Noether's theorems of a fractional Birkhoffian system within Riemann–Liouville derivatives*

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The Noether symmetry and the conserved quantity of a fractional Birkhoffian system are studied within the Riemann–Liouville fractional derivatives. Firstly, the fractional Birkhoff's equations and the corresponding transversality conditions are given. Secondly, from special to general forms, Noether's theorems of a standard Birkhoffian system are given, which provide an approach and theoretical basis for the further research on the Noether symmetry of the fractional Birkhoffian system. Thirdly, the invariances of the fractional Pfaffian action under a special one-parameter group of infinitesimal transformations without transforming the time and a general one-parameter group of infinitesimal transformations with transforming the time are studied, respectively, and the corresponding Noether's theorems are established. Finally, an example is given to illustrate the application of the results.

Keywords: fractional Birkhoffian system, Noether's theorem, fractional conserved quantity, Riemann–Liouville fractional derivative

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1. Introduction

The German mathematician Emmy Noether discussed the invariance problem under the Lie continuous group in her inaugural thesis of University of Gottingen in 1918.[1] Noether's theorem which links the symmetry, the invariance, and the conservation law in physics was given in her thesis. Noether's theorem is an important integration theory, and it reveals the relationship between the conserved quantities of a mechanical system and its symmetries. The study of Noether's theorem has been a hot subject in the circles of the mathematicians, physicists, and dynamics experts, and it has gained a series of results. Djukić and Vujanović[2] extended Noether's theorem to the case where the infinitesimal transformations contain generalized velocities, and established Noether's theorem of holonomic nonconservative system. Li[3] further extended Noether's theorem to linear nonholonomic systems, which came into being six years earlier than Bahar's similar results.[4] Liu[5] established Noether's theorem and its converse of nonholonomic nonconservative dynamical systems. Mei[6,7] further established Noether's theorem of the Birkhoffian system by applying the invariance of Pfaffian action under the infinitesimal transformations.

The fractional calculus first appeared in the letter that L'Hopital wrote to Leibniz in 1695, and then the fractional calculus came into use. The first book about fractional calculus was published in 1974.[8] In recent decades, some progresses have been made in the study of fractional calculus. In 1996, Riewe[9,10] applied the fractional calculus to the nonconservative mechanical system, and preliminarily studied the fractional variational problems. Later, Agrawal,[11–14] Baleanu,[15–18] Atanacković,[19–21] El-Nabulsi,[22–25] and others did a series of researches on the fractional variational problems. The studies of fractional symmetries and conserved quantities were carried out first by Frederico and Torres,[26–28] and they did the researches under the Riemann–Liouville,[26] Caputo,[27] and Riesz–Caputo[28] derivatives respectively, and the corresponding fractional Noether's theorems were established under the infinitesimal transformations without and with transforming the time respectively. Frederico and Torres further studied the fractional Noether's theorem of a Hamiltonian system.[29] In addition, based on the El-Nabulsi's model, Frederico and Torres[30–32] studied the invariances of the fractional action-like variational problems. Zhang and Zhou extended the approach to the fractional action-like Pfaffian variational problems[33] and the fractional variational problem from extended exponentially fractional integral.[34,35] In recent years, fractional symmetries and conserved quantities have become hot research subjects of mathematics, physics, and mechanics, and a series of important results has been achieved.[20,26–43] While, most of the studies were related to the fractional Lagrangian systems and the Hamiltonian systems, rather than the fractional Birkhoffian systems.

In the present paper, we study the Noether symmetry and the conserved quantity for the fractional Birkhoffian system within Riemann–Liouville fractional derivatives. Ac-

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cording to the invariance of the Pfaffian action under the infinitesimal transformations, the extremum problem of the fractional Pfaff–Birkhoff integral functional is studied under a one-parameter group of infinitesimal transformations without and with transforming the time respectively, then the corresponding invariance and the fractional conserved quantities are obtained.

2. Preliminaries

In this section we briefly review some basic definitions and properties of fractional derivatives used in the following sections, and present the fractional Birkhoff's equations within Riemann–Liouville derivatives.

First, we introduce the definitions of Riemann–Liouville fractional derivatives.[44,45]

Let f be a continuous and integrable function in the interval $[t_1,t_2]$, and then the left Riemann–Liouville fractional derivative will be defined as

$$ {}_{t_1}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \int_{t_1}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}} \mathrm{d}\tau, \quad (1) $$

and the right Riemann–Liouville fractional derivative is defined as

$$ {}_{t}D_{t_2}^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \int_{t}^{t_2} \frac{f(\tau)}{(\tau-t)^{\alpha-m+1}} \mathrm{d}\tau, \quad (2) $$

where $\Gamma(*)$ is the Euler Gamma function, α is the order of the fractional derivative satisfying $m-1 \le \alpha < m$, and m is a positive integer.

If α is an integer, then we have

$$ {}_{t_1}D_t^{\alpha} f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha} f(t), \quad (3) $$

$$ {}_{t}D_{t_2}^{\alpha} f(t) = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha} f(t). \quad (4) $$

Next, we introduce some properties of Riemann–Liouville fractional calculus.[44,45]

Let f and g be continuous functions in the interval $[t_1,t_2]$, a and b are arbitrary constants, and $\alpha > 0$, then $\forall t \in [t_1,t_2]$, we will have

$$ {}_{t_1}D_t^{\alpha} (af(t) + bg(t)) = a\,{}_{t_1}D_t^{\alpha} f(t) + b\,{}_{t_1}D_t^{\alpha} g(t). \quad (5) $$

If the function f is continuous in the interval $[t_1,t_2]$, and $\alpha > 0$, $\beta > 0$, then $\forall t \in [t_1,t_2]$, we have

$$ \begin{aligned} &{}_{t_1}D_t^{\alpha} \left({}_{t_1}D_t^{-\beta} f(t)\right) \\ &= {}_{t_1}D_t^{\alpha-\beta} f(t), \\ &{}_{t_1}D_t^{-\alpha} \left({}_{t_1}D_t^{\beta} f(t)\right) \\ &= {}_{t_1}D_t^{\beta-\alpha} f(t) - \sum_{j=1}^{p} \left[{}_{t_1}D_t^{\beta-j} f(t)\right]_{t=t_1} \\ &\quad \times \frac{(t-t_1)^{\alpha-j}}{\Gamma(1+\alpha-j)}, \quad (0 \le p-1 \le \beta < p). \end{aligned} \quad (6)\text{–}(7) $$

If function f is continuous in the interval $[t_1,t_2]$, and $0 \le m-1 \le \alpha < m$, $0 \le p-1 \le \beta < p$, then $\forall t \in [t_1,t_2]$, and we have

$$ \begin{aligned} &{}_{t_1}D_t^{\alpha} \left({}_{t_1}D_t^{\beta} f(t)\right) \\ &= {}_{t_1}D_t^{\alpha+\beta} f(t) - \sum_{j=1}^{p} \left[{}_{t_1}D_t^{\beta-j} f(t)\right]_{t=t_1} \frac{(t-t_1)^{-\alpha-j}}{\Gamma(1-\alpha-j)}, \end{aligned} \quad (8) $$

$$ \begin{aligned} &{}_{t_1}D_t^{\beta} \left({}_{t_1}D_t^{\alpha} f(t)\right) \\ &= {}_{t_1}D_t^{\alpha+\beta} f(t) - \sum_{j=1}^{m} \left[{}_{t_1}D_t^{\alpha-j} f(t)\right]_{t=t_1} \frac{(t-t_1)^{-\beta-j}}{\Gamma(1-\beta-j)}. \end{aligned} \quad (9) $$

Finally, we introduce the fractional Pfaffian variational problem and the fractional Birkhoff's equations within Riemann–Liouville derivatives.[46]

We study the extremum problem of the fractional integral functional

$$ \begin{aligned} S(a^{\nu}(\cdot)) = \int_{t_1}^{t_2} \Bigg\{ &\sum_{\mu=1}^{2n} R_{\mu}^{\alpha}(t,a^{\nu})\,{}_{t_1}D_t^{\alpha} a^{\mu} \\ &+ \sum_{\mu=1}^{2n} R_{\mu}^{\beta}(t,a^{\nu})\,{}_{t}D_{t_2}^{\beta} a^{\mu} - B(t,a^{\nu}) \Bigg\} \mathrm{d}t \end{aligned} \quad (10) $$

with the commutative relations

$$ \begin{aligned} &{}_{t_1}D_t^{\alpha} \delta a^{\mu} = \delta\,{}_{t_1}D_t^{\alpha} a^{\mu}, \\ &{}_{t}D_{t_2}^{\beta} \delta a^{\mu} = \delta\,{}_{t}D_{t_2}^{\beta} a^{\mu}, \quad (\mu = 1,2,\ldots,2n) \end{aligned} \quad (11) $$

and the boundary conditions

$$ \delta a^{\mu}|_{t=t_1} = \delta a^{\mu}|_{t=t_2} = 0, \quad (\mu = 1,2,\ldots,2n), \quad (12) $$

where $B(t,a^{\nu})$ is the Birkhoffian, $R_{\mu}^{\alpha}(t,a^{\nu})$ and $R_{\mu}^{\beta}(t,a^{\nu})$, $(\mu,\nu = 1,2,\ldots,2n)$ are the Birkhoff's functions, a^{μ} is the Birkhoff variable, and the functions $R_{\mu}^{\alpha}(t,a^{\nu})$, $R_{\mu}^{\beta}(t,a^{\nu})$, and B are C^2 functions with respect to all its arguments, $\delta(\cdot)$ is the isochronous variation, and $m-1 \le \alpha < m$, $p-1 \le \beta < p$ (m and p are the positive integers). The fractional calculus functional (10) can also be called fractional Pfaffian action, and then the fractional Birkhoff's equations can be written as follows:[46]

$$ \begin{aligned} &\sum_{\nu=1}^{2n} \left(\frac{\partial R_{\nu}^{\alpha}}{\partial a^{\mu}}\,{}_{t_1}D_t^{\alpha} a^{\nu} + \frac{\partial R_{\nu}^{\beta}}{\partial a^{\mu}}\,{}_{t}D_{t_2}^{\beta} a^{\nu} \right) - \frac{\partial B}{\partial a^{\mu}} + {}_{t}D_{t_2}^{\alpha} R_{\mu}^{\alpha} \\ &+ {}_{t_1}D_t^{\beta} R_{\mu}^{\beta} = 0, \quad (\mu = 1,2,\ldots,2n) \end{aligned} \quad (13) $$

with the transversality condition

$$ \begin{aligned} &\sum_{\nu=1}^{2n} \sum_{k=0}^{m-1} \left[{}_{t_1}D_t^{\alpha-k-1} (R_{\nu}^{\alpha}) \left(\frac{\mathrm{d}^k \delta a^{\nu}}{\mathrm{d}t^k} \right) \right] \Bigg|_{t=t_2} \\ &+ \sum_{\nu=1}^{2n} \sum_{k=0}^{p-1} (-1)^k \left[{}_{t}D_{t_2}^{\beta-k-1} \left(R_{\nu}^{\beta} \right) \left(\frac{\mathrm{d}^k \delta a^{\nu}}{\mathrm{d}t^k} \right) \right] \Bigg|_{t=t_1} \\ &= 0. \end{aligned} \quad (14) $$

In particular, when $0 \leq \alpha < 1$, and $0 \leq \beta < 1$, then we have $m = p = 1$, and the transversality condition (14) gives

$$\sum_{\nu=1}^{2n} \left[\left({}_{t_1}D_t^{\alpha-1} R_\nu^\alpha \right) (\delta a^\nu) \right] \Big|_{t=t_2} + \sum_{\nu=1}^{2n} \left[\left({}_{t}D_{t_2}^{\beta-1} R_\nu^\beta \right) (\delta a^\nu) \right] \Big|_{t=t_1} = 0. \tag{15}$$

From the boundary conditions (12), we know that the transversality condition (15) is always satisfied. So under the situation that $0 \leq \alpha$ and $\beta < 1$, the fractional Birkhoff's equation (13) is free. At the same time, the standard Birkhoff's equations[46] are the special case of the fractional Birkhoff's equations (13).

3. Noether's theorems of a standard Birkhoffian system

For a Birkhoffian system, the Noether symmetry is the invariance of the Pfaffian action under the group of infinitesimal transformations. Unlike in Ref. [47], in this section we will study the invariance of a Pfaffian action under a special one-parameter group of infinitesimal transformations without and with transforming the time respectively, and give Noether's theorems of a standard Birkhoffian system, which provide an approach and the theoretical basis for the following study of Noether's theorems for a fractional Birkhoffian system.

We study the extremum problem of the integral functional

$$S(a^\nu(\cdot)) = \int_{t_1}^{t_2} \left\{ \sum_{\mu=1}^{2n} R_\mu(t,a^\nu)\, \dot{a}^\mu - B(t,a^\nu) \right\} \mathrm{d}t \tag{16}$$

with the boundary conditions

$$\delta a^\mu|_{t=t_1} = \delta a^\mu|_{t=t_2} = 0, \quad (\mu = 1,2,\ldots,2n), \tag{17}$$

where R_μ $(\mu = 1,2,\ldots,2n)$ are Birkhoff's functions, and B is the Birkhoffian. Integral functional (16) is also called the Pfaffian action.

Definition 1 (invariance without transforming the time) For a standard Birkhoffian system, functional (16) is said to be invariant under a one-parameter group of infinitesimal transformations

$$\bar{a}^\mu(t) = a^\mu(t) + \varepsilon \xi_\mu(t,a^\nu) + o(\varepsilon), \quad (\mu = 1,2,\ldots,2n), \tag{18}$$

where ε is an infinitesimal parameter, ξ_μ is an infinitesimal generator or generating function of the infinitesimal transformation, if and only if

$$\int_{T_1}^{T_2} \left\{ \sum_{\mu=1}^{2n} R_\mu(t,a^\nu)\, \dot{a}^\mu - B(t,a^\nu) \right\} \mathrm{d}t = \int_{T_1}^{T_2} \left\{ \sum_{\mu=1}^{2n} R_\mu(t,\bar{a}^\nu)\, \dot{\bar{a}}^\mu - B(t,\bar{a}^\nu) \right\} \mathrm{d}t \tag{19}$$

for any subinterval $[T_1,T_2] \subseteq (t_1,t_2)$.

Theorem 1 (necessary condition of invariance) For a standard Birkhoffian system, if functional (16) is invariant under transformations (18), then the following condition:

$$\sum_{\mu=1}^{2n} \left[\sum_{\nu=1}^{2n} \frac{\partial R_\mu(t,a^\nu)}{\partial a^\nu} \xi_\nu \right] \dot{a}^\mu + \sum_{\mu=1}^{2n} R_\mu(t,a^\nu)\, \dot{\xi}_\mu - \sum_{\nu=1}^{2n} \frac{\partial B(t,a^\mu)}{\partial a^\nu} \xi_\nu = 0 \tag{20}$$

holds.

Proof Because of the arbitrariness of the interval $[T_1,T_2]$, equation (19) is equal to

$$\sum_{\mu=1}^{2n} R_\mu(t,a^\nu)\, \dot{a}^\mu - B(t,a^\nu) = \sum_{\mu=1}^{2n} R_\mu(t,\bar{a}^\nu)\, \dot{\bar{a}}^\mu - B(t,\bar{a}^\nu). \tag{21}$$

Taking the derivatives of both sides of Eq. (21) with respect to ε, and making $\varepsilon = 0$, we can obtain Eq. (20).

Definition 2 (conserved quantity) For a standard Birkhoffian system, the quantity $I(t,a^\nu)$ is said to be conserved if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t} I(t,a^\nu) = 0 \tag{22}$$

along all the solutions of standard Birkhoff's equations

$$\sum_{\nu=1}^{2n} \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \right) \dot{a}^\nu - \left(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t} \right) = 0, \quad (\mu = 1,2,\ldots,2n). \tag{23}$$

Theorem 2 (Noether's theorem without transforming time) For a standard Birkhoffian system, if functional (16) is invariant under a one-parameter group (18) of infinitesimal transformations, then

$$I(t,a^\nu) = \sum_{\mu=1}^{2n} R_\mu(t,a^\nu)\, \xi_\mu(t,a^\nu) \tag{24}$$

is a conserved quantity of the system.

Proof

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} I(t,a^\nu) &= \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \sum_{\mu=1}^{2n} R_\mu(t,a^\nu)\, \xi_\mu(t,a^\nu) \right\} \\
&= \sum_{\mu=1}^{2n} \left(\sum_{\nu=1}^{2n} \frac{\partial R_\mu}{\partial a^\nu} \dot{a}^\nu + \frac{\partial R_\mu}{\partial t} \right) \xi_\mu + \sum_{\mu=1}^{2n} R_\mu \dot{\xi}_\mu \\
&= \sum_{\mu=1}^{2n} \left(\sum_{\nu=1}^{2n} \frac{\partial R_\mu}{\partial a^\nu} \dot{a}^\nu + \frac{\partial R_\mu}{\partial t} \right) \xi_\mu + \sum_{\mu=1}^{2n} R_\mu \dot{\xi}_\mu \\
&\quad + \sum_{\mu=1}^{2n} \sum_{\nu=1}^{2n} \frac{\partial R_\mu}{\partial a^\nu} \xi_\nu \dot{a}^\mu - \sum_{\nu=1}^{2n} \sum_{\mu=1}^{2n} \frac{\partial R_\nu}{\partial a^\mu} \xi_\mu \dot{a}^\nu \\
&= \sum_{\mu=1}^{2n} \sum_{\nu=1}^{2n} \frac{\partial R_\mu}{\partial a^\nu} \xi_\nu \dot{a}^\mu + \sum_{\mu=1}^{2n} R_\mu \dot{\xi}_\mu
\end{aligned}$$

$$-\sum_{\mu=1}^{2n}\sum_{\nu=1}^{2n}\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu\xi_\mu + \sum_{\mu=1}^{2n}\frac{\partial R_\mu}{\partial t}\xi_\mu. \tag{25}$$

From Birkhoff's equation (23), we can obtain

$$\frac{\partial B}{\partial a^\mu}=\sum_{\nu=1}^{2n}\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial R_\mu}{\partial t}. \tag{26}$$

Substituting Eq. (26) into Eq. (25), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I(t,a^\nu) &= \sum_{\mu=1}^{2n}\sum_{\nu=1}^{2n}\frac{\partial R_\mu}{\partial a^\nu}\xi_\nu\dot{a}^\mu+\sum_{\mu=1}^{2n}R_\mu\dot{\xi}_\mu \\ &\quad -\sum_{\mu=1}^{2n}\frac{\partial B}{\partial a^\mu}\xi_\mu.\end{aligned} \tag{27}$$

From Eq. (20), we verify that equation (27) is equal to zero, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}I(t,a^\nu)=\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sum_{\mu=1}^{2n}R_\mu(t,a^\nu)\xi_\mu(t,a^\nu)\right\}=0. \tag{28}$$

Using the definition of conserved quantity, we verify that equation (24) is a conserved quantity of the standard Birkhoffian system (23).

Definition 3 (invariance) For a standard Birkhoffian system, functional (16) is said to be invariant under a one-parameter group of infinitesimal transformations

$$\begin{aligned}&\bar{t}=t+\varepsilon\zeta(t,a^\nu)+o(\varepsilon),\\ &\bar{a}^\mu(t)=a^\mu(t)+\varepsilon\xi_\mu(t,a^\nu)+o(\varepsilon),\\ &(\mu=1,2,\ldots,2n),\end{aligned} \tag{29}$$

where ε is an infinitesimal parameter, ζ and ξ_μ are the infinitesimal generators or generating functions of the infinitesimal transformation, if, and only if

$$\begin{aligned}&\int_{T_1}^{T_2}\left\{\sum_{\mu=1}^{2n}R_\mu(t,a^\nu)\dot{a}^\mu-B(t,a^\nu)\right\}\mathrm{d}t\\ &=\int_{\bar{T}_1}^{\bar{T}_2}\left\{\sum_{\mu=1}^{2n}R_\mu(\bar{t},\bar{a}^\nu)\dot{\bar{a}}^\mu-B(\bar{t},\bar{a}^\nu)\right\}\mathrm{d}\bar{t}\end{aligned} \tag{30}$$

for any subinterval $[T_1,T_2]\subseteq(t_1,t_2)$.

Theorem 3 (Noether's theorem) For a standard Birkhoffian system, if functional (16) is invariant in the sense of Definition 3, then

$$I(t,a^\nu)=\sum_{\mu=1}^{2n}R_\mu\xi_\mu-B\zeta \tag{31}$$

is a conserved quantity of the system.

Proof If we consider t as a dependent variable, then each non-autonomous problem (16) is equivalent to an autonomous one. For this case we consider a Lipschitzian one-to-one transformation

$$t\in[t_1,t_2]\mapsto\sigma\in[\sigma_1,\sigma_2], \tag{32}$$

then

$$\begin{aligned}S(a^\nu(\cdot)) &= \int_{t_1}^{t_2}\left\{\sum_{\mu=1}^{2n}R_\mu(t,a^\nu)\dot{a}^\mu-B(t,a^\nu)\right\}\mathrm{d}t\\ &=\int_{\sigma_1}^{\sigma_2}\left\{\sum_{\mu=1}^{2n}R_\mu(t(\sigma),a^\nu(t(\sigma)))\frac{\mathrm{d}a^\mu(t(\sigma))/\mathrm{d}\sigma}{\mathrm{d}t(\sigma)/\mathrm{d}\sigma}\right.\\ &\quad\left.-B(t(\sigma),a^\nu(t(\sigma)))\right\}\frac{\mathrm{d}t(\sigma)}{\mathrm{d}\sigma}\mathrm{d}\sigma\\ &=\int_{\sigma_1}^{\sigma_2}\left\{\sum_{\mu=1}^{2n}R_\mu(t(\sigma),a^\nu(t(\sigma)))\frac{\mathrm{d}a^\mu(t(\sigma))}{\mathrm{d}\sigma}\right.\\ &\quad\left.-B(t(\sigma),a^\nu(t(\sigma)))\frac{\mathrm{d}t(\sigma)}{\mathrm{d}\sigma}\right\}\mathrm{d}\sigma\\ &\doteq \bar{S}[t(\cdot),a^\nu(t(\cdot))],\end{aligned} \tag{33}$$

where $t(\sigma_1)=t_1$, and $t(\sigma_2)=t_2$. If functional $S(a^\nu(\cdot))$ is invariant in the sense of Definition 3, then functional $\bar{S}[t(\cdot),a^\nu(\cdot)]$ is invariant in the sense of Definition 1. By use of Theorem 2, we can obtain

$$I(t,a^\nu)=\sum_{\mu=1}^{2n}R_\mu\xi_\mu-B\zeta. \tag{34}$$

4. Noether's theorems of a fractional Birkhoffian system

In this section, we generalize Noether's theorems of a standard Birkhoffian system for problems having fractional derivatives. The Noether symmetry and the conserved quantity of a fractional Birkhoffian system are studied under a one-parameter group of infinitesimal transformations without and with transforming the time respectively.

The extremum problem of the fractional integral functional (10) under $0<\alpha,\beta\leq 1$ will be studied. From Ref. [46], we can know that the extremum problem of fractional integral functional (10) can lead to fractional Birkhoff's equation (13).

Definition 4 (invariance without transforming the time) For a fractional Birkhoffian system, functional (10) is said to be invariant under a one-parameter group (18) of infinitesimal transformations, if and only if

$$\begin{aligned}&\int_{T_1}^{T_2}\left\{\sum_{\mu=1}^{2n}R_\mu^\alpha(t,a^\nu)\,{}_{t_1}D_t^\alpha a^\mu\right.\\ &\quad\left.+\sum_{\mu=1}^{2n}R_\mu^\beta(t,a^\nu)\,{}_tD_{t_2}^\beta a^\mu-B(t,a^\nu)\right\}\mathrm{d}t\\ &=\int_{T_1}^{T_2}\left\{\sum_{\mu=1}^{2n}R_\mu^\alpha(t,\bar{a}^\nu)\,{}_{t_1}D_t^\alpha\bar{a}^\mu\right.\\ &\quad\left.+\sum_{\mu=1}^{2n}R_\mu^\beta(t,\bar{a}^\nu)\,{}_tD_{t_2}^\beta\bar{a}^\mu-B(t,\bar{a}^\nu)\right\}\mathrm{d}t,\end{aligned} \tag{35}$$

for any subinterval $[T_1,T_2]\subseteq(t_1,t_2)$.

Theorem 4 (necessary condition of invariance) For a fractional Birkhoffian system, if functional (10) is invariant under a one-parameter group (18) of infinitesimal transformations, then the following condition:

$$\begin{aligned}
&\sum_{\mu=1}^{2n}\left[\sum_{\nu=1}^{2n}\frac{\partial R_\mu^\alpha(t,a^\nu)}{\partial a^\nu}\xi_\nu(t,a^\mu)\right]{}_{t_1}D_t^\alpha a^\mu\\
&+\sum_{\mu=1}^{2n}\left[\sum_{\nu=1}^{2n}\frac{\partial R_\mu^\beta(t,a^\nu)}{\partial a^\nu}\xi_\nu(t,a^\mu)\right]{}_tD_{t_2}^\beta a^\mu\\
&+\sum_{\mu=1}^{2n}R_\mu^\alpha(t,a^\nu){}_{t_1}D_t^\alpha\xi_\mu(t,a^\nu)\\
&+\sum_{\mu=1}^{2n}R_\mu^\beta(t,a^\nu){}_tD_{t_2}^\beta\xi_\mu(t,a^\nu)\\
&-\sum_{\nu=1}^{2n}\frac{\partial B(t,a^\mu)}{\partial a^\nu}\xi_\nu(t,a^\mu)=0
\end{aligned}\tag{36}$$

holds.

Proof Because of the arbitrariness of the interval $[T_1,T_2]$, from Eq. (35), we can obtain

$$\begin{aligned}
&\sum_{\mu=1}^{2n}R_\mu^\alpha(t,a^\nu){}_{t_1}D_t^\alpha a^\mu\\
&+\sum_{\mu=1}^{2n}R_\mu^\beta(t,a^\nu){}_tD_{t_2}^\beta a^\mu-B(t,a^\nu)\\
=&\sum_{\mu=1}^{2n}R_\mu^\alpha(t,\bar a^\nu){}_{t_1}D_t^\alpha\bar a^\mu\\
&+\sum_{\mu=1}^{2n}R_\mu^\beta(t,\bar a^\nu){}_tD_{t_2}^\beta\bar a^\mu-B(t,\bar a^\nu).
\end{aligned}\tag{37}$$

Taking the derivatives of both sides of Eq. (37) with respect to ε, and making $\varepsilon=0$, we have

$$\begin{aligned}
0=&\sum_{\mu=1}^{2n}\left[\sum_{\nu=1}^{2n}\frac{\partial R_\mu^\alpha(t,a^\nu)}{\partial a^\nu}\xi_\nu(t,a^\mu)\right]{}_{t_1}D_t^\alpha a^\mu\\
&+\sum_{\mu=1}^{2n}\left[\sum_{\nu=1}^{2n}\frac{\partial R_\mu^\beta(t,a^\nu)}{\partial a^\nu}\xi_\nu(t,a^\mu)\right]{}_tD_{t_2}^\beta a^\mu\\
&+\sum_{\mu=1}^{2n}R_\mu^\alpha(t,a^\nu)\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\frac{1}{\Gamma(m-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m\right.\\
&\times\int_{t_1}^t(t-\tau)^{m-\alpha-1}a^\mu(\tau)\mathrm{d}\tau+\frac{\varepsilon}{\Gamma(m-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m\\
&\left.\times\int_{t_1}^t(t-\tau)^{m-\alpha-1}\xi_\mu(\tau,a^\nu)\mathrm{d}\tau\right]_{\varepsilon=0}\\
&+\sum_{\mu=1}^{2n}R_\mu^\beta(t,a^\nu)\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\frac{1}{\Gamma(m-\beta)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^m\right.\\
&\times\int_t^{t_2}(\tau-t)^{m-\beta-1}a^\mu(\tau)\mathrm{d}\tau+\frac{\varepsilon}{\Gamma(m-\beta)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^m\\
&\left.\times\int_t^{t_2}(\tau-t)^{m-\beta-1}\xi_\mu(\tau,a^\nu)\mathrm{d}\tau\right]_{\varepsilon=0}\\
&-\sum_{\nu=1}^{2n}\frac{\partial B(t,a^\mu)}{\partial a^\nu}\xi_\nu(t,a^\mu)\\
=&\sum_{\mu=1}^{2n}\left[\sum_{\nu=1}^{2n}\frac{\partial R_\mu^\alpha(t,a^\nu)}{\partial a^\nu}\xi_\nu(t,a^\mu)\right]{}_{t_1}D_t^\alpha a^\mu\\
&+\sum_{\mu=1}^{2n}\left[\sum_{\nu=1}^{2n}\frac{\partial R_\mu^\beta(t,a^\nu)}{\partial a^\nu}\xi_\nu(t,a^\mu)\right]{}_tD_{t_2}^\beta a^\mu\\
&+\sum_{\mu=1}^{2n}R_\mu^\alpha(t,a^\nu)\frac{1}{\Gamma(m-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m\\
&\times\int_{t_1}^t(t-\tau)^{m-\alpha-1}\xi_\mu(\tau,a^\nu)\mathrm{d}\tau\\
&+\sum_{\mu=1}^{2n}R_\mu^\beta(t,a^\nu)\frac{1}{\Gamma(m-\beta)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^m\\
&\times\int_t^{t_2}(\tau-t)^{m-\beta-1}\xi_\mu(\tau,a^\nu)\mathrm{d}\tau\\
&-\sum_{\nu=1}^{2n}\frac{\partial B(t,a^\mu)}{\partial a^\nu}\xi_\nu(t,a^\mu).
\end{aligned}\tag{38}$$

Formula (38) is equal to Eq. (36), so the theorem is proved.

Now we introduce the definition of the fractional conserved quantity for the fractional Birkhoffian system.

Definition 5[26] (conserved quantity) For a fractional Birkhoffian system, the quantity $I(t,a^\nu,{}_{t_1}D_t^\alpha a^\nu,{}_tD_{t_2}^\beta a^\nu)$ is said to be fractional-conserved if and only if it can be written in the form

$$\begin{aligned}
&I\left(t,a^\nu,{}_{t_1}D_t^\alpha a^\nu,{}_tD_{t_2}^\beta a^\nu\right)\\
=&\sum_{i=1}^m C_i^1\left(t,a^\nu,{}_{t_1}D_t^\alpha a^\nu,{}_tD_{t_2}^\beta a^\nu\right)\\
&\times C_i^2\left(t,a^\nu,{}_{t_1}D_t^\alpha a^\nu,{}_tD_{t_2}^\beta a^\nu\right),
\end{aligned}\tag{39}$$

for some values of $m\in N$ and some functions C_i^1 and C_i^2 $(i=1,\ldots,m)$, where each pair of C_i^1 and C_i^2 $(i=1,\ldots,m)$ satisfies

$$\begin{aligned}
&D_t^{\gamma_i}\left(C_i^{j_i^1}(t,a^\nu,{}_{t_1}D_t^\alpha a^\nu,{}_tD_{t_2}^\beta a^\nu),C_i^{j_i^2}(t,a^\nu,{}_{t_1}D_t^\alpha a^\nu,{}_tD_{t_2}^\beta a^\nu)\right)\\
=&\,0,
\end{aligned}\tag{40}$$

with $\gamma_i\in\{\alpha,\beta\}$, $j_i^1=1$ and $j_i^2=2$ or $j_i^1=2$ and $j_i^2=1$, along all the solutions of the fractional Birkhoff's equation (13).

In Eq. (40), the factor $D_t^\gamma(f,g)$ is defined as[26]

$$D_t^\gamma(f,g)=-g_tD_{t_2}^\gamma f+f_{t_1}D_t^\gamma g,\tag{41}$$

where f and g are C^1 functions in the interval $[t_1,t_2]$, with $t\in[t_1,t_2]$ and $\gamma\in R_0^+$. When $\gamma=1$, we have

$$D_t^\gamma(f,g)=-g_tD_{t_2}^1f+f_{t_1}D_t^1g=\dot fg+f\dot g=\frac{\mathrm{d}}{\mathrm{d}t}(fg).\tag{42}$$

In this case, $D_t^1(f,g)=D_t^1(g,f)$, but in general $D_t^1(f,g)\neq D_t^1(g,f)$.

Theorem 5 (fractional Noether's theorem without transforming time) For a fractional Birkhoffian system, if functional (10) is invariant in the sense of Definition 4, then

$$I\left(t,a^{\nu},{}_{t_1}D_t^{\alpha}a^{\nu},{}_{t}D_{t_2}^{\beta}a^{\nu}\right) = \sum_{\mu=1}^{2n}\left[R_{\mu}^{\alpha}\left(t,a^{\nu}\right)-R_{\mu}^{\beta}\left(t,a^{\nu}\right)\right]\xi_{\mu}\left(t,a^{\nu}\right) \tag{43}$$

is a fractional conserved quantity of the system.

Proof From the fractional Birkhoff's equation (13), we have

$$\frac{\partial B}{\partial a^{\mu}} = \sum_{\nu=1}^{2n}\left(\frac{\partial R_{\nu}^{\alpha}}{\partial a^{\mu}}{}_{t_1}D_t^{\alpha}a^{\nu}+\frac{\partial R_{\nu}^{\beta}}{\partial a^{\mu}}{}_{t}D_{t_2}^{\beta}a^{\nu}\right) + {}_{t}D_{t_2}^{\alpha}R_{\mu}^{\alpha}+{}_{t_1}D_t^{\beta}R_{\mu}^{\beta}. \tag{44}$$

Because functional (10) is invariant in the sense of Definition 4, then substituting Eq. (44) into Eq. (36), we have

$$\begin{aligned}&\sum_{\mu=1}^{2n}\left(\sum_{\nu=1}^{2n}\frac{\partial R_{\mu}^{\alpha}}{\partial a^{\nu}}\xi_{\nu}\right){}_{t_1}D_t^{\alpha}a^{\mu}+\sum_{\mu=1}^{2n}\left(\sum_{\nu=1}^{2n}\frac{\partial R_{\mu}^{\beta}}{\partial a^{\nu}}\xi_{\nu}\right){}_{t}D_{t_2}^{\beta}a^{\mu}\\&+\sum_{\mu=1}^{2n}R_{\mu t_1}^{\alpha}D_t^{\alpha}\xi_{\mu}+\sum_{\mu=1}^{2n}R_{\mu t}^{\beta}D_{t_2}^{\beta}\xi_{\mu}\\&-\sum_{\mu=1}^{2n}\sum_{\nu=1}^{2n}\left(\frac{\partial R_{\nu}^{\alpha}}{\partial a^{\mu}}{}_{t_1}D_t^{\alpha}a^{\nu}+\frac{\partial R_{\nu}^{\beta}}{\partial a^{\mu}}{}_{t}D_{t_2}^{\beta}a^{\nu}\right)\xi_{\mu}\\&-\sum_{\mu=1}^{2n}\xi_{\mu t}D_{t_2}^{\alpha}R_{\mu}^{\alpha}-\sum_{\mu=1}^{2n}\xi_{\mu t_1}D_t^{\beta}R_{\mu}^{\beta}=0.\end{aligned} \tag{45}$$

Equation (45) can be reduced to

$$\begin{aligned}&\sum_{\mu=1}^{2n}R_{\mu t_1}^{\alpha}D_t^{\alpha}\xi_{\mu}+\sum_{\mu=1}^{2n}R_{\mu t}^{\beta}D_{t_2}^{\beta}\xi_{\mu}\\&-\sum_{\mu=1}^{2n}\xi_{\mu t}D_{t_2}^{\alpha}R_{\mu}^{\alpha}-\sum_{\mu=1}^{2n}\xi_{\mu t_1}D_t^{\beta}R_{\mu}^{\beta}=0,\end{aligned} \tag{46}$$

i.e.,

$$\sum_{\mu=1}^{2n}\left\{D_t^{\alpha}\left(R_{\mu}^{\alpha},\xi_{\mu}\right)-D_t^{\beta}\left(\xi_{\mu},R_{\mu}^{\beta}\right)\right\}=0. \tag{47}$$

Then, from Definition 5, formula (43) is the fractional conserved quantity of this situation.

Definition 6 (invariance) For a fractional Birkhoffian system, functional (10) is said to be invariant under a one-parameter group (29) of infinitesimal transformations, if and only if

$$\begin{aligned}&\int_{T_1}^{T_2}\left\{\sum_{\mu=1}^{2n}R_{\mu}^{\alpha}\left(t,a^{\nu}\right){}_{t_1}D_t^{\alpha}a^{\mu}\right.\\&\left.+\sum_{\mu=1}^{2n}R_{\mu}^{\beta}\left(t,a^{\nu}\right){}_{t}D_{t_2}^{\beta}a^{\mu}-B\left(t,a^{\nu}\right)\right\}\mathrm{d}t\\&=\int_{\bar{T}_1}^{\bar{T}_2}\left\{\sum_{\mu=1}^{2n}R_{\mu}^{\alpha}\left(\bar{t},\bar{a}^{\nu}\right){}_{\bar{t}_1}D_{\bar{t}}^{\alpha}\bar{a}^{\mu}\right.\\&\left.+\sum_{\mu=1}^{2n}R_{\mu}^{\beta}\left(\bar{t},\bar{a}^{\nu}\right){}_{\bar{t}}D_{\bar{t}_2}^{\beta}\bar{a}^{\mu}-B\left(\bar{t},\bar{a}^{\nu}\right)\right\}\mathrm{d}\bar{t}\end{aligned} \tag{48}$$

for any subinterval $[T_1,T_2]\subseteq(t_1,t_2)$.

Theorem 6 (fractional Noether's theorem) For a fractional Birkhoffian system, if functional (10) is invariant under the Definition 6, then

$$\begin{aligned}&I\left(t,a^{\mu},{}_{t_1}D_t^{\alpha}a^{\mu},{}_{t}D_{t_2}^{\beta}a^{\mu}\right)\\&=\sum_{\mu=1}^{2n}\left(R_{\mu}^{\alpha}-R_{\mu}^{\beta}\right)\xi_{\mu}+\left\{\sum_{\mu=1}^{2n}\left[(1-\alpha)R_{\mu t_1}^{\alpha}D_t^{\alpha}a^{\mu}\right.\right.\\&\left.\left.+(1-\beta)R_{\mu t}^{\beta}D_{t_2}^{\beta}a^{\mu}\right]-B\right\}\zeta\end{aligned} \tag{49}$$

is a fractional-conserved quantity of the system.

Proof Consider a Lipschitzian one-to-one transformation about t

$$t\in[t_1,t_2]\mapsto\sigma f(\lambda)\in[\sigma_1,\sigma_2]. \tag{50}$$

Here t is a dependent variable. When $\lambda=0$, it satisfies

$$t'_{\sigma}=\frac{\mathrm{d}t(\sigma)}{\mathrm{d}\sigma}=f(\lambda)=1.$$

Under transformation (50), functional (10) can be written as

$$\begin{aligned}&\bar{S}\left(t(\cdot),a^{\nu}(\cdot)\right)\\&=\int_{\sigma_1}^{\sigma_2}\left\{\sum_{\mu=1}^{2n}R_{\mu}^{\alpha}\left(t(\sigma),a^{\nu}(t(\sigma))\right){}_{\sigma_1}D_{t(\sigma)}^{\alpha}a^{\mu}\left(t(\sigma)\right)\right.\\&+\sum_{\mu=1}^{2n}R_{\mu}^{\beta}\left(t(\sigma),a^{\nu}(t(\sigma))\right){}_{t(\sigma)}D_{\sigma_2}^{\beta}a^{\mu}\left(t(\sigma)\right)\\&\left.-B\left(t(\sigma),a^{\nu}(t(\sigma))\right)\right\}t'_{\sigma}\mathrm{d}\sigma,\end{aligned} \tag{51}$$

where $t(\sigma_1)=t_1$, $t(\sigma_2)=t_2$, and

$$\begin{aligned}&{}_{\sigma_1}D_{t(\sigma)}^{\alpha}a^{\nu}\left(t(\sigma)\right)\\&=\frac{1}{\Gamma(m-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t(\sigma)}\right)^{m}\\&\times\int_{t_1/f(\lambda)}^{\sigma f(\lambda)}(\sigma f(\lambda)-\tau)^{m-\alpha-1}a^{\nu}\left(\tau f^{-1}(\lambda)\right)\mathrm{d}\tau\\&=\frac{(t'_{\sigma})^{-\alpha}}{\Gamma(m-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)^{m}\int_{t_1/(t'_{\sigma})^2}^{\sigma}(\sigma-s)^{m-\alpha-1}a^{\nu}(s)\mathrm{d}s\\&=\left(t'_{\sigma}\right)^{-\alpha}{}_{t_1/(t'_{\sigma})^2}D_{\sigma}^{\alpha}a^{\nu}(\sigma).\end{aligned} \tag{52}$$

Similarly, we have

$${}_{t(\sigma)}D_{\sigma_2}^{\alpha}a^{\nu}\left(t(\sigma)\right)=\left(t'_{\sigma}\right)^{-\beta}{}_{\sigma}D_{t_2/(t'_{\sigma})^2}^{\beta}a^{\nu}(\sigma). \tag{53}$$

Substituting Eqs. (52) and (53) into Eq. (51), we have

$$\begin{aligned}&\bar{S}\left(t(\cdot),a^{\nu}(\cdot)\right)\\&=\int_{\sigma_1}^{\sigma_2}\left\{\sum_{\mu=1}^{2n}R_{\mu}^{\alpha}\left(t(\sigma),a^{\nu}(t(\sigma))\right)\left(t'_{\sigma}\right)^{-\alpha}{}_{t_1/(t'_{\sigma})^2}D_{\sigma}^{\alpha}a^{\mu}(\sigma)\right.\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\mu=1}^{2n} R_{\mu}^{\beta}\left(t\left(\sigma\right),a^{\nu}\left(t\left(\sigma\right)\right)\right)\left(t_{\sigma}^{\prime}\right)^{-\beta}{}_{\sigma}D_{{}_{t_2/(t_{\sigma}^{\prime})^2}}^{\beta}a^{\mu}\left(\sigma\right)\\
&-B\left(t\left(\sigma\right),a^{\nu}\left(t\left(\sigma\right)\right)\right)\Big\}t_{\sigma}^{\prime}\mathrm{d}\sigma\\
=&\int_{\sigma_1}^{\sigma_2}\Big\{\sum_{\mu=1}^{2n} R_{\mu}^{\alpha}\left(t\left(\sigma\right),a^{\nu}\left(t\left(\sigma\right)\right)\right)\left(t_{\sigma}^{\prime}\right)^{-\alpha+1}{}_{t_1/(t_{\sigma}^{\prime})^2}D_{\sigma}^{\alpha}a^{\mu}\left(\sigma\right)\\
&+\sum_{\mu=1}^{2n} R_{\mu}^{\beta}\left(t\left(\sigma\right),a^{\nu}\left(t\left(\sigma\right)\right)\right)\left(t_{\sigma}^{\prime}\right)^{-\beta+1}{}_{\sigma}D_{{}_{t_2/(t_{\sigma}^{\prime})^2}}^{\beta}a^{\mu}\left(\sigma\right)\\
&-B\left(t\left(\sigma\right),a^{\nu}\left(t\left(\sigma\right)\right)\right)t_{\sigma}^{\prime}\Big\}\mathrm{d}\sigma\\
\doteq&\int_{\sigma_1}^{\sigma_2}\Big\{\sum_{\mu=1}^{2n}\bar{R}_{\mu}^{\alpha}\left(t\left(\sigma\right),a^{\nu}\left(t\left(\sigma\right)\right),t_{\sigma}^{\prime}\right)_{t_1/(t_{\sigma}^{\prime})^2}D_{\sigma}^{\alpha}a^{\mu}\left(t\left(\sigma\right)\right)\\
&+\sum_{\mu=1}^{2n}\bar{R}_{\mu}^{\beta}\left(t\left(\sigma\right),a^{\nu}\left(t\left(\sigma\right)\right),t_{\sigma}^{\prime}\right){}_{\sigma}D_{{}_{t_2/(t_{\sigma}^{\prime})^2}}^{\beta}a^{\mu}\left(t\left(\sigma\right)\right)\\
&-B\left(t\left(\sigma\right),a^{\nu}\left(t\left(\sigma\right)\right),t_{\sigma}^{\prime}\right)\Big\}\mathrm{d}\sigma\\
=&\int_{t_1}^{t_2}\Big\{\sum_{\mu=1}^{2n} R_{\mu}^{\alpha}\left(t,a^{\nu}\right){}_{t_1}D_{t}^{\alpha}a^{\mu}+\sum_{\mu=1}^{2n} R_{\mu}^{\beta}\left(t,a^{\nu}\right){}_{t}D_{t_2}^{\beta}a^{\mu}\\
&-B\left(t,a^{\nu}\right)\Big\}\mathrm{d}t\\
=&\,S\left(a^{\nu}\left(\cdot\right)\right).
\end{aligned}
\tag{54}
$$

Making use of Theorem 5, we can obtain the fractional conserved quantity

$$
\begin{aligned}
&I\left(t,a^{\mu},{}_{t_1}D_{t}^{\alpha}a^{\mu},{}_{t}D_{t_2}^{\beta}a^{\mu}\right)\\
=&\sum_{\mu=1}^{2n}\left(R_{\mu}^{\alpha}-R_{\mu}^{\beta}\right)\xi_{\mu}+\Big\{\sum_{\mu=1}^{2n}\left(R_{\mu t_1}^{\alpha}D_{t}^{\alpha}a^{\mu}+R_{\mu t}^{\beta}D_{t_2}^{\beta}a^{\mu}\right.\\
&\left.-\alpha R_{\mu t_1}^{\alpha}D_{t}^{\alpha}a^{\mu}-\beta R_{\mu t}^{\beta}D_{t_2}^{\beta}a^{\mu}\right)-B\Big\}\zeta.
\end{aligned}
\tag{55}
$$

By means of associative law, formula (55) is the fractional-conserved quantity (49).

5. Example

Now, we come to study the extremum problem of fractional Pfaffian action functional

$$
\begin{aligned}
&S\left(a^{\nu}\left(\cdot\right)\right)\\
=&\int_{t_1}^{t_2}\left[a^{4}{}_{t_1}D_{t}^{\alpha}a^{1}+a^{5}{}_{t_1}D_{t}^{\alpha}a^{2}-\left(a^{1}-a^{2}\right)a^{3}\right]\mathrm{d}t
\end{aligned}
\tag{56}
$$

with the boundary condition

$$
\delta a^{\mu}|_{t=t_1}=\delta a^{\mu}|_{t=t_2}=0,\quad(\mu=1,\ldots,6).
\tag{57}
$$

Problem (56) is a six-order Pfaffian fractional variational problem, where

$$
B=\left(a^{1}-a^{2}\right)a^{3},
\tag{58}
$$

$$
R_1=a^{4},\quad R_2=a^{5},\quad R_3=R_4=R_5=R_6=0.
\tag{59}
$$

According to Definition 6, we can easily verify that for problem (56), condition (48) has the following solutions:

$$
\begin{aligned}
&\zeta^{1}=0,\quad \xi_{1}^{1}=a^{1},\quad \xi_{2}^{1}=a^{2},\quad \xi_{3}^{1}=-a^{3},\\
&\xi_{4}^{1}=-a^{4},\quad \xi_{5}^{1}=-a^{5},\quad \xi_{6}^{1}=0,
\end{aligned}
\tag{60}
$$

$$
\begin{aligned}
&\zeta^{2}=t,\quad \xi_{1}^{2}=-a^{1},\quad \xi_{2}^{2}=-a^{2},\quad \xi_{3}^{2}=0,\\
&\xi_{4}^{2}=0,\quad \xi_{5}^{2}=0,\quad \xi_{6}^{2}=0,
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
&\zeta^{3}=-t,\quad \xi_{1}^{3}=0,\quad \xi_{2}^{3}=0,\quad \xi_{3}^{3}=a^{3},\\
&\xi_{4}^{3}=0,\quad \xi_{5}^{3}=0,\quad \text{and}\quad \xi_{6}^{3}=0.
\end{aligned}
\tag{62}
$$

The generators (60), (61), and (62) all correspond to the Noether symmetries of the fractional Pfaffian action functional (56). Then making use of the Theorem 6, we have

$$
I^{1}=a^{4}a^{1}+a^{5}a^{2},
\tag{63}
$$

$$
\begin{aligned}
I^{2}=&\,(1-\alpha)(a^{4}{}_{t_1}D_{t}^{\alpha}a^{1}+a^{5}{}_{t_1}D_{t}^{\alpha}a^{2})t\\
&-(a^{1}-a^{2})a^{3}t-a^{4}a^{1}-a^{5}a^{2},
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
I^{3}=&-t[(1-\alpha)(a^{4}{}_{t_1}D_{t}^{\alpha}a^{1}+a^{5}{}_{t_1}D_{t}^{\alpha}a^{2})\\
&-(a^{1}-a^{2})a^{3}].
\end{aligned}
\tag{65}
$$

Equations (63), (64), and (65) are the fractional-conserved quantities corresponding to generators (60), (61), and (62) respectively.

6. Conclusions

The study of fractional variational problems has been a hot spot in various fields of science and engineering, for example, in physics, biology, chemistry, dynamics and control, chaotic system, mechanical system, etc. In this paper, we introduce the fractional calculus into the Birkhoffian system, and we propose and study the Noether symmetries and the conserved quantities of the fractional Birkhoffian system. Our results are very interesting, and they are the generalizations of the fractional Noether's theorems of Lagrangian system and Hamiltonian system. The researches on the fractional variational problems and their symmetries and conserved quantities are still in the initial stage, and there are many problems to be solved. For example, though fruitful research results have been achieved in the classical Noether symmetries and the conserved quantities, how to extend these results to fractional dynamical system and others remain to be investigated.

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