Dual solutions in boundary layer flow of Maxwell fluid over a porous shrinking sheet*

Krishnendu Bhattacharyya[a)], Tasawar Hayat[b)†], and Ahmed Alsaedi[c)]

[a)] *Department of Mathematics, The University of Burdwan, Burdwan-713104, West Bengal, India*
[b)] *Department of Mathematics, Quaid-I-Azam University 45320, Islamabad 44000, Pakistan*
[c)] *Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia*

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An analysis is carried out for dual solutions of the boundary layer flow of Maxwell fluid over a permeable shrinking sheet. In the investigation, a constant wall mass transfer is considered. With the help of similarity transformations, the governing partial differential equations (PDEs) are converted into a nonlinear self-similar ordinary differential equation (ODE). For the numerical solution of transformed self-similar ODE, the shooting method is applied. The study reveals that the steady flow of Maxwell fluid is possible with a smaller amount of imposed mass suction compared with the viscous fluid flow. Dual solutions for the velocity distribution are obtained. Also, the increase of Deborah number reduces the boundary layer thickness for both solutions.

Keywords: dual solutions, boundary layer flow, Maxwell fluid, porous shrinking sheet

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1. Introduction

Boundary layer flows due to stretching/shrinking sheet have aroused the interest of researches for their enormous applications in many engineering processes. The viscous flow over a linearly stretching plate was first studied by Crane.[1] The pioneering work of Crane[1] was extended by Gupta and Gupta,[2] Chen and Char,[3] Pavlov[4] and Zheng *et al.*[5–7] with considering various physical conditions. In the application of Crane problem in unsteady liquid film, Wang[8] observed the flow due to shrinking. The flow due to shrinking of flat sheet has unusual properties. The steady boundary layer shrinking flow does not occur, because the generated vorticity due to shrinking goes off the site of the boundary layer, i.e., boundary layer separation appears. Miklavčič and Wang[9] considered the viscous flow due to a shrinking sheet with suction and established the criteria for the existence and uniqueness of the similarity solution. They also found that the possibility of steady flow depends on externally imposed mass suction, i.e., the mass suction controls the boundary layer separation. Hayat *et al.*[10,11] obtained the analytic solution of magnetohydrohdynamic (MHD) flow of second grade fluid over a shrinking sheet by using homotopy analysis method (HAM). They reported the analytic solution for MHD rotating flow of second grade fluid over a shrinking surface. The boundary layer flow and heat transfer over an exponentially shrinking sheet were investigated by Bhattacharyya.[12] Bhattacharyya and Layek[13] showed the effects of suction/blowing on the steady boundary layer stagnation-point flow and heat transfer towards a shrinking sheet in the presence of thermal radiation. Bhattacharyya *et al.*[14] considered the slip effects on stagnation-point flow and heat transfer past a shrinking sheet. Bhattacharyya[15] explained the mass transfer in boundary layer stagnation-point flow over a shrinking/stretching sheet with chemical reaction.

In the past few decades, considerable attention has been paid to the study of the flows involving non-Newtonian fluids, i.e., fluids whose flow properties differ from those of Newtonian fluids. Many materials in real field, such as melts, muds, condensed milk, glues, printing ink, emulsions, soaps, sugar solution, paints, shampoos, tomato paste, etc. show the non-Newtonian flow characteristics. So, the main difficulty is to construct a single constitutive equation exhibiting all properties of such non-Newtonian fluids. Because of this, various non-Newtonian fluid models[16–20] have been proposed in the literature. The governing equations of non-Newtonian fluids are highly nonlinear and much more complicated than the governing equations of Newtonian fluids. Maxwell fluid model[21,22] is one of the non-Newtonian fluid models, where the relaxation phenomenon is taken into consideration. This model has applications in viscoelastic problems where the dimensionless relaxation time is small.

The flow of Maxwell fluid over a shrinking sheet is very interesting, because it is exciting to find out that with the change of nature of the fluid from Newtonian to Maxwell, the vorticity generation due to shrinking of the sheet either

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†Corresponding author. E-mail: pensy_t@yahoo.com

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changes or does not. Hayat *et al.*[23] investigated a similar problem quite differently and they obtained a series solution. In the present paper, an analysis is made to see the change in vorticity generation due to shrinking when the fluid is viscoelastic. The existence, uniqueness and non-existence are discussed numerically using the shooting method, and numerical results are presented in the figures.

2. Mathematical formulation of the problem

Consider a two-dimensional (2D) steady boundary layer flow of Maxwell fluid over a porous shrinking sheet with wall mass transfer. The sheet coincides with the plane $y=0$ and the flow occupies the region $y>0$. The x and y axes are taken to be along and perpendicular to the sheet, respectively. The governing equations of motion for Maxwell fluid may be written in usual notation as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\lambda\left[u^2\frac{\partial^2 u}{\partial x^2}+v^2\frac{\partial^2 u}{\partial y^2}+2uv\frac{\partial^2 u}{\partial x\partial y}\right]=\upsilon\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

where u and v are the velocity components in the x and y directions respectively, $\upsilon(=\mu/\rho)$ is the kinematic fluid viscosity, ρ is the fluid density, μ is the coefficient of fluid viscosity, and λ is the relaxation time.

The boundary conditions are given by

$$u=U_w(x)=-cx,\ \ v=v_{\rm w}\ \ \text{at}\ \ y=0;\ \ u\to 0\ \ \text{as}\ \ y\to\infty, \tag{3}$$

where $U_{\rm w}(x)=-cx$ is the shrinking velocity with $c>0$ and c is the shrinking constant and $v_{\rm w}$ is the wall mass transfer velocity with $v_{\rm w}<0$ for suction and $v_{\rm w}>0$ for blowing.

Now, the stream function $\psi(x,y)$ is introduced as

$$u=\frac{\partial\psi}{\partial y}\ \text{and}\ v=-\frac{\partial\psi}{\partial x}. \tag{4}$$

For relations in Eq. (4), equation (1) is identically satisfied and equation (2) is reduced to the following form:

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}+\lambda\Big[\Big(\frac{\partial\psi}{\partial y}\Big)^2\frac{\partial^3\psi}{\partial x^2\partial y}+\Big(\frac{\partial\psi}{\partial x}\Big)^2\frac{\partial^3\psi}{\partial y^3}-2\frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial x\partial y^2}\Big]=\upsilon\frac{\partial^3\psi}{\partial y^3}. \tag{5}$$

The boundary conditions in Eq. (3) become

$$\frac{\partial\psi}{\partial y}=U_{\rm w}(x),\ \ \frac{\partial\psi}{\partial x}=-v_{\rm w}\ \text{at}\ y=0;\ \ \frac{\partial\psi}{\partial y}\to 0\ \text{as}\ y\to\infty. \tag{6}$$

Next, the dimensionless variable for ψ is brought in as

$$\psi=\sqrt{\upsilon c}\,xf(\eta), \tag{7}$$

where η is the similarity variable and is defined as $\eta=y\sqrt{c/\upsilon}$.

Using relation (7) and the similarity variable, finally the following nonlinear self-similar equation is obtained:

$$f'''+ff''-f'^2+\beta(2ff'f''-f^2f''')=0, \tag{8}$$

where $\beta=\lambda c$ is the Deborah number.

The boundary conditions (6) reduce to the following forms:

$$f(\eta)=S,\ f'(\eta)=-1\ \text{at}\ \eta=0;\ \ f'(\eta)\to 0\ \text{as}\ \eta\to\infty, \tag{9}$$

where $S=-v_{\rm w}/\sqrt{\upsilon c}$ is the mass transfer parameter, $S>0$ (i.e. $v_{\rm w}<0$) corresponds to mass suction, and $S<0$ (i.e. $v_{\rm w}>0$) corresponds to mass blowing.

3. Numerical method for solution

The nonlinear ordinary differential equation (8) along with the boundary conditions (9) form a two-point boundary value problem (BVP) and is solved by using the shooting method through converting it into an initial value problem (IVP). In this method it is necessary to choose a suitable finite value of $\eta(\to\infty)$, say, η_∞. The first-order system is set as follows:

$$f'=p,\ p'=q,\ q'=(p^2-fq-2\beta fpq)/(1-\beta f^2) \tag{10}$$

with the boundary conditions

$$f(0)=S,\ p(0)=-1. \tag{11}$$

To solve Eq. (10) with boundary conditions (11) serving as an IVP, the values for $q(0)$ i.e., $f''(0)$ must be needed but no such value is given. The initial guess for $f''(0)$ is chosen and the fourth order Runge–Kutta method is used to obtain a solution. Then the calculated value of $f'(\eta)$ at $\eta_\infty(=20)$ is compared with the given boundary condition $f'(\eta_\infty)=0$ and the value of $f''(0)$ is adjusted by using the "secant method" to give a better approximation for the solution. The step-size is taken to be $\Delta\eta=0.01$. The process is repeated until the results are correct up to the desired accuracy of level 10^{-6}.

4. Results and discussion

The existence, uniqueness and non-existence of the 2D steady flow of Maxwell fluid over a permeable shrinking sheet with wall mass transfer are explored by the numerical scheme described above. According to Miklavčič and Wang,[8] as well as Fang and Zhang[12] the steady flow of Newtonian fluid past a porous shrinking sheet is possible only when the mass transfer (suction) parameter is greater than or equal to 2. In the present problem, due to the change of the fluid type from Newtonian ($\beta=0$) to Maxwell ($\beta>0$) the generation of velocity, owing to shrinking of the sheet, is reduced, and consequently to confine the generated vorticity the requirement for the amount of mass suction through the porous sheet is also

reduced. So, for the steady flow of Maxwell fluid the essential quantity of mass suction is smaller than that of Newtonian fluid. Actually, the Deborah number (β) characterizes the intrinsic fluidity of a material: large Deborah number signifies less fluid material and for Maxwell fluid the Deborah number is positive; so the vorticity generation due to shrinking for Maxwell fluid is not so considerable as that for Newtonian fluid. From the numerical investigation, it is found that for Deborah number, $\beta = 0.1$, the value of mass transfer parameter S needs to be greater than or equal to 1.907045 to maintain the boundary layer structure. For a larger value of Deborah number, i.e., $\beta = 0.2$, the steady flow is possible when $S \geq 1.801921$. So, for Maxwell ($\beta > 0$) fluid the boundary layer separation is delayed and accordingly the similarity solution exists for less amount of mass suction through the porous sheet. In this regard, the values of $f''(0)$ against S for different values of β are plotted in Fig. 1. It is clear from the figure that for $\beta = 0.1$, dual similarity solutions exist when $S \geq 1.907045$ and no similarity solution is found for $S < 1.907045$, i.e., boundary layer separation occurs. Also, for $\beta = 0.2$, the dual solutions are obtained when $S \geq 1.801921$ and boundary layer solution disappears for $S < 1.801921$. Furthermore, from the figure it is also observed that the value of $f''(0)$ increases with the augment of β for first solution and it decreases for second solution.

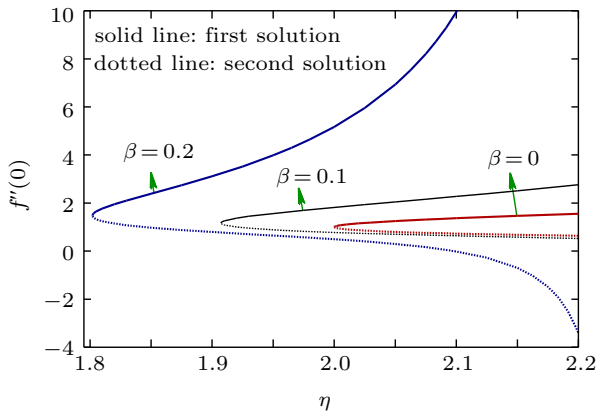


Fig. 1. (color online) Values of $f''(0)$ versus S for different values of β.

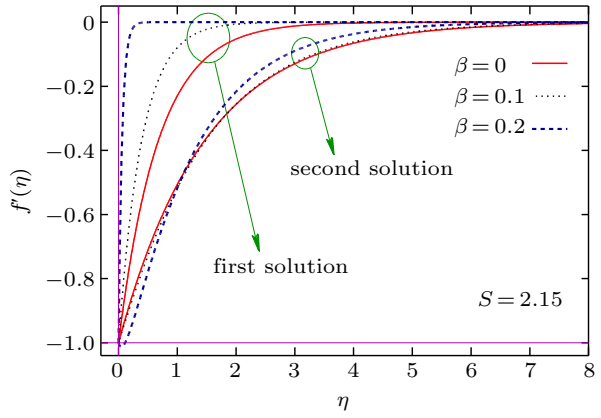


Fig. 2. (color online) Dual velocity profiles $f'(\eta)$ for different values of β.

The variations in dimensionless velocity profiles for different values of Deborah number β are presented in Fig. 2. Here the dual solutions can be seen to be different. It is also noted that the boundary layer thickness for second solution is always bigger than the thickness for first solution. The velocity $f'(\eta)$ at a point increases with the increase of β in first solution and so the boundary layer thickness decreases. On the other hand, in the case of second solution the velocity initially decreases with the increase of β and for large η it increases with the augment of β; accordingly the boundary layer thickness decreases. So, the increase of Deborah number causes the decrease of thickness of the boundary layer for both solutions, because the increase of Deborah number leads to the cutback of fluid material from the flow field.

The effect of mass suction on velocity distribution is very important because the mass suction plays a central role in controlling the generated vorticity due to shrinking and in preserving the boundary layer smooth fluid flow. The velocity profiles for different values of mass suction parameter $S(> 0)$ are exhibited in Fig.3. For the first solution, the velocity $f'(\eta)$ increases with increasing the value of S and so the boundary layer thickness decreases. The reverse picture is shown for the second solution, i.e., the velocity decreases with the increase of S and the boundary layer thickness increases.

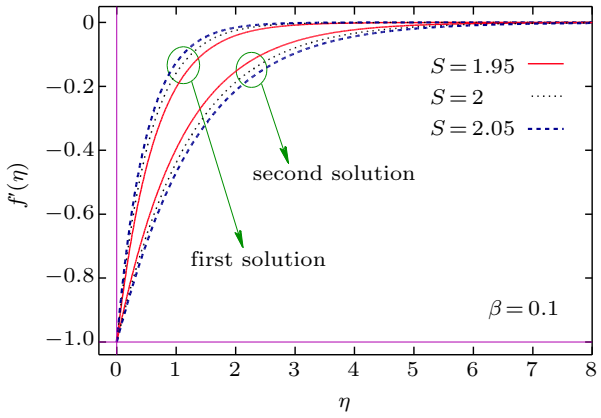


Fig. 3. (color online) Dual velocity profiles $f'(\eta)$ for different values of S.

5. Conclusions

In this paper, the boundary layer flow of Maxwell fluid over a permeable shrinking sheet with wall mass transfer is studied. The transformed self-similar equations are solved numerically by using the shooting method. The study explores the conditions of steady flow of Maxwell fluid. For increasing Deborah number the necessity of the amount of mass suction becomes less to control the boundary layer separation. So, for Maxwell fluid the steady flow takes place with a smaller amount of mass suction. Furthermore, dual solutions for the similarity equation are observed.

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