On the exact solutions to the long–short-wave interaction system*

Fan Hui-Ling(范慧玲)†, Fan Xue-Fei(范雪飞), and Li Xin(李 欣)

School of Science, Heilongjiang Bayi Agriculture University, Daqing 163319, *China*

(Received 11 June 2013; revised manuscript received 21 July 2013; published online 10 December 2013)

The complete discrimination system for polynomial method is applied to the long–short-wave interaction system to obtain the classifications of single traveling wave solutions. Compared with the solutions given by the (G'/G)-expansion method, we gain some new solutions.

Keywords: single traveling wave solution, complete discrimination system for polynomial method, long–short-wave interaction system

PACS: 02.30.Jr, 02.70.Wz, 05.45.Yv, 94.05.Fg **DOI:** 10.1088/1674-1056/23/2/020201

1. Introduction

In Ref. [1], Ahmet Bekir *et al.* gave three kinds of solutions of the long–short-wave interaction system by the (G'/G)-expansion method. The long–short-wave interaction system reads as

$$\begin{aligned} &\mathrm{i}\psi_t + \psi_{xx} - \psi v = 0,\\ &v_t + v_x + (|\psi|^2)_x = 0, \end{aligned} \tag{1}$$

where v and ψ are the amplitude of the long wave and the envelope of the short wave. Equation (1) is important in wave dynamics and other physical problems.

In the present paper, we first use the complete discrimination system for polynomial method[2–9] to Eq. (1) and give the classification of its all single traveling wave solutions. Next, we compare both of the solutions given by the two methods and certify that our solutions include the solutions given by the (G'/G)-expansion method. Moreover, we also obtain new solutions.

The complete discrimination system for polynomial method was recently proposed by Liu. By Liu's method, many nonlinear partial differential equations have been solved. For example, Wang and Li[10] used Liu's method and other methods to investigate the single and multi-solitary solutions to some nonlinear evolution equations; Wang *et al.*[11] obtained the classification of traveling wave solutions to CH–DP equation; and, Fan[12] gave the classification of the single traveling wave solutions to the generalized equal width equation, and so on.

There are also a lot of other powerful methods of finding exact solutions to nonlinear differential equations; for example, trial equation method,[13] Darboux transformation, and variable separation approach,[14–16] and so on.

2. Classification and comparison

2.1. Classification

Taking the following transformation to Eq. (1)

$$\begin{aligned} &\psi(x,t) = \mathrm{e}^{\mathrm{i}\theta}u(\xi),\ \ v(x,t) = v(\xi),\\ &\theta = px+qt,\ \ \xi = x-ct, \end{aligned} \tag{2}$$

where p, q are real constants, and $u(x,t), v(x,t)$ are real functions. Substituting Eq. (2) into Eq. (1), we find $c = 2p$ and equation (1) turns into the following form

$$u'' - (q+p^2)u - uv = 0, \tag{3}$$

$$(2p-1)v' = 2uu'. \tag{4}$$

Integrating Eq. (4), we have

$$v = \frac{1}{2p-1}u^2 + g_1, \tag{5}$$

where g_1 is an integration constant.

Substituting Eq. (5) into Eq. (3), we have

$$u'' = \frac{1}{2p-1}u^3 + (q+p^2+g_1)u. \tag{6}$$

Integrating Eq. (6), we have

$$(u')^2 = \frac{1}{4p-2}u^4 + (q+p^2+g_1)u^2 + g_2, \tag{7}$$

where g_2 is an integration constant. According to the above Eq. (7), we will give the traveling wave solutions to Eq. (1).

When $p > 1/2$, we take the transformation

$$W_1 = \left(\frac{1}{4p-2}\right)^{1/4}u,\ \ \xi_1 = \left(\frac{1}{4p-2}\right)^{1/4}\xi,$$

equation (7) turns into the following form

$$W_{1\xi_1}^2 = W_1^4 + p_1W_1^2 + p_2, \tag{8}$$

where $p_1 = \sqrt{4p-2}(q+p^2+g_1)$, $p_2 = g_2$.

*Project supported by the Scientific Research Fund of Education Department of Heilongjiang Province of China (Grant No. 12531475).

†Corresponding author. E-mail: huilingnepu@126.com

© 2014 Chinese Physical Society and IOP Publishing Ltd

When $p < 1/2$, we take the transformation

$$W_1 = \left(\frac{1}{2-4p}\right)^{1/4} u, \quad \xi_1 = \left(\frac{1}{2-4p}\right)^{1/4} \xi,$$

equation (7) turns into the following form

$$W_{1\xi_1}^2 = -(W_1^4 + p_1 W_1^2 + p_2), \tag{9}$$

where $p_1 = -\sqrt{2-4p}(q+p^2+g_1)$, $p_2 = -g_2$. We continue to take the transformation $W_1^2 = W$, then equation (8) and equation (9) become

$$W_{\xi_1}^2 = \epsilon[4W(W^2 + p_1 W + p_2)]. \tag{10}$$

where $\epsilon = \pm 1$, ($\epsilon = 1$ and $p > 1/2$ are equivalent, $\epsilon = -1$ and $p < 1/2$ are equivalent). Integrating Eq. (10) yields

$$\pm 2(\xi_1 - \xi_0) = \int \frac{\mathrm{d}W}{\sqrt{\epsilon W F(W)}}, \tag{11}$$

where $F(W) = W^2 + p_1 W + p_2$, and ξ_0 is an integration constant. Write the discrimination of $F(W)$ as follows:

$$\Delta = p_1^2 - 4p_2. \tag{12}$$

According to the discrimination, we give the corresponding traveling wave solutions to Eq. (1).

Case 1 $\Delta = 0$. Then, we have

$$F(W) = \left(W + \frac{p_1}{2}\right)^2. \tag{13}$$

When $\epsilon = 1, p_1 < 0$, the solutions of Eq. (10) are

$$W = -\frac{p_1}{2}\tanh^2\left[\sqrt{-\frac{p_1}{2}}(\xi_1 - \xi_0)\right]; \tag{14}$$

$$W = -\frac{p_1}{2}\coth^2\left[\sqrt{-\frac{p_1}{2}}(\xi_1 - \xi_0)\right]. \tag{15}$$

According to solutions (14) and (15), we can obtain the corresponding traveling wave solutions of Eq. (1)

$$\begin{aligned}\psi(x,t) &= \pm \mathrm{e}^{\mathrm{i}(px+qt)}\sqrt{(2p-1)(-q-p^2-g_1)} \\ &\quad \times \tanh\left[\sqrt{\frac{-q-p^2-g_1}{2}}(x-2pt-\xi_0)\right]; \end{aligned}\tag{16}$$

$$\begin{aligned}v(x,t) &= -(q+p^2+g_1)\tanh^2\left[\sqrt{\frac{-q-p^2-g_1}{2}}\right. \\ &\quad \left.\times (x-2pt-\xi_0)\right] + g_1; \end{aligned}\tag{17}$$

$$\begin{aligned}\psi(x,t) &= \pm \mathrm{e}^{\mathrm{i}(px+qt)}\sqrt{(2p-1)(-q-p^2-g_1)} \\ &\quad \times \coth\left[\sqrt{\frac{-q-p^2-g_1}{2}}(x-2pt-\xi_0)\right]; \end{aligned}\tag{18}$$

$$\begin{aligned}v(x,t) &= -(q+p^2+g_1)\coth^2\left[\sqrt{\frac{-q-p^2-g_1}{2}}\right. \\ &\quad \left.\times (x-2pt-\xi_0)\right] + g_1. \end{aligned}\tag{19}$$

When $\epsilon = 1$, $p_1 > 0$, the solutions of Eq. (10) are

$$W = \frac{p_1}{2}\tan^2\left[\sqrt{\frac{p_1}{2}}(\xi_1 - \xi_0)\right]. \tag{20}$$

According to solution (20), we can obtain the corresponding solutions of Eq. (1)

$$\begin{aligned}\psi(x,t) &= \pm \mathrm{e}^{\mathrm{i}(px+qt)}\sqrt{(2p-1)(q+p^2+g_1)} \\ &\quad \times \tan\left[\sqrt{\frac{q+p^2+g_1}{2}}(x-2pt-\xi_0)\right]; \end{aligned}\tag{21}$$

$$\begin{aligned}v(x,t) &= (q+p^2+g_1)\tan^2\left[\sqrt{\frac{q+p^2+g_1}{2}}\right. \\ &\quad \left.\times (x-2pt-\xi_0)\right] + g_1. \end{aligned}\tag{22}$$

When $\epsilon = 1, p_1 = 0$, the solutions of Eq. (10) are

$$W = \frac{1}{(\xi_1 - \xi_0)^2}. \tag{23}$$

According to solutions (23), we can obtain the corresponding solutions of Eq. (1) as

$$\psi(x,t) = \pm\sqrt{4p-2}\,\mathrm{e}^{\mathrm{i}(px+qt)}/x - 2pt - \xi_0; \tag{24}$$

$$v(x,t) = \frac{2}{(x-2pt-\xi_0)^2} + g_1. \tag{25}$$

Case 2 $\Delta > 0, p_2 = 0$. We have

$$F(W) = W(W + p_1). \tag{26}$$

When $\epsilon = 1, p_1 > 0$, the solutions of Eq. (10) are

$$W = p_1\tanh^2[\sqrt{p_1}(\xi_1 - \xi_0)] - p_1; \tag{27}$$

$$W = p_1\coth^2[\sqrt{p_1}(\xi_1 - \xi_0)] - p_1. \tag{28}$$

When $\epsilon = -1, p_1 < 0$, the solutions of Eq. (10) are

$$W = p_1\tanh^2[\sqrt{-p_1}(\xi_1 - \xi_0)] - p_1; \tag{29}$$

$$W = p_1\coth^2[\sqrt{-p_1}(\xi_1 - \xi_0)] - p_1. \tag{30}$$

According to solutions (27)–(30), when $q + p^2 + g_1 > 0, \epsilon = \pm 1$, the corresponding solutions of Eq. (1) are

$$\begin{aligned}\psi(x,t) &= \pm \mathrm{e}^{\mathrm{i}(px+qt)}\left\{(4p-2)(q+p^2+g_1)\right. \\ &\quad \times \tanh^2\left[\sqrt{q+p^2+g_1}(x-2pt-\xi_0)\right] \\ &\quad \left.-(4p-2)(q+p^2+g_1)\right\}^{1/2}; \end{aligned}\tag{31}$$

$$\begin{aligned}v(x,t) &= 2(q+p^2+g_1)\tanh^2\left[\sqrt{q+p^2+g_1}\right. \\ &\quad \left.\times (x-2pt-\xi_0)\right] - 2(q+p^2) - g_1; \end{aligned}\tag{32}$$

$$\begin{aligned}\psi(x,t) &= \pm \mathrm{e}^{\mathrm{i}(px+qt)}\left\{(4p-2)(q+p^2+g_1)\right. \\ &\quad \times \coth^2\left[\sqrt{q+p^2+g_1}(x-2pt-\xi_0)\right] \end{aligned}$$

$$-(4p-2)(q+p^2+g_1)\Big\}^{1/2}; \tag{33}$$

$$v(x,t) = 2(q+p^2+g_1)\coth^2\Big[\sqrt{q+p^2+g_1} \times (x-2pt-\xi_0)\Big] - 2(q+p^2) - g_1. \tag{34}$$

When $\epsilon = 1$, $p_1 < 0$, the solutions of Eq. (10) are

$$W = -p_1\tan^2[\sqrt{-p_1}(\xi_1-\xi_0)] - p_1. \tag{35}$$

According to solutions (35), the corresponding solutions of Eq. (1) are

$$\psi(x,t) = \pm e^{i(px+qt)}\Big\{(2-4p)(q+p^2+g_1) \times \tan^2\Big[\sqrt{-(q+p^2+g_1)}(x-2pt-\xi_0)\Big] - (4p-2)(q+p^2+g_1)\Big\}^{1/2}; \tag{36}$$

$$v(x,t) = -2(q+p^2+g_1)\tan^2\Big[\sqrt{-(q+p^2+g_1)} \times (x-2pt-\xi_0)\Big] - 2(q+p^2) - g_1. \tag{37}$$

Case 3 $\Delta > 0, p_2 \neq 0$. When $\epsilon = 1$, we assume $\alpha < \beta < \gamma$, one of them is zero, the rest of them are roots of $F(W)$.

If $\alpha < W < \beta$, we take the transformation $W = \alpha + (\beta - \alpha)\sin^2\varphi$. According to Eq. (11), we have

$$\pm 2(\xi_1-\xi_0) = \frac{2}{\sqrt{\gamma-\alpha}}\int \frac{d\varphi}{\sqrt{1-m^2\sin^2\varphi}}, \tag{38}$$

where $m^2 = (\beta-\alpha)/(\gamma-\alpha)$. According to Eq. (11) and the definition of Jacobi elliptic function sn, we have

$$\mathrm{sn}^2(\pm\sqrt{\gamma-\alpha}(\xi_1-\xi_0),m) = \sin^2\varphi = \frac{W-\alpha}{\beta-\alpha}, \tag{39}$$

the solutions of Eq. (10) are

$$W = \alpha + (\beta-\alpha)\mathrm{sn}^2(\sqrt{\gamma-\alpha}(\xi_1-\xi_0),m); \tag{40}$$

the corresponding solutions of Eq. (1) are

$$\psi(x,t) = \pm e^{i(px+qt)}\Big\{\sqrt{4p-2}\Big[\alpha + (\beta-\alpha) \times \mathrm{sn}^2\Big(\sqrt{\frac{\gamma-\alpha}{\sqrt{4p-2}}}(x-2pt-\xi_0),m\Big)\Big]\Big\}^{1/2}; \tag{41}$$

$$v(x,t) = \frac{1}{\sqrt{4p-2}}\Big[\alpha + (\beta-\alpha)\mathrm{sn}^2\Big(\sqrt{\frac{\gamma-\alpha}{\sqrt{4p-2}}} \times (x-2pt-\xi_0),m\Big)\Big] + g_1. \tag{42}$$

If $W > \gamma$, we take the following transformation: $W = (-\beta\sin^2\varphi + \gamma)/\cos^2\varphi$, then we obtain the solutions of Eq. (10)

$$W = \frac{-\beta\,\mathrm{sn}(\sqrt{\gamma-\alpha}(\xi_1-\xi_0),m) + \gamma}{\mathrm{cn}(\sqrt{\gamma-\alpha}(\xi_1-\xi_0),m)}; \tag{43}$$

the corresponding solutions of Eq. (1) are

$$\psi(x,t) = \pm e^{i(px+qt)}\Big\{\Big[-\beta\sqrt{4p-2}\,\mathrm{sn}\Big(\sqrt{\frac{\gamma-\alpha}{\sqrt{4p-2}}} \times (x-2pt-\xi_0),m\Big)\Big]\Big/\mathrm{cn}\Big(\sqrt{\frac{\gamma-\alpha}{\sqrt{4p-2}}} \times (x-2p-\xi_0),m\Big)\Big\}^{1/2}; \tag{44}$$

$$v(x,t) = -\beta\,\mathrm{sn}\Big(\sqrt{\frac{\gamma-\alpha}{\sqrt{4p-2}}}(x-2pt-\xi_0),m\Big) \Big/\sqrt{4p-2}\,\mathrm{cn}\Big(\sqrt{\frac{\gamma-\alpha}{\sqrt{4p-2}}}(x-2p-\xi_0),m\Big) + g_1. \tag{45}$$

Case 4 $\Delta < 0$. According to Eq. (10), when $\epsilon = 1$, $p_2 > 0$, we take the transformation $W = \sqrt{p_2}\tan^2(\varphi/2)$, according to Eq. (11), we have

$$\pm 2(\xi_1-\xi_0) = (p_2)^{-1/4}\int \frac{d\varphi}{\sqrt{1-m^2\sin^2\varphi}}, \tag{46}$$

where

$$m^2 = \frac{1}{2}\left(1 - \frac{\sqrt{4p-2}(q+p^2+g_1)}{2\sqrt{g_2}}\right),$$

the solutions of Eq. (10) are

$$W = \frac{2\sqrt{p_2}}{1+\mathrm{cn}(2p_2^{1/4}(\xi_1-\xi_0),m)} - \sqrt{p_2}. \tag{47}$$

The corresponding solutions of Eq. (1) are

$$\psi(x,t) = \pm e^{i(px+qt)}\Big\{2\sqrt{(4p-2)g_2} \Big/\Big[1+\mathrm{cn}\Big(2\Big(\frac{g_2}{4p-2}\Big)^{1/4}(x-2pt-\xi_0),m\Big)\Big] - \sqrt{(4p-2)g_2}\Big\}^{1/2}; \tag{48}$$

$$v(x,t) = 2\sqrt{g_2}\Big/\Big[\sqrt{(4p-2)}\Big(1+\mathrm{cn}(2\Big(\frac{g_2}{4p-2}\Big)^{1/4} \times (x-2pt-\xi_0),m)\Big)\Big] - \frac{\sqrt{g_2}}{\sqrt{4p-2}} + g_1. \tag{49}$$

Remark 1 All solutions given are envelop traveling wave solutions, among these: solutions (16)–(19) and (31)–(34) contain hyperbolic functions; The solutions (21), (22), (36), and (37) contain trigonometric functions; solutions (24) and (25) contain rational functions; and, solutions (41), (42), (44), (45), (48), and (49) contain elliptic functions.

2.2. Comparison

The solutions (21) and (22) in Ref. [1] can be expressed as

$$\psi(x,t) = \mp e^{i(px+qt)}\sqrt{\Big(\alpha^2 - \frac{\alpha}{2}\Big)(\lambda^2-4\mu)}$$

$$\times \tanh\frac{\sqrt{\lambda^2-4\mu}}{2}(x-2\alpha t-\xi_0); \qquad (50)$$

$$v(x,t) = \alpha^2\frac{\lambda^2-4\mu}{2}\tanh^2\frac{\sqrt{\lambda^2-4\mu}}{2} \times (x-2\alpha t-\xi_0)-\alpha^2\frac{2-\lambda^2+4\mu}{2}; \qquad (51)$$

where

$$\xi_0=\frac{-2\tanh^{-1}(-C_2/C_1)}{\sqrt{\lambda^2-4\mu}}.$$

They are just the solutions (16) and (17) in the present paper.

The solutions (28) and (29) in Ref. [1] can be expressed as

$$\psi(x,t) = \mp e^{i(px+qt)}\sqrt{\left(\alpha^2-\frac{\alpha}{2}\right)(4\mu-\lambda^2)} \times \tan\frac{\sqrt{\lambda^2-4\mu}}{2}(x-2\alpha t-\xi_0); \qquad (52)$$

$$v(x,t) = \alpha^2\frac{4\mu-\lambda^2}{2}\tan^2\frac{\sqrt{\lambda^2-4\mu}}{2} \times (x-2\alpha t-\xi_0)-\alpha^2\frac{2+\lambda^2-4\mu}{2}; \qquad (53)$$

where

$$\xi_0=\frac{-2\arctan(-C_2/C_1)}{\sqrt{4\mu-\lambda^2}}.$$

They are just the solutions (21) and (22) in the present paper.

The solutions (35) and (36) in Ref. [1] can be expressed as

$$\psi(x,t)=\mp e^{i(px+qt)}\frac{\sqrt{\alpha^2-\alpha/2}}{x-2\alpha t-\xi_0}; \qquad (54)$$

$$v(x,t)=\frac{2\alpha^2}{(x-2\alpha t-\xi_0)^2}+\alpha^2; \qquad (55)$$

where $\xi_0=-C_2/C_1$. They are just the solutions (24) and (25) in the present paper.

The solutions given by the (G'/G)-expansion method are merely the solutions of Case 1 in the present paper.

3. Conclusion

In the present paper, we use the complete discrimination system for polynomial method to the long–short-wave interaction system, and we obtain the classification of envelop traveling wave solutions. These solutions contain trigonometric functions and elliptic functions. By our verification, the solutions given by the (G'/G)-expansion method in Ref. [1] are merely part of our solutions.

Acknowledgments

We offer our thanks to our reviewers for their helpful comments and suggestions.

References

[1] Ahmet Bekir, Burcu Ayhan and M Naci Ozer 2013 *Chin. Phys. B* **22** 010202
[2] Liu C S 2007 *Commun. Theor. Phys.* **48** 601
[3] Liu C S 2006 *Commun. Theor. Phys.* **45** 991
[4] Liu C S 2007 *Chin. Phys.* **16** 1832
[5] Liu C S 2008 *Commun. Theor. Phys.* **49** 153
[6] Liu C S 2008 *Commun. Theor. Phys.* **49** 291
[7] Liu C S 2010 *Comput. Phys. Commun.* **181** 317
[8] Liu C S 2004 *Chin. Phys. Lett.* **21** 2369
[9] Liu C S 2005 *Chin. Phys.* **14** 1710
[10] Wang D S and Li H B 2008 *Math. Anal. Appl.* **343** 273
[11] Wang C Y, Guan J and Wang B Y 2011 *Pramana-J. Phys.* **77** 759
[12] Fan H L 2012 *Appl. Math. Comput.* **219** 748
[13] Liu C S 2005 *Acta Phys. Sin.* **54** 2505 (in Chinese)
[14] Hu H C, Lou S Y and Liu Q P 2003 *Chin. Phys. Lett.* **20** 1413
[15] Ying J P and Lou S Y 2003 *Chin. Phys. Lett.* **20** 1448
[16] Fan E G, Zhang H Q and Lin G 1998 *Chin. Phys.* **7** 649