^2\upsilon^2-8\mu\upsilon\lambda^2-32\mu^2\upsilon+16\mu^2+8\mu\lambda^2+\lambda^4}{(\upsilon-1)^2}\Big)\Big\}\\&\times\Big\{\frac{1}{2(\nu-1)}(\sqrt{\Omega}\tanh(\sqrt{\Omega}\xi/2))\Big\}^{-2}+\Big(-\frac{4}{3}\mu\upsilon+\frac{1}{3}\lambda^2+\frac{4}{3}\mu-\frac{1}{6}+\frac{1}{6}V^2\Big), \quad (49)\end{aligned}$$

where $\xi=x-Vt$, and λ, μ, and ν are arbitrary constants;

$$\begin{aligned}u_{3_2}(x,t)=&(4\upsilon-2\upsilon^2-2)\Big\{\frac{1}{2(\nu-1)}(\sqrt{\Omega}\coth(\sqrt{\Omega}\xi/2))\Big\}^2+\Big\{-\frac{1}{8}\Big(\frac{16\mu^2\upsilon^2-8\mu\upsilon\lambda^2-32\mu^2\upsilon+16\mu^2+8\mu\lambda^2+\lambda^4}{(\upsilon-1)^2}\Big)\Big\}\\&\times\Big\{\frac{1}{2(\nu-1)}(\sqrt{\Omega}\coth(\sqrt{\Omega}\xi/2))\Big\}^{-2}+\Big(-\frac{4}{3}\mu\upsilon+\frac{1}{3}\lambda^2+\frac{4}{3}\mu-\frac{1}{6}+\frac{1}{6}V^2\Big), \quad (50)\\u_{3_3}(\xi)=&(4\upsilon-2\upsilon^2-2)\Big[\frac{1}{2(\nu-1)}\times\Big\{\sqrt{\Omega}(\tanh(\sqrt{\Omega}\xi)\pm\mathrm{i}\,\mathrm{sech}(\sqrt{\Omega}\xi))\Big\}\Big]^2+\Big\{-\frac{1}{8}\Big(\frac{16\mu^2\upsilon^2-8\mu\upsilon\lambda^2-32\mu^2\upsilon+16\mu^2+8\mu\lambda^2+\lambda^4}{(\upsilon-1)^2}\Big)\Big\}\\&\times\Big[\frac{1}{2(\nu-1)}\Big\{\sqrt{\Omega}(\tanh(\sqrt{\Omega}\xi)\pm\mathrm{i}\,\mathrm{sech}(\sqrt{\Omega}\xi))\Big\}\Big]^{-2}+\Big(-\frac{4}{3}\mu\upsilon+\frac{1}{3}\lambda^2+\frac{4}{3}\mu-\frac{1}{6}+\frac{1}{6}V^2\Big). \quad (51)\end{aligned}$$

For simplicity, others families of exact solutions are omitted.

When $\Omega = \lambda^2 - 4\mu\nu + 4\mu < 0$ and $\lambda(\nu-1) \neq 0$ (or $\mu(\nu-1) \neq 0$), we have

$$u_{3_{12}}(\xi) = (4\upsilon - 2\upsilon^2 - 2)\left\{\frac{1}{2(\nu-1)}(\sqrt{-\Omega}\tan(\sqrt{-\Omega}\xi/2))\right\}^2 + \left\{-\frac{1}{8}\left(\frac{16\mu^2\upsilon^2 - 8\mu\upsilon\lambda^2 - 32\mu^2\upsilon + 16\mu^2 + 8\mu\lambda^2 + \lambda^4}{(\upsilon-1)^2}\right)\right\} \times \left\{\frac{1}{2(\nu-1)}(\sqrt{-\Omega}\tan(\sqrt{-\Omega}\xi/2))\right\}^{-2} + \left(-\frac{4}{3}\mu\upsilon + \frac{1}{3}\lambda^2 + \frac{4}{3}\mu - \frac{1}{6} + \frac{1}{6}V^2\right), \quad (52)$$

$$u_{3_{13}}(\xi) = (4\upsilon - 2\upsilon^2 - 2)\left\{\frac{1}{2(\nu-1)}(\sqrt{-\Omega}\cot(\sqrt{-\Omega}\xi/2))\right\}^2 + \left\{-\frac{1}{8}\left(\frac{16\mu^2\upsilon^2 - 8\mu\upsilon\lambda^2 - 32\mu^2\upsilon + 16\mu^2 + 8\mu\lambda^2 + \lambda^4}{(\upsilon-1)^2}\right)\right\} \times \left\{\frac{1}{2(\nu-1)}(\sqrt{-\Omega}\cot(\sqrt{-\Omega}\xi/2))\right\}^{-2} + \left(-\frac{4}{3}\mu\upsilon + \frac{1}{3}\lambda^2 + \frac{4}{3}\mu - \frac{1}{6} + \frac{1}{6}V^2\right), \quad (53)$$

$$u_{3_{14}}(\xi) = (4\upsilon - 2\upsilon^2 - 2) \left\{\frac{1}{2(\nu-1)}\left\{\sqrt{-\Omega}(\tan(\sqrt{-\Omega}\xi) \pm \sec(\sqrt{-\Omega}\xi))\right\}\right\}^2 + \left\{-\frac{1}{8}\left(\frac{16\mu^2\upsilon^2 - 8\mu\upsilon\lambda^2 - 32\mu^2\upsilon + 16\mu^2 + 8\mu\lambda^2 + \lambda^4}{(\upsilon-1)^2}\right)\right\} \times \left\{\frac{1}{2(\nu-1)}\left\{\sqrt{-\Omega}(\tan(\sqrt{-\Omega}\xi) \pm \sec(\sqrt{-\Omega}\xi))\right\}\right\}^{-2} + \left(-\frac{4}{3}\mu\upsilon + \frac{1}{3}\lambda^2 + \frac{4}{3}\mu - \frac{1}{6} + \frac{1}{6}V^2\right). \quad (54)$$

When $(\nu-1) \neq 0$ and $\lambda = \mu = 0$, the solution of Eq. (8) is

$$u_{3_{25}}(\xi) = (4\upsilon - 2\upsilon^2 - 2)\left\{\frac{\lambda}{2(\nu-1)} - \frac{1}{(\nu-1)\xi + c_1}\right\}^2 + \left\{-\frac{1}{8}\left(\frac{16\mu^2\upsilon^2 - 8\mu\upsilon\lambda^2 - 32\mu^2\upsilon + 16\mu^2 + 8\mu\lambda^2 + \lambda^4}{(\upsilon-1)^2}\right)\right\} \times \left\{\frac{\lambda}{2(\nu-1)} - \frac{1}{(\nu-1)\xi + c_1}\right\}^{-2} + \left(-\frac{4}{3}\mu\upsilon + \frac{1}{3}\lambda^2 + \frac{4}{3}\mu - \frac{1}{6} + \frac{1}{6}V^2\right), \quad (55)$$

where c_1 is an arbitrary constant.

The other families of exact solutions of Eq. (8) are omitted for convenience.

4. Physical explanation

Solutions (17), (19), (42), and (44) are cuspon of the Boussinesq equation. Cuspons are different forms of solitons where solution exhibits cusps at their crests. Unlike peakons, where the derivatives at the peak differ only by a sign, the derivatives at the jump of a cuspon diverge. It is important to note that the soliton solution $u(x,t)$, along with its derivatives, tends to be zero as $|x| \to \infty$. The assumption is that cuspon can be represented as $u(x,t) = e^{-|x-a|^{1/2}}, n > 1$. It can easily be shown that $u_\xi = \infty$ at the cusp, and $u_\xi, u_{\xi\xi} \cdots \to 0$ to characterize the soliton property. Figure 1 below shows the cuspon, obtained from solution (17) of the Boussinesq equation (8). The shapes of figure of solutions (19), (42), and (44) are similar to the figures of solution (17), and so the figures of these solutions are omitted here for convenience.

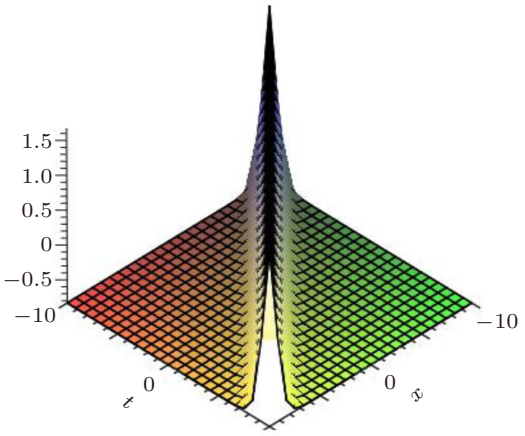


Fig. 1. (color online) Graph of cuspon solution of Eq. (17) for $\lambda = 1, \mu = -1, \nu = 2, d = 1, \alpha = 1$ with $-10 \le x,t \le 10$.

Solutions (39) and (40) are bell-shaped sech^2 solitary traveling wave solutions. Figure 2 shows the shape of the bell-shaped sech^2 solitary traveling wave solution (Shown here is only the shape of solution of Eq. (21) with $V = 1, k = 1, \lambda = 1, \mu = 0, \nu = -2, d = 1$, and $\alpha = 1$ with $-10 \le x,t \le 10$.).

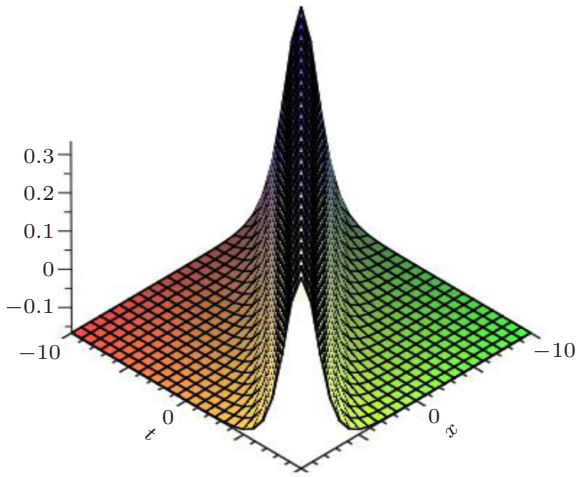


Fig. 2. (color online) Bell-shaped sech^2 solitary traveling wave solution of Eq. (39) with $-10 \le x,t \le 10$.

Solutions (18), (20), (49)–(51), and (55) are singular soliton solutions. Figure 3 shows the shape of the exact singular soliton solution of Eq. (18) of the Boussinesq equation. The shapes of figures of solutions (20), (48)–(51), and (55) are similar to the shape of the figure of solution (18).

The solution of Eq. (21) is a singular kink solution. Figure 4 shows the shape of the exact singular kink-type solution (shown here is only the shape of solution of Eq. (21) with $\lambda = 1, \mu = -1, \nu = 2, d = 3, \alpha = 1$ with $-10 \le x,t \le 10$).

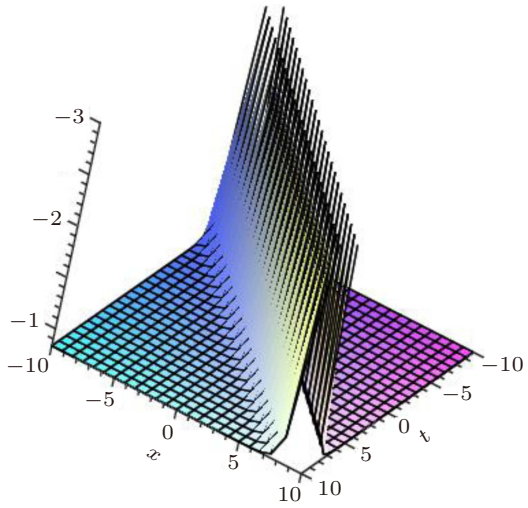


Fig. 3. (color online) Graph of singular soliton solution of Eq. (18) with $-10 \le x,t \le 10$.

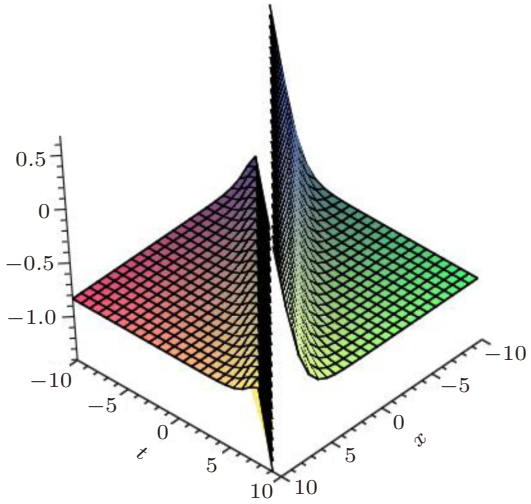


Fig. 4. (color online) Graph of the singular kink traveling wave solution of Eq. (21) with $-10 \le x,t \le 10$.

Solutions (22)–(27), (39)–(41), (43), and (48) describe the soliton, which is a special kind of solitary wave. The soliton solution is a specially localized solution; hence, $u'(\xi)$, $u''(\xi)$, $u'''(\xi) \to 0$ as $\xi \to \pm\infty$ and $\xi = x - ct$. The soliton has a remarkable property in that it keeps its identity upon interacting with other solitons. Figure 5 shows the soliton obtained from solution (22).

Solutions of Eqs. (28), (30), (31), (33), (35), (37), (45)–(47), and (54) represent the exact periodic traveling wave solutions. The periodic solutions are traveling wave solutions that are periodic, such as $\cos(x-t)$. Figure 6 below shows the periodic solution of Eq. (28). The graph of periodic solution of Eq. (28), for $V = 1$, $\lambda = 1$, $\mu = 1$, $\nu = 2$, $d = 2$, and $\alpha = 1$ with $-1 \le x,t \le 1$ is omitted for convenience.

Solutions of Eqs. (29), (32), (34), (36), (38), and (52)–(53) are the exact singular periodic traveling wave solutions. Figure 7 shows the shape of $u_{2_{13}}(x,t)$ with $\lambda = 1$, $\mu = 1$, $\nu = 2$, $d = 2$, $\alpha = 1$, and $-1 \le x,t \le 1$.

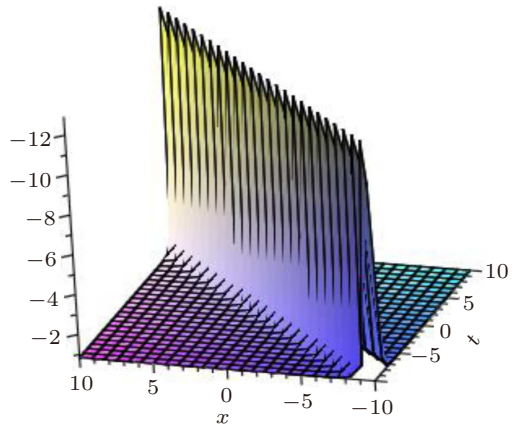


Fig. 5. (color online) Graph of the soliton traveling wave solution of Eq. (22) with $-10 \le x,t \le 10$.

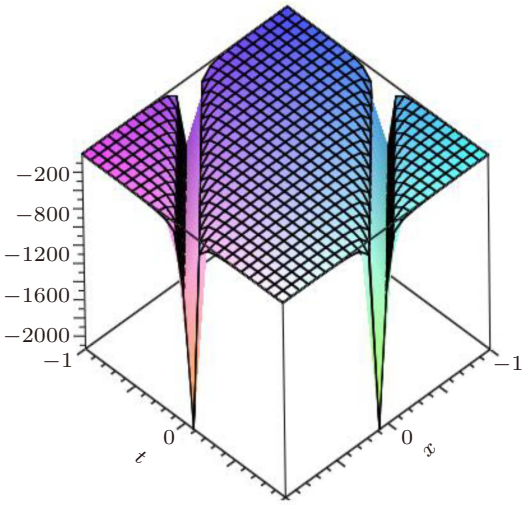


Fig. 6. (color online) Graph of the periodic traveling wave solution of Eq. (28) with $-10 \le x,t \le 10$.

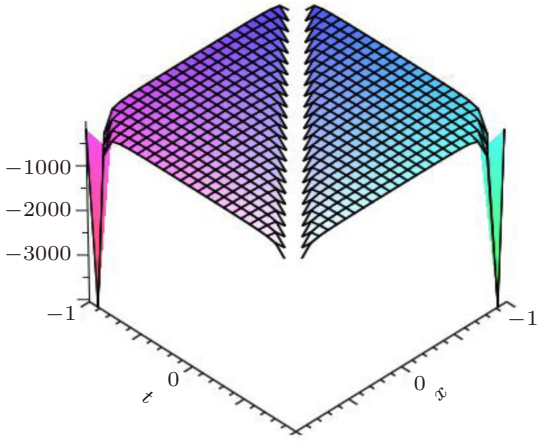


Fig. 7. (color online) Graph of the singular periodic traveling wave solution of Eq. (29) with $-10 \le x,t \le 10$.

Remark 1 We have checked the obtained solutions by putting them back into the original equation and have found them to be correct.

5. Discussion

From the above solutions we observe that if we set $\nu = 0$, $d = 0$,

$$\alpha_0 = \frac{4\lambda^2}{3} - \frac{3\lambda}{2} - \frac{2}{3} - \frac{16\mu}{3} + \frac{4V^2}{6},$$

and λ and μ are replaced by $-\lambda$ and $-\mu$, respectively, in our solutions. Then, Case 1 in Ref. [25] by Neyrame *et al.* is identical to our solution u_{1_1} when $C_2 = 0$ and solution (13) is identical to our solution u_{1_2} when $C_1 = 0$. Similarly, Case 2 in Ref. [25] by Neyrame *et al.* is identical to our solutions $u_{1_{12}}$ and $u_{1_{13}}$ when $C_2 = 0$ and $C_1 = 0$ respectively. On the other hand Case 3 in Ref. [25] by Neyrame *et al.* is identical to our solution $u_{1_{25}}$. Although Neyrame *et al.*[25] did not find any more solutions, by using the proposed new (G'/G)-expansion method, for Set 1, apart from these solutions, we obtain a further twenty new solutions. The solutions obtained in this article for Set 2 and for Set 3 are not obtained by Neyrame *et al.* It can be shown that the solutions obtained by the improved (G'/G)-expansion method[21] and the basic (G'/G)-expansion method[14] are only special cases of our proposed (G'/G)-expansion method.

6. Conclusions

A novel (G'/G)-expansion method has been proposed and applied to the Boussinesq equation in this article. Abundant exact traveling wave solutions of this equation are constructed by the proposed method. The obtained solutions are more general, and many known solutions are only a special cases of these. Furthermore, this study shows that the proposed method is quite efficient and practically well suited to be used in finding exact solutions of NLEEs. It is noteworthy to observe that the basic (G'/G)-expansion method, the improve (G'/G)-expansion method and the generalized and improved (G'/G)-expansion method are only special cases of the proposed new (G'/G)-expansion method and, thus, the new (G'/G)-expansion method would be a powerful mathematical tool for solving NLEEs.

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