Fractional cyclic integrals and Routh equations of fractional Lagrange system with combined Caputo derivatives*

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In this paper, we develop a fractional cyclic integral and a Routh equation for fractional Lagrange system defined in terms of fractional Caputo derivatives. The fractional Hamilton principle and the fractional Lagrange equations of the system are obtained under a combined Caputo derivative. Furthermore, the fractional cyclic integrals based on the Lagrange equations are studied and the associated Routh equations of the system are presented. Finally, two examples are given to show the applications of the results.

Keywords: fractional cyclic integral, fractional Routh equation, combined Caputo fractional derivative

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1. Introduction

Fractional calculus plays a main role in various fields of science and engineering,[1–9] and it is one of the generalizations of classical calculus. There are several definitions of fractional derivatives and fractional integrals, like Riemann–Liouville (RL), Caputo, Riesz, Grunwald–Letnikov, etc. Fractional calculus is a popular subject of current research. Riewe[10,11] formulated the problem of the calculus of variations with fractional derivatives and obtained the respective Lagrange equations. Agrawal[12] further presented the fractional variational problems with right and left RL fractional derivatives. Zhou *et al.*[13] studied the fractional Lagrange equations of nonholonomic systems. Zhang[14] gave the fractional differential equations in terms of combined Riemann–Liouville derivatives.

RL fractional derivatives have been popular largely among mathematicians. However, the RL derivative of a constant is not zero, and in many applications, it is difficult to solve the problems. In contrast, the Caputo derivative of a constant is zero, it can be used to solve some physical problems conveniently, like calculating the conserved quantity (or first integral). Baleanu and Agrawal,[15] gave the fractional Hamilton formalism within Caputo's derivative. Malinowska and Torres,[16] developed the fractional variational calculus in terms of a combined Caputo derivative. However, the fractional cyclic integrals and Routh equations for dynamic systems have not been studied.

In this paper, the fractional Hamilton principle and the Lagrange equations with a combined Caputo derivative are established. Further, we also give the fractional cyclic integrals and Routh equations of different cases by the Caputo derivatives. The results can be used to reduce the order of the equations of motion, thus simplifying the problems.

2. Definitions and properties of Caputo's derivative

In this section, we briefly present some fundamental definitions of Caputo derivatives and some main properties, which will be used in the following sections.[15,16]

The left Caputo fractional derivative is defined as

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n f(\tau)\,\mathrm{d}\tau. \quad (1) $$

The right Caputo fractional derivative is defined as

$$ {}_t^C D_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} \left(-\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n f(\tau)\,\mathrm{d}\tau, \quad (2) $$

where $n \in N$, $n-1 \le \alpha < n$, and Γ is the Gamma function.

If α is an integer, from Eqs. (1) and (2), we obtain

$$ {}_a^C D_t^\alpha f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha f(t), \ {}_t^C D_b^\alpha f(t) = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha f(t). \quad (3) $$

The formulas of fractional integration by part are as follows:

$$ \int_a^b g(t)\,{}_a^C D_t^\alpha f(t)\,\mathrm{d}t = \int_a^b f(t)\,{}_t^C D_b^\alpha g(t)\,\mathrm{d}t, \quad (4) $$

$$ \int_a^b g(t)\,{}_t^C D_b^\alpha f(t)\,\mathrm{d}t = \int_a^b f(t)\,{}_a^C D_t^\alpha g(t)\,\mathrm{d}t, \quad (5) $$

where we assume that $0 < \alpha < 1$, and $f(a) = f(b) = 0$.

We use the fractional derivative operator ${}^C D_\gamma^{\alpha,\beta}$ defined by

$$ {}^C D_\gamma^{\alpha,\beta} = \gamma\,{}_a^C D_t^\alpha + (1-\gamma)\,{}_t^C D_b^\beta, \quad (6) $$

where $\gamma \in [0,1]$.

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Let $f \in AC([a,b])$, where $AC([a,b])$ represents the space of absolutely continuous functions in $[a,b]$. Then we have

$$ {}^{C}D_{\gamma}^{\alpha,\beta} f(t) = \gamma {}_{a}^{C}D_{t}^{\alpha} f(t) + (1-\gamma) {}_{t}^{C}D_{b}^{\beta} f(t). \tag{7} $$

Note that

$$ \gamma = 0, {}^{C}D_{0}^{\alpha,\beta} f(t) = {}_{t}^{C}D_{b}^{\beta} f(t), \tag{8} $$

$$ \gamma = 1, {}^{C}D_{1}^{\alpha,\beta} f(t) = {}_{a}^{C}D_{t}^{\alpha} f(t). \tag{9} $$

The formula of fractional integration by part for the combined Caputo fractional derivatives is as follows:

$$ \int_{a}^{b} g(t) {}^{C}D_{\gamma}^{\alpha,\beta} f(t)\,\mathrm{d}t = \int_{a}^{b} f(t) {}^{C}D_{1-\gamma}^{\beta,\alpha} g(t)\,\mathrm{d}t, \tag{10} $$

where $0 < \alpha, \beta < 1$, $f(a) = f(b) = 0$.

3. Fractional Hamilton principle

At first, we discuss the commutation relation between isochronous variation and combined Caputo derivatives. In Ref. [13], we have studied that the isochronous variation and fractional derivatives are commutable, that is,

$$ \delta {}_{a}^{C}D_{t}^{\alpha} q_s = {}_{a}^{C}D_{t}^{\alpha} \delta q_s, \tag{11} $$

$$ \delta {}_{t}^{C}D_{b}^{\beta} q_s = {}_{t}^{C}D_{b}^{\beta} \delta q_s. \tag{12} $$

Calculating $\delta {}^{C}D_{\gamma}^{\alpha,\beta} q_s$ and using the exchange relations (11) and (12), we have

$$ \begin{aligned} \delta {}^{C}D_{\gamma}^{\alpha,\beta} q_s &= \gamma \delta {}_{a}^{C}D_{t}^{\alpha} q_s + (1-\gamma) \delta {}_{t}^{C}D_{b}^{\beta} q_s \\ &= \gamma {}_{a}^{C}D_{t}^{\alpha} \delta q_s + (1-\gamma) {}_{t}^{C}D_{b}^{\beta} \delta q_s. \end{aligned} \tag{13} $$

According to Eq. (7), we obtain

$$ {}^{C}D_{\gamma}^{\alpha,\beta} \delta q_s = \gamma {}_{a}^{C}D_{t}^{\alpha} \delta q_s + (1-\gamma) {}_{t}^{C}D_{b}^{\beta} \delta q_s. \tag{14} $$

Comparing Eq. (13) with Eq. (14), we obtain the commutation relation

$$ \delta {}^{C}D_{\gamma}^{\alpha,\beta} q_s = {}^{C}D_{\gamma}^{\alpha,\beta} \delta q_s. \tag{15} $$

The terminal conditions are given below

$$ \delta q_s|_{t=a} = \delta q_s|_{t=b} = 0. \tag{16} $$

Now we consider a mechanical system with n generalized coordinates $q_s (s = 1,2,\ldots,n)$, the Lagrangian of the system is

$$ L(t, q_s, {}^{C}D_{\gamma}^{\alpha,\beta} q_s), \quad (s = 1,2,\ldots,n), $$

and we define the fractional Hamilton action as

$$ S = \int_{a}^{b} L(t, q_s, {}^{C}D_{\gamma}^{\alpha,\beta} q_s)\,\mathrm{d}t. \tag{17} $$

Fractional Hamilton principle can be written as

$$ \delta S = \delta \int_{a}^{b} L(t, q_s, {}^{C}D_{\gamma}^{\alpha,\beta} q_s)\,\mathrm{d}t = 0. \tag{18} $$

4. Fractional Lagrange equations

The fractional Lagrange equations are based on the fractional Hamilton principle. From Eq. (18), we obtain

$$ \delta S = \delta \int_{a}^{b} L\,\mathrm{d}t = \int_{a}^{b} \delta L\,\mathrm{d}t = 0. \tag{19} $$

Calculating δL, we can obtain

$$ \begin{aligned} \delta L &= \frac{\partial L}{\partial q_s} \delta q_s + \frac{\partial L}{\partial {}^{C}D_{\gamma}^{\alpha,\beta} q_s} \delta {}^{C}D_{\gamma}^{\alpha,\beta} q_s \\ &= \frac{\partial L}{\partial q_s} \delta q_s + \frac{\partial L}{\partial {}^{C}D_{\gamma}^{\alpha,\beta} q_s} {}^{C}D_{\gamma}^{\alpha,\beta} \delta q_s. \end{aligned} \tag{20} $$

Performing integration by part for Eq. (10), and taking the conditions (16) into consideration, we have

$$ \int_{a}^{b} \frac{\partial L}{\partial {}^{C}D_{\gamma}^{\alpha,\beta} q_s} {}^{C}D_{\gamma}^{\alpha,\beta} \delta q_s\,\mathrm{d}t = \int_{a}^{b} \delta q_s {}^{C}D_{1-\gamma}^{\beta,\alpha} \frac{\partial L}{\partial {}^{C}D_{\gamma}^{\alpha,\beta} q_s}\,\mathrm{d}t. \tag{21} $$

Substituting the variation (20) and Eq. (21) into Eq. (19), we obtain

$$ \int_{a}^{b} \left\{ \left(\frac{\partial L}{\partial q_s} + {}^{C}D_{1-\gamma}^{\beta,\alpha} \frac{\partial L}{\partial {}^{C}D_{\gamma}^{\alpha,\beta} q_s} \right) \delta q_s \right\} \mathrm{d}t = 0. \tag{22} $$

This identity holds for arbitrary integral interval $[a,b]$, and therefore the integrand should be zero, i.e.

$$ \left(\frac{\partial L}{\partial q_s} + {}^{C}D_{1-\gamma}^{\beta,\alpha} \frac{\partial L}{\partial {}^{C}D_{\gamma}^{\alpha,\beta} q_s} \right) \delta q_s = 0. \tag{23} $$

Considering that δq_s $(s = 1,2,\ldots,n)$ are independent, we obtain the following equation:

$$ \frac{\partial L}{\partial q_s} + {}^{C}D_{1-\gamma}^{\beta,\alpha} \frac{\partial L}{\partial {}^{C}D_{\gamma}^{\alpha,\beta} q_s} = 0, \quad (s = 1,2,\ldots,n). \tag{24} $$

Equation (24) is called the fractional Lagrange equations within combined Caputo fractional derivatives.

Two important cases of Eq. (24) are considered below.

Case 4.1 If $\gamma = 0$, the Lagrangian is $L(t, q_s, {}_{t}^{C}D_{b}^{\beta} q_s)$, then equation (24) gives

$$ \frac{\partial L}{\partial q_s} + {}_{a}^{C}D_{t}^{\beta} \frac{\partial L}{\partial {}_{t}^{C}D_{b}^{\beta} q_s} = 0, \quad (s = 1,2,\ldots,n). \tag{25} $$

Case 4.2 If $\gamma = 1$, the Lagrangian is $L(t, q_s, {}_{a}^{C}D_{t}^{\alpha} q_s)$, then equation (24) gives

$$ \frac{\partial L}{\partial q_s} + {}_{t}^{C}D_{b}^{\alpha} \frac{\partial L}{\partial {}_{a}^{C}D_{t}^{\alpha} q_s} = 0, \quad (s = 1,2,\ldots,n). \tag{26} $$

5. Fractional cyclic integrals

In this section, we study the fractional cyclic integrals of the fractional Lagrange system. If the Lagrangian does not contain q_j $(j=1,2,\ldots,k)$, then we have

$$\frac{\partial L}{\partial q_j}=0,\quad (j=1,2,\ldots,k), \tag{27}$$

where q_j is called the cyclic coordinates. According to the previous results, we discuss the cyclic integrals in two cases.

Case 5.1 If $\gamma=0$, substituting Eq. (27) into Eq. (25), and taking notice that the Caputo derivative of a constant is zero, we obtain

$$\frac{\partial L}{\partial {}_t^C D_b^\beta q_j}=\beta_j,\quad (j=1,2,\ldots,k), \tag{28}$$

where β_j $(j=1,2,\ldots,k)$ are constants.

Case 5.2 If $\gamma=1$, substituting Eq. (27) into Eq. (26), and taking notice that the Caputo derivative of a constant is zero, we obtain

$$\frac{\partial L}{\partial {}_a^C D_t^\alpha q_j}=\beta_j,\quad (j=1,2,\ldots,k), \tag{29}$$

where β_j $(j=1,2,\ldots,k)$ are constants.

Equations (28) and (29) are called the fractional cyclic integrals.

6. Fractional Routh equations

In this section, we further discuss the fractional Routh equations in two cases according to the fractional cyclic integrals.

Case 6.1 For $\gamma=0$,we define the Routh function

$$R=L-\sum_{j=1}^{k}{}_t^C D_b^\beta q_j\frac{\partial L}{\partial {}_t^C D_b^\beta q_j}, \tag{30}$$

where $q_j(j=1,2,\ldots,k)$ are the cyclic coordinates, the Lagrangian is

$$L(q_{k+1},q_{k+2},\ldots,q_n,{}_t^C D_b^\beta q_1,{}_t^C D_b^\beta q_2,\ldots,{}_t^C D_b^\beta q_n,t) \tag{31}$$

and the cyclic integrals are

$$\frac{\partial L}{\partial {}_t^C D_b^\beta q_j}=\beta_j,\quad (j=1,2,\ldots,k). \tag{32}$$

From Eq. (32), we can obtain

$$\begin{aligned}{}_t^C D_b^\beta q_j &= f_j(\beta_1,\beta_2,\ldots,\beta_k,q_{k+1},q_{k+2},\ldots,q_n,{}_t^C D_b^\beta q_{k+1},\\ &\quad {}_t^C D_b^\beta q_{k+2},\ldots,{}_t^C D_b^\beta q_n,t),\quad (j=1,2,\ldots,k).\end{aligned} \tag{33}$$

Substituting Lagrangian (31) and Eq. (33) into Eq. (30), we obtain

$$\begin{aligned}R &= R(q_{k+1},q_{k+2},\ldots,q_n,{}_t^C D_b^\beta q_{k+1},\ldots,{}_t^C D_b^\beta q_n,\\ &\quad \beta_1,\beta_2,\ldots,\beta_k,t).\end{aligned} \tag{34}$$

According to Eq. (30), we derive the variation below

$$\begin{aligned}\delta R &= \delta\left(L-\sum_{j=1}^{k}{}_t^C D_b^\beta q_j\frac{\partial L}{\partial {}_t^C D_b^\beta q_j}\right)\\ &= \delta L-\sum_{j=1}^{k}{}_t^C D_b^\beta q_j\delta\frac{\partial L}{\partial {}_t^C D_b^\beta q_j}\\ &\quad -\sum_{j=1}^{k}\frac{\partial L}{\partial {}_t^C D_b^\beta q_j}\delta {}_t^C D_b^\beta q_j,\end{aligned} \tag{35}$$

then, considering cyclic integrals (32), we obtain

$$\delta R=\delta L-\sum_{j=1}^{k}{}_t^C D_b^\beta q_j\delta\beta_j-\sum_{j=1}^{k}\frac{\partial L}{\partial {}_t^C D_b^\beta q_j}\delta {}_t^C D_b^\beta q_j. \tag{36}$$

Using Lagrangian (31), we have

$$\begin{aligned}\delta L &= \sum_{r=k+1}^{n}\frac{\partial L}{\partial q_r}\delta q_r+\sum_{j=1}^{k}\frac{\partial L}{\partial {}_t^C D_b^\beta q_j}\delta {}_t^C D_b^\beta q_j\\ &\quad +\sum_{r=k+1}^{n}\frac{\partial L}{\partial {}_t^C D_b^\beta q_r}\delta {}_t^C D_b^\beta q_r.\end{aligned} \tag{37}$$

From Eq. (34), we acquire

$$\begin{aligned}\delta R &= \sum_{r=k+1}^{n}\frac{\partial R}{\partial q_r}\delta q_r+\sum_{r=k+1}^{n}\frac{\partial R}{\partial {}_t^C D_b^\beta q_r}\delta {}_t^C D_b^\beta q_r\\ &\quad +\sum_{j=1}^{k}\frac{\partial R}{\partial \beta_j}\delta\beta_j.\end{aligned} \tag{38}$$

Inserting Eqs. (37) and (38) into Eq. (36), we obtain

$$\begin{aligned}&\sum_{r=k+1}^{n}\left(\frac{\partial R}{\partial {}_t^C D_b^\beta q_r}-\frac{\partial L}{\partial {}_t^C D_b^\beta q_r}\right)\delta {}_t^C D_b^\beta q_r\\ &+\sum_{r=k+1}^{n}\left(\frac{\partial R}{\partial q_r}-\frac{\partial L}{\partial q_r}\right)\delta q_r\\ &+\sum_{j=1}^{k}\left(\frac{\partial R}{\partial \beta_j}+{}_t^C D_b^\beta q_j\right)\delta\beta_j=0.\end{aligned} \tag{39}$$

Considering that $\delta {}_t^C D_b^\beta q_r$ $(r=k+1,k+2,\ldots,n)$, δq_r $(r=k+1,k+2,\ldots,n)$, $\delta\beta_j$ $(j=1,2,\ldots,k)$ are independent, we obtain

$$\frac{\partial R}{\partial {}_t^C D_b^\beta q_r}-\frac{\partial L}{\partial {}_t^C D_b^\beta q_r}=0,\quad (r=k+1,k+2,\ldots,n), \tag{40}$$

$$\frac{\partial R}{\partial q_r}-\frac{\partial L}{\partial q_r}=0,\quad (r=k+1,k+2,\ldots,n), \tag{41}$$

$$\frac{\partial R}{\partial \beta_j}+{}_t^C D_b^\beta q_j=0,\quad (j=1,2,\ldots,k). \tag{42}$$

Using Eqs. (40) and (41), we change Eq. (25) into

$$\frac{\partial R}{\partial q_r}+{}_a^C D_t^\beta\frac{\partial R}{\partial {}_t^C D_b^\beta q_r}=0,(r=k+1,k+2,\ldots,n), \tag{43}$$

and the first k equations of Eq. (25) become the cyclic integrals as Eq. (32).

From Eq. (42), we have

$$ {}_t^C D_b^\beta q_j = -\frac{\partial R}{\partial \beta_j}, \quad (j = 1,2,\ldots,k). \tag{44} $$

Case 6.2 For $\gamma = 1$, the same as case 1, the Routh function is

$$ R = L - \sum_{j=1}^{k} {}_a^C D_t^\alpha q_j \frac{\partial L}{\partial {}_a^C D_t^\alpha q_j}, \tag{45} $$

we can easily obtain

$$ \frac{\partial R}{\partial q_r} + {}_t^C D_b^\alpha \frac{\partial R}{\partial {}_a^C D_t^\alpha q_r} = 0, \quad (r = k+1, k+2, \ldots, n), \tag{46} $$

$$ {}_a^C D_t^\alpha q_j = -\frac{\partial R}{\partial \beta_j}, \quad (j = 1,2,\ldots,k). \tag{47} $$

Equations (43) and (46) are in accordance with Eqs. (25) and (26), and the Routh function R is equivalent to the Lagrangian L. We can see that the order of the equation is reduced to $n-k$, which can simplify the problems. Equations (43) and (46) are called the fractional Routh Equations. Finally, we can compute q_j by integrating Eqs. (44) and (47).

7. Examples

Example 1 Considering a mechanical system with two free degrees, the Lagrangian is given by

$$ L = \frac{({}_t^C D_b^\beta q_2)^2}{4q_2} + q_2 ({}_t^C D_b^\beta q_1)^2 + l^2 ({}_t^C D_b^\beta q_1)^2, \tag{48} $$

where l is a constant.

Obviously, L does not contain q_1, and q_1 is the cyclic coordinate.

Then equation (28) becomes

$$ \frac{\partial L}{\partial {}_t^C D_b^\beta q_1} = \beta_1. \tag{49} $$

Substituting the Lagrangian (48) into Eq. (49) we obtain the cyclic integral

$$ 2q_2 {}_t^C D_b^\beta q_1 + 2l^2 {}_t^C D_b^\beta q_1 = \beta_1. \tag{50} $$

We construct the Routh function as

$$ R = L - \frac{\partial L}{\partial {}_t^C D_b^\beta q_1} \cdot {}_t^C D_b^\beta q_1 = \frac{({}_t^C D_b^\beta q_2)^2}{4q_2} - \frac{1}{2}\beta_1, \tag{51} $$

using Eq. (43), we obtain the Routh equation

$$ {}_a^C D_t^\beta \frac{{}_t^C D_b^\beta q_2}{2q_2} - \frac{({}_t^C D_b^\beta q_2)^2}{4q_2^2} = 0. \tag{52} $$

From Eq. (44), we have

$$ {}_t^C D_b^\beta q_1 = -\frac{1}{2}. \tag{53} $$

Example 2 The Lagrangian of a system is given by

$$ L = \frac{1}{2}\frac{({}_a^C D_t^\alpha q_1)^2}{a + bq_2^2} + \frac{1}{2}({}_a^C D_t^\alpha q_2)^2 - a_1 - b_1 q_2^2, \tag{54} $$

where a, b, a_1, and b_1 are nonzero constants.

We can see that L does not contain q_1, and q_1 is the cyclic coordinate.

Equation (29) can be written as

$$ \frac{\partial L}{\partial {}_a^C D_t^\alpha q_1} = \beta. \tag{55} $$

According to the Lagrangian (54), we obtain

$$ \frac{\partial L}{\partial {}_a^C D_t^\alpha q_1} = \frac{{}_a^C D_t^\alpha q_1}{a + bq_2^2}. \tag{56} $$

Substituting Eq. (56) into Eq. (55), we obtain the cyclic integral

$$ \frac{{}_a^C D_t^\alpha q_1}{a + bq_2^2} = \beta. \tag{57} $$

We construct the Routh function as

$$ \begin{aligned} R &= L - \frac{\partial L}{\partial {}_a^C D_t^\alpha q_1} \cdot {}_a^C D_t^\alpha q_1 \\ &= -\frac{1}{2}\beta^2 (a + bq_2^2) + \frac{1}{2}({}_a^C D_t^\alpha q_2)^2 - a_1 - b_1 q_2^2, \end{aligned} \tag{58} $$

then, from Eq. (46), we obtain the Routh equation

$$ {}_t^C D_b^\alpha {}_a^C D_t^\alpha q_2 - b\beta^2 q_2 - 2b_1 q_2 = 0. \tag{59} $$

Using Eq. (47), we have

$$ {}_a^C D_t^\alpha q_1 = (a + bq_2^2)\beta. \tag{60} $$

8. Conclusion

Recently the traditional calculus of variations has been extended to the systems containing fractional derivatives. In this paper, the fractional cyclic integrals and Routh equations of fractional Lagrange system with Caputo fractional derivative are achieved. Using this approach, we can reduce the order of the systems and simplify the problems.

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