Exact solutions of the nonlinear differential–difference equations associated with the nonlinear electrical transmission line through a variable-coefficient discrete (G'/G)-expansion method

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We investigated exact traveling soliton solutions for the nonlinear electrical transmission line. By applying a concise and straightforward method, the variable-coefficient discrete (G'/G)-expansion method, we solve the nonlinear differential–difference equations associated with the network. We obtain some exact traveling wave solutions which include hyperbolic function solution, trigonometric function solution, rational solutions with arbitrary function, bright as well as dark solutions.

Keywords: nonlinear transmission line, discrete (G'/G)-expansion method, solitary waves

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1. Introduction

In recent years, the behavior of nonlinear discrete systems has received considerable attention. The linear properties of this class of systems are strongly modified and as a result, their nonlinear response is known to exhibit features that are otherwise impossible in the bulk/continuous regime.[1] There are two main reasons for this growth: the development of experimental techniques making it possible to realize experiments in complicated periodic nonlinear structure and the potential for all-optical switching applications.[1] These systems are intrinsically discrete and a correct microscopic description involves a set of coupled ordinary differential equations instead of a partial-differential equation.

The dynamics of nonlinear lattices are drastically different in generic nonintegrable systems, a paradigmatic example being the discrete nonlinear Schrödinger (DNLS) equation and its exceptional integrable models the Ablowitz Ladik (AL) equation.[2] Only in the latter case are exact solutions for soliton families available. In nonintegrable systems, solitons are sought in a numerical form[3,4] or by means of a variational approximation.[5]

Finding exact discrete solutions of nonlinear differential–difference equations (NDDEs) has attracted wide interest in recent years. Nonlinear differential–difference equations play a crucial role in many branches of applied sciences, such as condensed matter physics, biophysics, atomic chains, molecular crystals, discretization in solid-state, and quantum physics. Unlike difference equations which are fully discretized, nonlinear differential–difference equations are semi-discretized with some (or all) of their space variables discretized while time is usually kept continuous. Generally, it is a very hard task to find exact discrete solutions of nonlinear partial differential–discrete equations, including soliton equations. Moreover, even if one manages to find a method for solving one nonlinear equation, in general such a method will not be applicable to other equations. These are probably the reasons why, since pioneering works by Hirota and Suzuki[6] on electrical transmission lines simulating Toda lattice, a growing interest has been devoted to the use of nonlinear transmission lines, in particular, for studying nonlinear waves and nonlinear modulated waves: pulse solitons, envelope pulse (bright), hole (dark) solitons and kink and anti-kink solitons,[7–9] intrinsic localized modes (also called discrete breathers),[10–12] modulational instability.[13–17]

Several methods for finding the exact solutions of nonlinear evolution equations for mathematical physics have been proposed,[18–34] such as Jacobian elliptic function method,[35,36] Fibonacci tanh function for NDDEs,[37] Exp-function method,[38,39] variable-coefficient discrete (G'/G)-expansion method for NDDEs,[40] and so on. In order to establish the effectiveness and reliability of the (G'/G)-expansion method and to expand the possibility of its application, further research has been carried out by a considerable number of researchers. Akbar *et al.*[41] proposed the modified alternative (G'/G)-expansion method to find traveling wave solutions of Drinfel'd–Sokolov–Wilson equation, Shakeel *et al.*[42] applied an improved (G'/G)-expansion method for Burger's, Zakharov–Kuznetsov (ZK) and Boussinesq equations. The new approach of generalized (G'/G)-expansion method has been applied in Ref. [43] to construct general traveling wave solutions of the strain wave equation in microstructured solids. The expansion method and its application have also been applied to solve nonlinear evolution equations.[44,45] In the present work, our desire is to construct general exact solutions for a nonlinear transmission line (NLTL) using a variable-coefficient discrete (G'/G)-expansion method. These solu-

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tions can be useful to understand the dynamical process described through the NLTLs.

The remainder of the paper is organized as follows: in Section 2 we present the model and we derive the equation governing modulated waves in the network. In Section 3 we briefly describe the method. Then, we derive exact discrete solutions in Section 4. The paper is concluded in the last section.

2. Model description and equation of motion

We consider a nonlinear electrical lattice made of N identical unit cells, as illustrated in Fig. 1. Each cell contains a linear inductor L and a nonlinear capacitor $C(V_n)$. This capacitor consists of a reverse-biased diode with a differential capacitance function of the voltage V_n across the n-th capacitor. The nonlinearity is introduced in the line by a varicap diode for which the capacitance varies with the applied tension. $Q_n(t)$ denotes the nonlinear electrical charge of the n-th cell and $V_n(t)$ is the corresponding voltage.[46] So, we can write

$$Q_n = C_0 V_0 \ln\left(1+\frac{V_n}{V_0}\right), \tag{1}$$

where C_0 and V_0 are constant. During computations, the following values of the parameters are used: $V_0 = 3.5$ V, $C_0 = 143$ pF, and $L = 22$ μH.

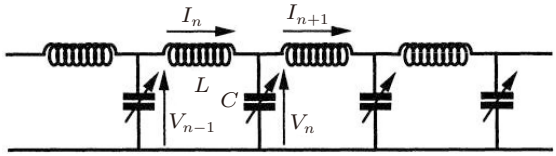


Fig. 1. Electrical Toda lattice with linear inductance L and a nonlinear capacitance.

Applying the Kirchhoff's laws to the network leads to the following equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\ln\left(1+\frac{V_n}{V_0}\right)\right] = \frac{1}{LC_0V_0}(V_{n-1}-2V_n+V_{n+1}). \tag{2}$$

This equation governs the propagation of modulated waves through the line. For solving it we set a variable s_n such as

$$V_n = \frac{\mathrm{d}s_n}{\mathrm{d}t}, \quad V_{n\pm1} = \frac{\mathrm{d}s_{n\pm1}}{\mathrm{d}t}. \tag{3}$$

Substituting Eq. (3) into Eq. (2) yields

$$LC_0V_0\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\ln\left(1+\frac{\mathrm{d}s_n/\mathrm{d}t}{V_0}\right)\right] = \frac{\mathrm{d}}{\mathrm{d}t}(s_{n-1}-2s_n+s_{n+1}). \tag{4}$$

Integrating the previous equation, we obtain

$$\frac{\mathrm{d}^2 s_n}{\mathrm{d}t^2} = \frac{1}{LC_0V_0}\left(V_0+\frac{\mathrm{d}s_n}{\mathrm{d}t}\right)(s_{n-1}-2s_n+s_{n+1}). \tag{5}$$

This is the NDDE associated with the NLTL.

3. Summary of the variable-coefficient discrete (G'/G)-expansion method

We now describe the variable-coefficient discrete (G'/G)-expansion method.[40] We consider a system of M polynomial nonlinear differential–difference equations in the form

$$P\big(s_{n+p_1}(t),\ldots,s_{n+p_k}(t),\ldots,s'_{n+p_1}(t),\ldots,s'_{n+p_k}(t),\ldots, s^{(r)}_{n+p_1}(t),\ldots,s^{(r)}_{n+p_k}(t)\big) = 0, \tag{6}$$

where the dependent variables s_n have M components $s_{i,n}$, the continuous variable t has N components t_i, the discrete variable n has Q components n_j, the k shift vectors $p_i \epsilon Z^Q$, and $s^{(r)}(t)$ denotes the collection of mixed derivative terms of order r. The main steps of the variable-coefficient discrete (G'/G)-expansion method are outlined as follows:

Step 1 A traveling wave transformation

$$s_{n+p_h}(t) = S_{n+p_h}(\xi_n), \quad \xi_n = \sum_{i=1}^{Q} d_i n_i + \eta(T), \tag{7}$$

where the coefficients d_i, c_j are all constants and $\eta(T)$ are under-determined differentiable functions of T, equation (6) can be reduced to

$$P\big(S_{n+p_1}(\xi_n),\ldots,S_{n+p_k}(\xi_n),\ldots,S'_{n+p_1}(\xi_n),\ldots,S'_{n+p_k}(\xi_n),\ldots, S^{(r)}_{n+p_1}(\xi_n),\ldots,S^{(r)}_{n+p_k}(\xi_n)\big) = 0. \tag{8}$$

Step 2 One supposes that equation (8) has the following solution:

$$S_n(\xi_n) = \sum_{l=-m}^{m} a_l(T)\left(\frac{G'(\xi_n)}{G(\xi_n)}\right)^l, \tag{9}$$

where $a_l(T)$ $(l = 0,1,2,\ldots,m)$ are all functions of T to be determined later, $G(\xi_n)$ satisfies a second-order linear ordinary differential equation:

$$\frac{\mathrm{d}^2G(\xi_n)}{\mathrm{d}\xi_n^2} + \lambda\frac{\mathrm{d}G(\xi_n)}{\mathrm{d}\xi_n} + \mu G(\xi_n) = 0, \tag{10}$$

where λ and μ are arbitrary constants. Using the general solutions of Eq. (10), we find

$$\frac{G'(\xi_n)}{G(\xi_n)} = \begin{cases} -\dfrac{\lambda}{2} + \dfrac{\sqrt{\lambda^2-4\mu}}{2}\left(\dfrac{C_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)+C_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}{C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)+C_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}\right), & \lambda^2-4\mu>0, \\ -\dfrac{\lambda}{2} + \dfrac{\sqrt{4\mu-\lambda^2}}{2}\left(\dfrac{-C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)+C_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}{C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)+C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}\right), & \lambda^2-4\mu<0, \\ \dfrac{C_2}{C_1+C_2\xi_n} - \dfrac{\lambda}{2}, & \lambda^2-4\mu=0. \end{cases} \tag{11}$$

Step 3 Through an easy computation, we can get the identity

$$\xi_{n+p_h} = \xi_n + \varphi_{p_h}, \quad \varphi_{p_h} = p_{h_1} d_1 + p_{h_2} d_2 + \cdots + p_{h_Q} d_Q, \tag{12}$$

where p_{s_j} is the j-th component of the shift vector p_s. Hence, we derive the following expression:

$$\frac{G'(\xi_{n\pm p_h})}{G(\xi_{n\pm p_h})} = \frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2} \left(\frac{\frac{2}{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right) \pm \epsilon f\left(\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2}\varphi_h\right)}{1 \pm \left(\frac{2}{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right)\right) f\left(\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2}\varphi_h\right)} \right) - \frac{\lambda}{2}, \tag{13}$$

where $\epsilon = \{1,-1,0\}$, $\delta = \{0,4\}$ and

$$f\left(\frac{\sqrt{\epsilon(\lambda^2-4\mu)}}{2}\varphi_h\right) = \begin{cases} \tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\varphi_h\right), & \epsilon = 1, \ \ \delta = 0, \ \ \lambda^2-4\mu > 0, \\ \tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\varphi_h\right), & \epsilon = -1, \ \ \delta = 0, \ \ \lambda^2-4\mu < 0, \\ \varphi_h, & \epsilon = 0, \ \ \delta = 4, \ \ \lambda^2-4\mu = 0. \end{cases} \tag{14}$$

Thus, we have

$$S_{n\pm p_h}(\xi_n) = \sum_{l=-m}^{m} a_l(T) \left[\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2} \left(\frac{\frac{2}{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right) \pm \epsilon f\left(\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2}\varphi_h\right)}{1 \pm \left(\frac{2}{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right)\right) f\left(\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2}\varphi_h\right)} \right) - \frac{\lambda}{2} \right]^l. \tag{15}$$

Step 4 Now, one determines the degree m of Eqs. (9) and (15) by balancing the highest-order nonlinear term(s) and the highest-order partial derivative of $S_n(\xi_n)$ in Eq. (9). Because $S_{n\pm p_h}$ can be interpreted as being of degree zero in $(G'(\xi_n)/(G(\xi_n)))$, the leading terms of $S_{n\pm p_h}$ $(p_h \neq 0)$ will not affect the balancing procedure.

Step 5 Substituting Eqs. (9) and (15) given the value of m determined in Step 4 along with Eq. (10) into Eq. (8) and clearing the denominator and collecting all terms with the same order of $(G'(\xi_n)/(G(\xi_n)))$ together, then setting each coefficient of $(G'(\xi_n)/(G(\xi_n)))^l$ to zero, we will derive a set of underdetermined partial differential equations, from which $a_l(T)$ and $\eta(T)$ can be determined using Maple or Mathematica.

Step 6 Using the results obtained in the above steps, we can finally obtain exact solutions of Eq. (6).

4. Applications

In this section, we examine the lattice-soliton solution of Eq. (5) by applying the proposed method. We make the transformation

$$s_n(t) = S_n(\xi_n), \quad \xi_n = d_1 n + \eta(t). \tag{16}$$

Substituting Eq. (16) into Eq. (5) we obtain

$$\left(\frac{\mathrm{d}\eta}{\mathrm{d}t}\right)^2 \ddot{S}_n + \frac{\mathrm{d}^2\eta}{\mathrm{d}t^2}\dot{S}_n = \frac{1}{LC_0V_0}\left(V_0 + \frac{\mathrm{d}\eta}{\mathrm{d}t}\dot{S}_n\right)(S_{n-1} - 2S_n + S_{n+1}), \tag{17}$$

where

$$\dot{S}_n = \frac{\mathrm{d}S_n}{\mathrm{d}\xi_n}(\xi_n) \quad \text{and} \quad \ddot{S}_n = \frac{\mathrm{d}^2S_n}{\mathrm{d}\xi_n^2}(\xi_n).$$

By balancing the highest order derivative terms and nonlinear terms in Eq. (17), we suppose that equation (16) can be written as follows:

$$S_n = a_0(t) + a_1(t)\left(\frac{G'(\xi_n)}{G(\xi_n)}\right) + a_{-1}(t)\left(\frac{G'(\xi_n)}{G(\xi_n)}\right)^{-1}, \tag{18}$$

$$S_{n+1} = \sum_{l=-1}^{1} a_l(t) \left[\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2} \left(\frac{\frac{2}{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right) + \epsilon f\left(\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2}\varphi_h\right)}{1 + \left(\frac{2}{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right)\right) f\left(\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2}\varphi_h\right)} \right) - \frac{\lambda}{2} \right]^l, \tag{19}$$

$$S_{n-1} = \sum_{l=-1}^{1} a_l(t) \left[\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2} \left(\frac{\frac{2}{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right) - \epsilon f\left(\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2}\varphi_h\right)}{1 - \left(\frac{2}{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right)\right) f\left(\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2}\varphi_h\right)} \right) - \frac{\lambda}{2} \right]^l, \tag{20}$$

where $a_0(t)$, $a_1(t)$, $a_{-1}(t)$, d_1, and $\eta(t)$ are functions to be determined later. Substituting Eqs. (18)–(20) along with Eq. (10) into Eq. (17), clearing the denominator and setting each of the coefficients of $(G'(\xi_n)/(G(\xi_n)))^l$ to zero, we get a system of under-determined partial equations for $a_0(t)$, $a_1(t)$, $a_{-1}(t)$, and $\eta(t)$. To avoid tediousness, we omit the over-determined, highly nonlinear and parameterized partial differential equations. Solving the system, we get the following solution set:

Case 1

$$a_0(t) = -V_0\Big(\epsilon LC_0\Big(\frac{\delta+\epsilon(\lambda^2-4\mu)}{4}\Big)k^2+1\Big)t + \frac{V_0LC_0k^2(\delta+\epsilon(\lambda^2-4\mu))}{4f^2\Big(\frac{\sqrt{\delta+\epsilon(\lambda^2-4\mu)}}{2}\varphi_h\Big)}t + c_3,$$
$$a_1(t) = V_0LC_0k;\ \ a_{-1}=0;\ \ \eta(t)=kt+c;\ \ d_1=d_1, \quad (21)$$

where c, d_1, and k are real constants.

Case 2

$$a_0 = c_4t+c_5;\ \ a_1(t)=0;\ \ a_{-1}=-\mu V_0LC_0k;$$
$$\eta(t)=kt+c;\ \ d_1=d_1, \quad (22)$$

where c, c_4, c_5, d_1, and k are real constants. When $\lambda^2-4\mu>0$, we obtain hyperbolic function solution of Eq. (17)

$${}^1S_n = c_3-\frac{V_0LC_0\lambda k}{2}-V_0t+\frac{V_0LC_0(\lambda^2-4\mu)k^2t}{4\sinh^2(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}+\frac{V_0LC_0k\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\Big)+C_2\cosh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\Big)}{C_1\cosh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\Big)+C_2\sinh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\Big)}\right), \quad (23)$$

where $\xi_n = d_1n+kt+c$.

$${}^2S_n = c_5+c_4t-\mu V_0LC_0k\left[-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\Big(\frac{C_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)+C_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}{C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)+C_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}\Big)\right]^{-1}, \quad (24)$$

where $\xi_n = d_1n+kt+c$. When $\lambda^2-4\mu<0$, we obtain trigonometric function solution of Eq. (17):

$${}^3S_n = c_3-\frac{V_0LC_0\lambda k}{2}-V_0t+\frac{V_0LC_0(4\mu-\lambda^2)k^2t}{4\sin^2\Big(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1\Big)}+\frac{V_0LC_0k\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin\Big(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\Big)+C_2\cos\Big(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\Big)}{C_1\cos\Big(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\Big)+C_2\sin\Big(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\Big)}\right), \quad (25)$$

$${}^4S_n = c_5+c_4t-\mu V_0LC_0k\left[-\frac{\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin\Big(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\Big)+C_2\cos\Big(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\Big)}{C_1\cos\Big(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\Big)+C_2\sin\Big(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\Big)}\right)\right]^{-1}, \quad (26)$$

where $\xi_n = d_1n+kt+c$. We deduce these expressions of V_n for each case:

$${}^1V_n = -V_0+\frac{V_0LC_0(\lambda^2-4\mu)k^2}{4\sinh^2\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1\Big)}+\frac{V_0LC_0k^2(\lambda^2-4\mu)}{4}\left[1-\left(\frac{C_1\sinh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\Big)+C_2\cosh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\Big)}{C_1\cosh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\Big)+C_2\sinh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\Big)}\right)^2\right], \quad (27)$$

$${}^2V_n = c_4-\mu V_0LC_0k^2\frac{\lambda^2-4\mu}{4}\left[\frac{\left(\frac{C_1\sinh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\Big)+C_2\cosh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\Big)}{C_1\cosh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\Big)+C_2\sinh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\Big)}\right)^2-1}{\left(-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\Big)+C_2\cosh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\Big)}{C_1\cosh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\Big)+C_2\sinh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\Big)}\right)\right)^2}\right]. \quad (28)$$

In particular, if $C_1\neq0$, $C_2=0$, then equations (27) and (28) can be written as the following solutions:

$${}^1V_n' = -V_0+\frac{V_0LC_0(\lambda^2-4\mu)k^2}{4\sinh^2\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1\Big)}+\frac{V_0LC_0k^2(\lambda^2-4\mu)}{4}\operatorname{sech}^2\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\Big), \quad (29)$$

$$^{2}V_n' = c_4 + \mu V_0 L C_0 k^2 \frac{\lambda^2 - 4\mu}{4} \left[\frac{\operatorname{sech}^2\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\right)}{\left(-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\right)\right)^2} \right]. \tag{30}$$

Similarly, if $C_1 = 0$, $C_2 \neq 0$, then equations (29) and (30) can be written as the following solutions:

$$^{1}V_n' = -V_0 + \frac{V_0 L C_0 (\lambda^2 - 4\mu) k^2}{4\sinh^2\left(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1\right)} - \frac{V_0 L C_0 k^2 (\lambda^2 - 4\mu)}{4} \operatorname{csch}^2\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\right), \tag{31}$$

$$^{2}V_n' = c_4 - \mu V_0 L C_0 k^2 \frac{\lambda^2 - 4\mu}{4} \left[\frac{\operatorname{csch}^2\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\right)}{\left(-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n\right)\right)^2} \right]. \tag{32}$$

For $\lambda^2 - 4\mu < 0$, we obtain

$$^{3}V_n = -V_0 + \frac{V_0 L C_0 (4\mu - \lambda^2) k^2}{4\sin^2\left(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1\right)} - \frac{V_0 L C_0 k^2 (4\mu - \lambda^2)}{4} \left[1 + \left(\frac{-C_1 \sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n) + C_2 \cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}{C_1 \cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n) + C_2 \sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)} \right)^2 \right], \tag{33}$$

$$^{4}V_n = c_4 - \mu V_0 L C_0 k^2 \frac{4\mu - \lambda^2}{4} \left[\frac{1 + \left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\right)} \right)^2}{\left(-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\right)} \right) \right)^2} \right]. \tag{34}$$

In particular, if $C_1 \neq 0$, $C_2 = 0$ then equations (33) and (34) can be written as the following solutions:

$$^{3}V_n' = -V_0 + \frac{V_0 L C_0 (4\mu - \lambda^2) k^2}{4\sin^2\left(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1\right)} - \frac{V_0 L C_0 k^2 (4\mu - \lambda^2)}{4} \times \sec^2\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\right), \tag{35}$$

$$^{4}V_n' = c_4 - \mu V_0 L C_0 k^2 \frac{4\mu - \lambda^2}{4} \times \left[\frac{\sec^2(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}{\left(-\frac{\lambda}{2} - \frac{\sqrt{4\mu-\lambda^2}}{2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\right)\right)^2} \right]. \tag{36}$$

Similarly, if $C_1 = 0$, $C_2 \neq 0$, then equations (35) and (36) can be written as the following solutions:

$$^{3}V"_n = -V_0 + \frac{V_0 L C_0 (4\mu - \lambda^2) k^2}{4\sin^2\left(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1\right)} - \frac{V_0 L C_0 k^2 (4\mu - \lambda^2)}{4} \times \csc^2\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\right), \tag{37}$$

$$^{4}V"_n = c_4 - \mu V_0 L C_0 k^2 \frac{4\mu - \lambda^2}{4} \times \left[\frac{\csc^2\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\right)}{\left(-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n\right)\right)^2} \right]. \tag{38}$$

When $\lambda^2 - 4\mu = 0$, we obtain the following rational solutions of Eq. (17)

$$^{5}V_n = -V_0 - \frac{V_0 L C_0 k^2 c_2}{(c_1 + c_2\xi_n)^2}, \tag{39}$$

$$^{6}V_n = c_4 - \frac{\mu V_0 L C_0 k^2 c_2}{\left(-\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2\xi_n}\right)^2 (c_1 + c_2\xi_n)^2}. \tag{40}$$

Remark 1 Setting $d_1 = (1/\Omega)\operatorname{arc}(\sinh(\pm\Omega)) = P/\Omega$ and $k^2 = (1/LC_0)$, where, $\Omega = \sqrt{\lambda^2 - 4\mu}/2$; equations (29)–(32) can be obtained as follows:

$$^{1}V_n' = V_0\Omega^2 \operatorname{sech}^2(Pn \pm \Omega u_0 t + c), \tag{41}$$

$$^{2}V_n' = c_4 + \frac{\mu V_0 \Omega^2 \operatorname{sech}^2(Pn \pm \Omega u_0 t + c)}{\left(-\frac{\lambda}{2} + \Omega\tanh(Pn \pm \Omega u_0 t + c)\right)^2}, \tag{42}$$

$$^{1}V_n' = -V_0\Omega^2 \operatorname{csch}^2(Pn \pm \Omega u_0 t + c), \tag{43}$$

$$^{2}V_n' = c_4 - \frac{\mu V_0 \Omega^2 \operatorname{csch}^2(Pn \pm \Omega u_0 t + c)}{\left(-\frac{\lambda}{2} + \Omega\coth(Pn \pm \Omega u_0 t + c)\right)^2}, \tag{44}$$

with $u_0 = 1/\sqrt{LC_0}$. Note that the solution equation (41) has been obtained by Remoissenet[46] after applying the long-wavelength approximation. He called it a voltage-pulse soliton. This lattice soliton has a pulse profile; it has amplitude $V_m = V_0\Omega^2$ and travels with velocity

$$u_S = \Omega u_0/P = \sinh(P)/(P\sqrt{LC_0}). \tag{45}$$

Its width at half amplitude is proportional to $1/P$. The parameters of the line are $V_0 = 3.9$ V; $C_0 = 470$ pF; $L = 680$ μH.

Figure 2 depicts the time evolution of solutions obtained from Eqs. (27) and (28) with a bright-like soliton profile, while the solutions described through Eqs. (41) and (42) have been plotted in Fig. 3. One can observe the propagation of two pulse solitons. The solutions of Figs. 2 and 3 have the same shape (bright-like shape) and almost the same width ($V_m/2$) as well as the same amplitude. This form of soliton arises from a suitable compensation between the nonlinear and dispersive effects in the line. The pulse solitons are used to transmit loads of information across trans-continental and trans-oceanic distances. This kind of solution has been numerically investigated in Refs. [47] and [48]. Our solutions also cover dark-like solitons as depicted in Fig. 4. A dark soliton appears as an intensity dip in an infinitely extended constant background. It has been investigated in many theoretical and experimental papers.[49,50] The dark solitons have attracted much attention due to their potential applications.[51] For example, using structures created during the propagation and interaction of the dark solitons, many types of all-optical switches may be written.[52] Other applications of the dark solitons involve the optical logic devices[53] and waveguide optics as dynamic switches and junctions.[54] They are also considered in signal processing and communication because of their inherent stability.[55] Compared with the pulse solitons, they have better stability against various perturbations such as fiber loss, mutual interaction between neighboring pulses, the Raman effect, and the superposition of noise emitted from optical amplifiers.[56]

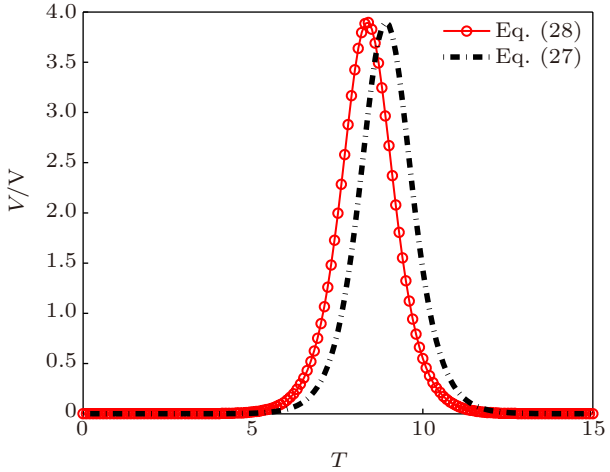


Fig. 2. (color online) Evolution of the solitons given by Eqs. (27) (dashed line) and (28) (circle) at $n = 10$, for $\Omega = 1$, $\lambda = 4$, $\mu = 3$, $k = 1/\sqrt{LC_0}$.

Note that for $d_1 = (1/\Omega)\,\mathrm{arc}(\sin(\pm\Omega)) = P/\Omega$ and $k = \pm u_0$, where, $\Omega = \sqrt{4\mu - \lambda^2}/2$; the solutions given by Eqs. (35) and (37) are reduced.

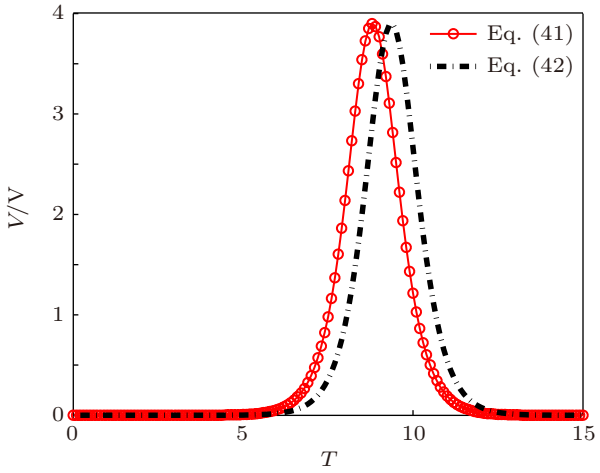


Fig. 3. (color online) Profile of the solitons given by Eqs. (41) and (42) at $n = 10$, $\Omega = 1$.

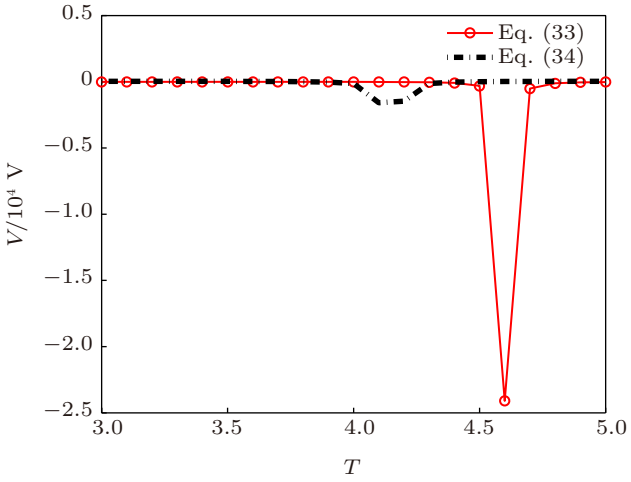


Fig. 4. (color online) The nonlinear evolution behavior of the dark solitons given by Eqs. (33) (dashed line) and (34) (circle) at $n = 5$, for $\Omega = 1$, $\lambda = 4$, $\mu = 5$, $k = 1/\sqrt{LC_0}$.

5. Conclusion

In this work, we have obtained an exact solitary wave solution for wave propagation in a nonlinear electrical transmission line. The variable-coefficient discrete (G'/G)-expansion method has been used to construct these solutions. As a result, bright and dark-like solutions, trigonometric function, and rational solutions with arbitrary functions have been obtained.

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