Cluster synchronization of uncertain complex networks with desynchronizing impulse*

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Cluster synchronization of nonlinear uncertain complex networks with desynchronizing impulse is explored. First of all, a feedback controller is employed, based on the Lyapunov stability theorem and Lipschitz condition, to guarantee that the uncertain complex networks with desynchronizing impulse synchronize with an object trajectory. Furthermore, a synchronizing impulse controller is presented, which is more efficiently and directly used to achieve the cluster synchronization. Finally, numerical examples are examined to show the effectiveness of the proposed methods.

Keywords: uncertain complex networks, desynchronizing impulse, cluster synchronization

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1. Introduction

Synchronization is not only a ubiquitous phenomenon in nature, but also an important research branch in physics.[1] The earliest studies on synchronization phenomena can be traced back to 1673 by Huygens about coupled pendulum. In the 1990s, Peroca and Carroll achieved the synchronization of chaotic systems by some drivers (or coupling), and first observed the phenomenon of chaos synchronization in electronic circuits. Since then, the research on synchronization has aroused considerable interest due to its many other applications in areas of biology systems, secure communication and cryptography.

Synchronization in a complex network as an important branch, can be achieved by virtue of controllers to drive the complex network to an arbitrary state. Up to now, many synchronization patterns have been reported, including complete synchronization, phase synchronization, general synchronization, projective synchronization, etc.[2–8]

Cluster synchronization, as a particular synchronization phenomenon, has been a hot spot of research. It is achieved when the dynamical nodes reach complete synchronization in each subgroup called a cluster (group, community), but no synchronization appears among different clusters. Based on these special properties, many results about this synchronization pattern of complex networks have been proposed. Wu and Fu[9] deliberated the cluster projective synchronization of community networks with non-identical nodes. Su *et al.*[10] proposed a novel decentralized feedback pinning control scheme for the cluster synchronization of undirected complex networks. Ma and Lu[11] employed directed complex networks, which are different from the assumed network topology in Ref. [10], in the cluster synchronization problem under pinning control scheme. Tang and Feng[12] studied the cluster synchronization problems for general networks with non-delayed and delayed coupling based on the pinning control scheme. Wang and Cao[13] exploited nonlinearly coupled networks with community structure and non-identical nodes with time-varying delay by applying pinning control to a fraction of network nodes. The obtained results move the investigation of cluster synchronization in nonlinear science fields an important step forward.

In the present paper, we concentrate on the effect of impulse on complex networks in this paper. In the above mentioned research studies, only the non-delayed or delayed effect on complex networks was considered. However, in the real world, the parameter fluctuation, external disturbance, and parameter uncertainties are unavoidable, and they can make the synchronization more difficult to realize due to measurement errors, or may destroy the networks' stability.[14–18] On the other hand, the states of many complex systems and realistic networks are often subjected to instantaneous perturbations and experience abrupt changes at certain instants, which may be caused by switching phenomenon, frequency change or other sudden noise, that is, they exhibit impulsive effects.

In general, there are two kinds of impulses in terms of synchronization in complex networks: synchronizing impulse and desynchronizing impulse.[19] An impulsive sequence is said to be desynchronized if it can suppress the synchronization of complex dynamical networks. Conversely, an impulsive sequence is said to be synchronized if it can enhance the synchronization. In the literature,[20–24] most of studies were devoted to investigating the desynchronizing impulse. All of them have a common feature that the network is syn-

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chronous and the synchronization is related to the topology structure, system parameter, impulse interval, and impulse strength. However, as is well known, networks cannot always be synchronous; there are some factors that will lead to an unstable network such as the general topology structure, uncertain parameters, and a random impulse.

Inspired by the foregoing analysis, what we are going to investigate in our present study is whether the cluster synchronization can be achieved in complex networks with a random impulse and parameter uncertainties, and whether there is any analytical argument for this phenomenon besides the numerical method.

We propose theoretical analysis and numerical simulation for cluster synchronization of complex networks with impulse disturbance and parameter uncertainties. The results show that the proposed controller scheme can control the complex networks with desynchronizing impulses and parameter uncertainties to a homogenous solution. The most important highlights of this paper are: (i) we study the cluster synchronization of uncertain complex networks with more representative impulse (a desynchronizing impulse) instead of a random impulse; (ii) owing to the control energy consumption, the cluster synchronization is achieved by utilizing feedback control and synchronizing impulse control separately; (iii) based on Lyapunov stability theorem, a detailed theoretical analysis is given to explore sufficient conditions of cluster synchronization for the complex networks.

The remainder of this paper is organized as follows. In Section 2, we formulate and describe the cluster synchronization problem of complex networks with impulse disturbance and parameter uncertainties. In Section 3, two outer controllers are designed and sufficient conditions for cluster synchronization are derived analytically. Two numerical simulations are given in Section 4 to illustrate the effectiveness and reliability of the proposed technique. In Section 5 some conclusions are drawn from the present study.

2. Network model and preliminaries

In this paper, we consider complex dynamical networks composed of N linearly coupled non-identical nodes. The dynamic behaviors of nodes in the presence of uncertain parameters and noise are described as

$$\dot{x}_i(t) = (C+\Delta C(t))x_i(t)+(B+\Delta B(t))f(x_i(t))+w(t), \quad (1)$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t)]^{\mathrm{T}}$ $(i = 1,2,\ldots,N)$ is the state vector of the i-th node at time t; $B, C \in R^{n\times n}$ are the nominal constant matrixes; $\Delta B(t)$ and $\Delta C(t)$ denote the time-varying uncertain parameters; $f(x_i(t)) = [f_1(x_{i1}(t)), f_2(x_{i2}(t)), \ldots, f_n(x_{in}(t))]^{\mathrm{T}}$ is the nonlinear part of the complex dynamical system; $w(t) \in R^n$ is the noise that is always described as an external input vector.

The model of an uncertain complex dynamical network can be described by the following differential equations:

$$\begin{aligned}\dot{x}_i(t) &= (C+\Delta C(t))x_i(t)+(B+\Delta B(t))f(x_i(t))+w(t)\\ &\quad + v\sum_{j=1}^{N} a_{ij}\Gamma x_j(t), \quad i \in \widehat{U}, \end{aligned} \quad (2)$$

where $\widehat{U} = \{\kappa_{l-1}+1, \ldots, \kappa_l\}$ represents the index set of all the nodes in the κ-th cluster (community), $\kappa = 1,2,\ldots,m$, $\kappa_m = N, l_{k-1} < l_\kappa$; v is the coupling strength; the $A = (a_{ij})_{N\times N}$ denotes the network topology, where $a_{ij} > 0$ if there is a connection from node j to node i ($j \neq i$), and $a_{ij} = 0$ otherwise.

Remark 1 The topology in the complex dynamical networks (2) represents the entire network if $\kappa = 1$. So our potential results in this paper can be extended to other networks, i.e., neural networks (NNs), economic networks, etc.

In the process of signal transmission, frequency change or other sudden noise, the states of nodes are often subjected to instantaneous perturbations and experience abrupt changes at discrete times t_k. That is, $x_i(t_k^+) - x_j(t_k^+) = S_k(x_i(t_k^-) - x_j(t_k^-))$. For simplicity, without loss of generality, constant impulse gain is chosen as $S_k = b_k I$, where I is an $n\times n$ identity matrix. Therefore, the uncertain complex networks with impulse disturbances are obtained as

$$\begin{cases} \dot{x}_i(t) = (C+\Delta C(t))x_i(t)+(B+\Delta B(t))f(x_i(t))\\ \quad +w(t)+v\sum\limits_{j=1}^{N} a_{ij}\Gamma x_j(t), \quad t \neq t_k, t \geq t_0,\\ x_j(t_k^+) - x_i(t_k^+) = b_k(x_j(t_k^-) - x_i(t_k^-)),\\ \quad t = t_k, k \in Z^+, i \in \widehat{U},\\ x_i(t) = \phi_i(t), \quad -\tau \leq t \leq 0, \end{cases} \quad (3)$$

where $Z^+ = \{1,2,\ldots\}$ denotes a set of positive integers, $\{t_k\}$ is a sequence of strictly increasing impulse moments, and $|1+b_k| > 1$ denotes the strength of impulse disturbance between the coupling of the connected nodes. Without loss of generality, we always assume that $x_i(t)$ is left continuous at $t = t_k$, i.e., $\lim\limits_{t\to t_k} x_i(t_k) = x_i(t_k^-)$. Therefore, the solutions of Eq. (3) are piecewise left-hand continuous functions with discontinuities at $t = t_k$ for $k \in Z^+$.

Remark 2 From previous impulse research,[16,22] we can easily conclude that the impulses are desynchronizing impulses when $|1+b_k| > 1$, which means that the impulses may destroy the synchronization. In this paper, it should be pointed out that we just consider desynchronizing impulse disturbances of complex networks. Therefore, we mainly focus on how to achieve the cluster synchronization of uncertain complex networks.

Now, we give some useful definitions, assumptions, and lemmas.

Definition 1[25] A network with N nodes is said to realize cluster synchronization, if the N nodes split into several clusters, such as $\{(1,2,\ldots,m_1), (m_1+1, m_1+2, \ldots, m_1+m_2), \ldots,$

$(m_1+m_2+\cdots+m_{k-1}+1,\ m_1+m_2+\cdots+m_{k-1}+2,\ldots,$ $m_1+m_2+\cdots+m_k)$, $m_1+m_2+\cdots+m_k=N\}$ such that the nodes in the same cluster synchronize with each other, i.e., for the states $x_i(t)$ and $x_j(t)$ of the arbitrary nodes i and j in the same cluster, $\lim_{t\to\infty}\|x_i-x_j\|=0$ holds.

Assumption 1[26] If the activation function $f(\cdot)$ satisfies Lipschitz condition, i.e., there exist constants $l_i>0$, $(i=1,2,\ldots,n)$ such that for $\forall x(t),y(t)\in R^n$, we have $|f(x(t))-f(y(t))|\le L|x(t)-y(t)|$,where $L=\mathrm{diag}(l_1,l_2,\ldots,l_n)$.

Assumption 2[27] The nodes belonging to the same cluster have the same function $f(\cdot)$, i.e., for the κ-th cluster $(1\le\kappa\le m)$, $f_{l_{\kappa-1}+1}=f_{l_{\kappa-1}+2}=\cdots=f_{l_\kappa}\equiv f_\kappa$.

Assumption 3[17] The parametric uncertainties $\Delta C(t)$ and $\Delta B(t)$ are in the form of $[\Delta C(t),\Delta B(t)]=EH(t)[D_1,D_2]$, where E and D_i $(i=1,2)$ are known real constant matrices with appropriate dimensions, and the uncertain matrix $H(t)$ is an unknown real time-varying matrix satisfying $H^{\mathrm{T}}(t)H(t)\le I$.

Assumption 4[27] The coupling matrix $A=(a_{ij})_{n\times n}$ of the network (2) satisfies

$$A=\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m}\\ A_{21} & A_{22} & \cdots & A_{2m}\\ \vdots & \vdots & \ddots & \vdots\\ A_{m1} & A_{m2} & \cdots & A_{mm}\end{pmatrix},$$

where $(A_{kk})_{(l_k-l_{k-1})\times(l_k-l_{k-1})}$ belongs to A_1 and $(A_{kq})_{(l_k-l_{k-1})\times(l_q-l_{q-1})}$ belongs to A_2, with $k,q=1,2,\ldots,m$.

Lemma 1[28] For any two n-dimensional real vectors X and Y and a positive-definite matrix $U\in R^{n\times n}$, the matrix inequality $2X^{\mathrm{T}}Y\le X^{\mathrm{T}}UX+Y^{\mathrm{T}}U^{-1}Y$ holds.

We present the expression of synchronization errors. Define $y(t)$ as an isolated complex dynamic network, that is,

$$\dot y(t)=(C+\Delta C(t))y(t)+(B+\Delta B(t))f(y(t))+w(t),\quad(4)$$

where $y(t)$ may be an equilibrium point, a limit cycle, or even a chaotic attractor, then error vectors are defined as $e_i(t)=x_i(t)-y_\kappa(t), i\in\widehat{U}=\{\kappa_{l-1}+1,\ldots,\kappa_l\}$, $\kappa=1,2,\ldots,m$ and we have

$$\begin{cases}\dot e_i(t)=(C+\Delta C(t))e_i(t)+(B+\Delta B(t))\tilde f(e_i(t))\\ \quad+v\sum\limits_{j=1}^{N}a_{ij}\Gamma x_j(t)+u(t),\\ t\ne t_k, t\ge t_0,\\ \Delta e_i=(1+b_k)e_i(t_k),\quad t=t_k,k\in Z^+,i\in\widehat{U},\end{cases}\quad(5)$$

where $\tilde f(e_i(t))=f(x_i(t))-f(y_\kappa(t))$. The main objective of this paper is to design a continuous and a discrete controller such that the uncertain complex networks (3) will be cluster-synchronized with the state of the system (4). Obviously the cluster synchronization of the uncertain complex networks (3) is achieved if and only if the error system (5) is robustly asymptotically stable at zero point.

3. Main result

In this section, we will separately investigate the cluster synchronization of the uncertain complex networks under two kinds of controllers. Detailed discussion will be given to verify that our obtained results are less conservative than previous results.

Before giving the main results, we state that the Assumptions 1–4 hold in the following theorem and proof.

3.1. Sufficient conditions for cluster synchronization with feedback controller

In this subsection, we design a feedback cluster synchronization controller in the following form:

$$u_i(t)=-k_i(x_i(t)-y_\kappa(t)),\quad i\in\widehat{U}.\quad(6)$$

Throughout this paper, in interval $(t_0,t]$, let b_i^{desy} denote the desynchronizing impulse strength, o_i represent the number of desynchronizing impulse times of the impulse sequence $\zeta=\{t_1,t_2,\cdots\}$, and $T_{\min}=\min\left|\{t_i-t_j\}\right|>0, i\ne j,i,j=1,2,\ldots$ refer to the minimum impulse interval, then it will not be hard to obtain that $(o_i-1)T_{\min}\le t-t_0$.

Theorem 1 The uncertain complex dynamical network (3) is globally exponentially cluster-synchronized if there exist a positive-definite matrix $\Gamma>0$ and a controller gain k_i such that

$$\frac{\ln(1+b_*^{\mathrm{desy}})^2}{T_{\min}}+\omega<0,\quad(7)$$

where

$$\omega=\lambda_{\max}2\Big[\Big(C+EE^{\mathrm{T}}+\frac{1}{2}(D_1^{\mathrm{T}}D_1+BB^{\mathrm{T}}+(1+\|D_2\|^2)L^{\mathrm{T}}L)-k_iI\Big)\otimes I+vA\otimes\Gamma\Big]$$

and $b_*^{\mathrm{desy}}=\max\{b_i^{\mathrm{desy}}\}$.

Proof Define a Lyapunov function candidate for system (5) as

$$V(t,e(t))=\sum_{i=1}^{N}e_i^{\mathrm{T}}(t)e_i(t).$$

For $t\in(t_{k-1},t_k]$, $k\in Z$, calculating the time derivative of $V(t,e(t))$ along the trajectories of system (5), we have

$$\begin{aligned}&D^+V(t,e(t))\\ &=2\sum_{i=1}^{N}e_i^{\mathrm{T}}(t)\dot e_i(t)\\ &=2\sum_{i=1}^{N}e_i^{\mathrm{T}}(t)\Big[(C+\Delta C(t))e_i(t)+(B+\Delta B(t))\tilde f(e_i(t))\\ &\quad+v\sum_{j=1}^{N}a_{ij}\Gamma x_j(t)+u(t)\Big].\end{aligned}\quad(8)$$

By Assumptions 1 and 3 and Lemma 1, we have the following three inequations:

$$\begin{aligned} & 2\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\Delta C(t)e_i(t) \\ = & \ 2\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)EH(t)D_1e_i(t) \\ \le & \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)EE^{\mathrm{T}}e_i(t)+\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)D_1^{\mathrm{T}}D_1e_i(t), \end{aligned} \tag{9}$$

$$2\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)B\tilde{f}(e_i(t)) \le \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)(BB^{\mathrm{T}}+L^{\mathrm{T}}L)e_i(t), \tag{10}$$

and

$$2\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\Delta B\tilde{f}(e_i(t)) \le \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)(EE^{\mathrm{T}}+\|D_2\|^2L^{\mathrm{T}}L)e_i(t). \tag{11}$$

From Assumption 4, we can obtain $\sum_{j\in\widehat{U}} a_{ij}y_\kappa(t)=0$ for $i=1,2,\ldots,N$ and $\kappa=1,2,\ldots,m$. Thus

$$\begin{aligned} \sum_{j=1}^{N} a_{ij}x_j(t) &= \sum_{\kappa=1}^{m}\sum_{j\in\widehat{U}} a_{ij}x_j(t)-\sum_{\kappa=1}^{m}\sum_{\kappa\in\widehat{U}} a_{ij}y_\kappa(t) \\ &= \sum_{\kappa=1}^{m}\sum_{j\in\widehat{U}} a_{ij}e_j(t)=\sum_{j=1}^{N} a_{ij}e_j(t). \end{aligned} \tag{12}$$

Using expressions (8)–(12) and performing more detailed calculations, we obtain

$$\begin{aligned} & D^+V(t,e(t)) \\ \le & \ 2e^{\mathrm{T}}(t)[(C+EE^{\mathrm{T}}+\frac{1}{2}(D_1^{\mathrm{T}}D_1+BB^{\mathrm{T}} \\ & +(1+\|D_2\|^2)L^{\mathrm{T}}L)-k_iI)\otimes I+\nu A\otimes\Gamma]e(t) \\ \le & \ \omega V(t), \ \text{ for } \ t\in(t_{k-1},t_k],\ k\in Z. \end{aligned}$$

Hence, $V(t)\le \mathrm{e}^{\omega(t-t_{k-1})}V(t_{k-1}^+)$, for $t\in(t_{k-1},t_k]$, $k\in Z$.

On the other hand, for $t=t_k^+, k\in Z$. Performing more detailed calculations, we can obtain

$$\begin{aligned} V(t_k^+) &= \sum_{i=1}^{N} e_i^{\mathrm{T}}(t_k^+)e_i(t_k^+) \\ &= \sum_{i=1}^{N}(1+b_k)^2e_i^{\mathrm{T}}(t_k)e_i(t_k)=(1+b_k)^2V(t_k). \end{aligned}$$

For $\forall t\in(t_0,t_1]$, we have $V(t)\le V(t_0^+)\mathrm{e}^{\omega(t-t_0)}$. Hence $V(t_1)\le V(t_0^+)\mathrm{e}^{\omega(t_1-t_0)}$, and $V(t_1^+)=(1+b_1)^2V(t_1)\le(1+b_1)^2V(t_0^+)\mathrm{e}^{\omega(t_1-t_0)}$. Similarly, for $\forall t\in(t_1,t_2]$, we have $V(t)\le V(t_1^+)\mathrm{e}^{\omega(t-t_1)}\le(1+b_1)^2V(t_0^+)\mathrm{e}^{\omega(t-t_0)}$, which implies $V(t_2)\le(1+b_1)^2V(t_0^+)\mathrm{e}^{\omega(t_2-t_0)}$, and $V(t_2^+)=(1+b_2)^2V(t_2)\le(1+b_1)^2(1+b_2)^2V(t_0^+)\mathrm{e}^{\omega(t_2-t_0)}$. In general, for $\forall t\in(t_{k-1},t_k]$, $V(t_k)\le\prod_{i=1}^{k}(1+b_i)^2V(t_0^+)\mathrm{e}^{\omega(t_k-t_0)}$.

Thus it is easy to deduce that $V(t)\le\prod_{i=1}^{o_i}(1+b_i)^2V(t_0^+)\mathrm{e}^{\omega(t-t_0)}$. From the supposing parameters, we can further obtain

$$\begin{aligned} V(t) &\le V(t_0^+)\exp\left[\ln(\prod_{i=1}^{o_i}(1+b_i)^2)+\omega(t-t_0)\right] \\ &\le V(t_0^+)\exp\left[\left(\frac{t-t_0}{T_{\min}}+1\right)\ln(1+b_*^{\mathrm{desy}})^2+\omega(t-t_0)\right] \\ &= (1+b_*^{\mathrm{desy}})^2V(t_0^+)\exp\left[\left(\frac{\ln(1+b_*^{\mathrm{desy}})^2}{T_{\min}}+\omega\right)(t-t_0)\right]. \end{aligned}$$

Since $\ln(1+b_*^{\mathrm{desy}})^2/T_{\min}+\omega<0$, we can draw the conclusion that the uncertain complex networks (3) are globally exponentially synchronized with the convergence rate $(\ln(1+b_*^{\mathrm{desy}})^2/T_{\min}+\omega)$. Thus the proof of Theorem 1 is completed. In particular, by letting $\Gamma=I$ in Theorem 1, we have the following corollary.

Corollary 1 The uncertain complex networks (3) are globally exponentially cluster-synchronized, if there exists a controller gain k_i such that

$$\frac{\ln(1+b_*^{\mathrm{desy}})^2}{T_{\min}}+\omega<0, \tag{13}$$

where

$$\begin{aligned} \omega = \lambda_{\max}\Big[\Big(C+EE^{\mathrm{T}}+\frac{1}{2}(D_1^{\mathrm{T}}D_1+BB^{\mathrm{T}} \\ +(1+\|D_2\|^2)L^{\mathrm{T}}L)\Big)\otimes I+\nu A\otimes\Gamma\Big]-k_i=\gamma-k_i. \end{aligned}$$

Remark 3 In Corollary 1, by deducing inequality (13), we obtain the relationship among desynchronizing impulse strength, impulses interval, and feedback gain as

$$k_i\ge\frac{\ln(1+b_*^{\mathrm{desy}})^2}{T_{\min}}+\gamma.$$

This inequality allows us to determine the feedback gain when desynchronizing impulse strength and impulse interval boundaries are given.

When the effect of desynchronizing impulse is small, the feedback controller is a simple way to achieve the synchronization. While the desynchronizing impulse has a great influence on the complex networks, the feedback controller may need a large feedback gain, which costs much control energy. Hence the following discussion is required.

3.2. Sufficient conditions for cluster synchronization with impulse controller

In this subsection, we employ a discrete controller including an impulse controller of the type $\{t_k, I_{ik}(x_i-y_\kappa)\}$ such that the uncertain complex networks can synchronize the trajectory of system (4).[21] For simplicity, we choose the discrete controller as

$$u_i(t) = \sum_{k=1}^{+\infty} I_{ik}(x_i-y_\kappa)\delta(t-\hat{t}_k)$$

$$= \sum_{k=1}^{+\infty} d_i(x_i - y_\kappa)\delta(t-\hat{t}_k), \quad i \in \widehat{U}, \tag{14}$$

where $d_i \in (-2,0)$ is the synchronization impulse gain, $\{\hat{t}_k\}_{k=1}^{\infty}$ denotes the time sequence and satisfies $\hat{t}_1 < \hat{t}_2 < \cdots < \hat{t}_k < \cdots$, and $\lim_{k\to\infty} \hat{t}_k = \infty$, $\delta(\cdot)$ is the Dirac impulse function.

For uncertain complex networks (3) with controller (14), there are two types of impulses: desynchronizing impulses and synchronizing impulses, according to impulsive strength. However, the effects of impulses are related to the time sequence. Clearly, $\{t_k\}_{k=1}^{\infty}$ is the time sequence of desynchronizing impulse existing in the uncertain complex networks (3); $\{\hat{t}_k\}_{k=1}^{\infty}$ is that of synchronizing impulse in the controlled networks. We arrange the impulsive sequence $\{t_k\}_{k=1}^{\infty}$ and $\{\hat{t}_k\}_{k=1}^{\infty}$ in chronological order, then a new time sequence $\{\bar{t}_k\}_{k=1}^{\infty}$ is obtained. There are two cases in the time sequence

$$\bar{t}_k = \begin{cases} t_i, & t_i \geq \hat{t}_j, \\ \hat{t}_j, & \hat{t}_j > t_i. \end{cases}$$

Simultaneously, the impulsive strength μ_k is obtained, i.e.

$$\mu_k = \begin{cases} b_k, & t = t_k, t \neq \hat{t}_k, \\ d_k, & t = \hat{t}_k, t \neq t_k, \\ b_k + d_k, & t = \hat{t}_k, t = t_k. \end{cases}$$

By the above analysis of the impulse, now the error system (5) is equivalent to

$$\begin{cases} \dot{e}_i(t) = (C+\Delta C(t))e_i(t) + (B+\Delta B(t))\tilde{f}(e_i(t)) \\ \quad + v\sum\limits_{j=1}^{N} a_{ij}\Gamma x_j(t), \\ t \neq t_k, t \geq t_0, \\ \Delta e_i = (1+\mu_k)e_i(t_k), \quad t = \bar{t}_k, k \in Z^+, i \in \widehat{U}. \end{cases} \tag{15}$$

Let o_j be the number of synchronizing impulse times in interval $(t_0,t]$, b_i^{desy} the synchronizing impulse strength, and $T'_{\max} = \max\left|\{\hat{t}_i - \hat{t}_j\}\right| > 0$, $i \neq j, i, j = 1,2,\ldots$ the maximum impulse interval, then we will have that $(o_j+1)T'_{\max} \geq t - t_0$.

Theorem 2 The uncertain complex networks (3) are globally exponentially cluster-synchronized if there exists a positive-definite matrix $\Gamma > 0$ such that

$$\frac{\ln(1+b_*^{\text{desy}})^2}{T_{\min}} + \frac{\ln(1+d_*^{sy})^2}{T'_{\max}} + \zeta < 0, \tag{16}$$

where

$$\zeta = \lambda_{\max} 2\left[\left(C + EE^{\mathrm{T}} + \frac{1}{2}(D_1^{\mathrm{T}}D_1 + BB^{\mathrm{T}} + (1+\|D_2\|^2)L^{\mathrm{T}}L)\right) \otimes I + vA \otimes \Gamma\right],$$

and $d_*^{sy} = \max\{d_i^{sy}\}$.

Proof Let $e_i(t) = (e_{i1}(t), e_{i2}(t), \ldots, e_{in}(t))^{\mathrm{T}} \in R^n$ for $i \in \widehat{U}$ and consider the following Lyapunov function:

$$V(t,e(t)) = 2\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t).$$

Just like the proof of Theorem 1, a brief proof is given below.

For $t \in (t_{k-1}, t_k]$, $k \in Z$,

$$D^+V(t,e(t)) = 2\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\dot{e}_i(t),$$

$$\begin{aligned} D^+V(t,e(t)) &\leq 2e^{\mathrm{T}}(t)\left[\left(C + EE^{\mathrm{T}} + \frac{1}{2}(D_1^{\mathrm{T}}D_1 + BB^{\mathrm{T}} + (1+\|D_2\|^2)L^{\mathrm{T}}L)\right) \otimes I + vA \otimes \Gamma\right] e(t) \\ &\leq \zeta V(t). \end{aligned}$$

When $t = t_k^+, k \in Z$, we have

$$V(t_k^+) = \sum_{i=1}^{N} e_i^{\mathrm{T}}(t_k^+)e_i(t_k^+) = (1+\mu_k)^2 V(t_k).$$

Like the analysis of $V(t)$ in time interval $(t_0,t]$ in the proof of Theorem 1, then

$$\begin{aligned} V(t) &\leq \prod_{i=1}^{o_i}(1+b_*^{\text{desy}})^2 \prod_{i=1}^{o_j}(1+d_*^{sy})^2 V(t_0^+)\, e^{\omega(t-t_0)} \\ &\leq (1+b_*^{\text{desy}})^2(1+d_*^{sy})^{-2}V(t_0^+) \\ &\quad \times \exp\left[\left(\frac{\ln(1+b_*^{\text{desy}})^2}{T_{\min}} + \frac{\ln(1+d_*^{sy})^2}{T'_{\max}} + \zeta\right)(t-t_0)\right]. \end{aligned}$$

From Theorem 2, we know

$$\frac{\ln(1+b_*^{\text{desy}})^2}{T_{\min}} + \frac{\ln(1+d_*^{sy})^2}{T'_{\max}} + \zeta < 0.$$

Then the uncertain complex networks (3) surely achieve exponentially cluster-synchronization with the discrete control law Eq. (7).

Remark 4 Theorem 2 provides a globally exponentially cluster-synchronizing criterion for uncertain complex networks (3). It should be stressed that we add new impulses to the networks where there have already existed desynchronizing impulses. This is distinguished from previous studies[20–24] either in the mathematical sense or in practical application, where the two kinds of impulses are assumed to occur separately, or there are two kinds of impulses in one impulse sequence.

Remark 5 It is a new scheme to investigate the uncertain complex networks (3) with desynchronizing impulses. The essential difference between Theorems 1 and 2 lies in different controllers which lead to different conditions for achieving cluster synchronization. When the effect of desynchronizing impulse is small, the feedback controller is a simple and valid way to achieve synchronization. However, when the effect becomes large, which leads to the feedback controller consuming much control energy, the synchronizing impulse controller shows its high efficiency and low energy consumption.

4. Numerical example

In this section, two examples are presented and numerical simulations are performed to verify the effectiveness and feasibility of the results derived.

Example 1 Consider a unified chaotic system, i.e.,[29]

$$\dot{x}(t) = (C + \Delta C(t))x(t) + (B + \Delta B(t))f(x(t)) + w(t), \quad (17)$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}} \in R^3$ is the state vector; $w(t) = \Delta C = \Delta B = 0$;

$$C = \begin{bmatrix} -(25\alpha+1) & 25\alpha+1 & 0 \\ 28-35\alpha & -(29\alpha-1) & 0 \\ 0 & 0 & -\dfrac{8+\alpha}{3} \end{bmatrix},$$

with $\alpha \in [0,1]$ being a system parameter; $B = (0,-1,1)^{\mathrm{T}}$; $f(x) = (0, x_1x_3, x_1x_2)^{\mathrm{T}}$ is a nonlinear function. Inspired by the colored networks,[30] one considers a color topology graph with three clusters (communities), where each cluster (community) has a different dynamical behavior (see Fig. 1). We fix the coupling strength $v = 1$ and the inner coupling matrix $\Gamma = I$ and let $L = 1$. Thus from Corollary 1 and Remark 4, we obtain the range of the feedback gain k_i as shown in Fig. 2. So we choose $b^{\mathrm{desy}} = 0.6$, and impulsive interval $\Delta t = 0.1$. Then it can be concluded that the uncertain complex networks (3) realize globally exponential cluster-synchronization if

$$k_i \geq \frac{\ln(1+b_*^{\mathrm{desy}})^2}{T_{\min}} + \gamma > 18.59.$$

Thus we can choose $k_i = 18.60$. Figures 3–5 show the trajectories of the synchronization errors in communities I, II, and III, respectively.

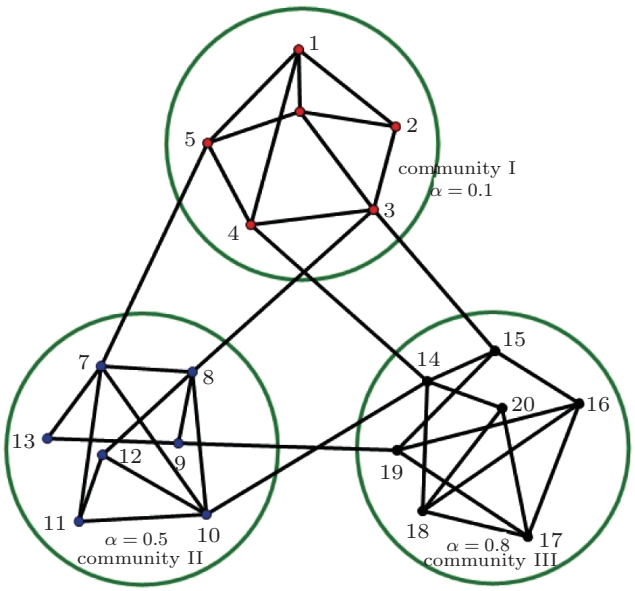


Fig. 1. (color online) Network with three communities showing different dynamical behaviors.

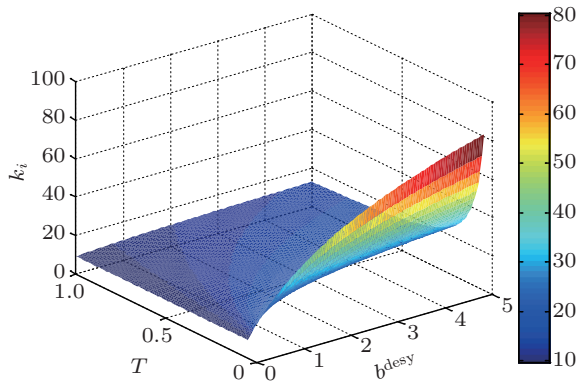


Fig. 2. (color online) Relationship between k_i, $T_{\min}$, and b^{desy}.

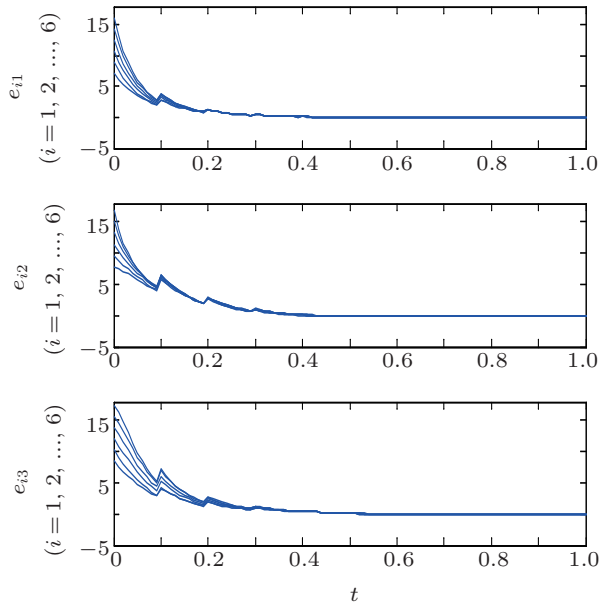


Fig. 3. (color online) Synchronization errors in community I.

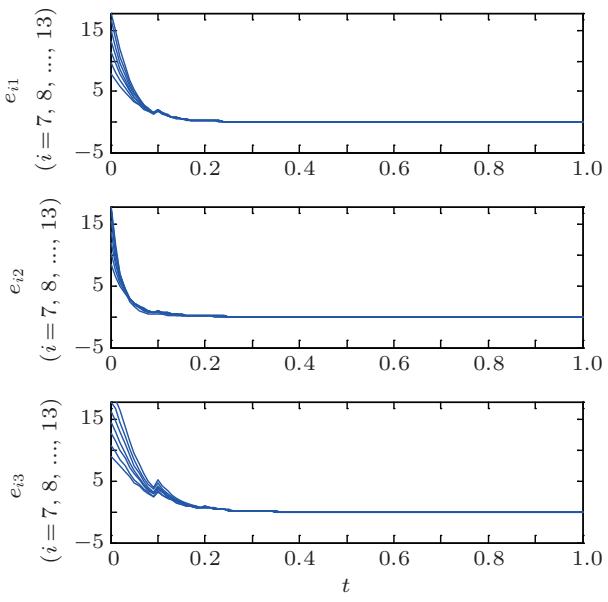


Fig. 4. (color online) Synchronization errors in community II.

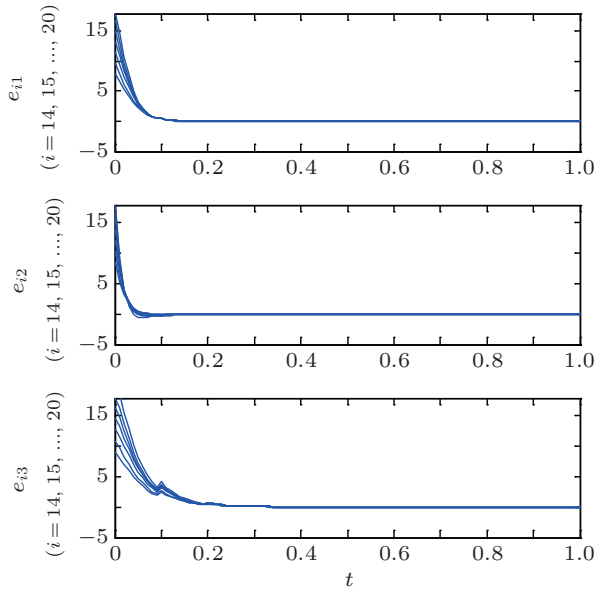


Fig. 5. (color online) Synchronization errors in community III.

Example 2 Consider an isolated node of the dynamical network,[16] i.e.,

$$\dot{x}(t) = (C+\Delta C(t))x(t) + (B+\Delta B(t))f(x(t)) + w(t), \quad (18)$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}} \in R^3$ is the state vector. $w(t) = 0$, $C = -I$,

$$B = \begin{bmatrix} 1.16 & -1.5 & -1.5 \\ -1.5 & 1.16 & -2.0 \\ -1.2 & 2.0 & 1.16 \end{bmatrix}$$

and

$$f(x(t)) = \Big(\frac{1}{2}(|x_1(t)+1| - |x_1(t)-1|), \frac{1}{2}(|x_2(t)+1| - |x_2(t)-1|), \frac{1}{2}(|x_3(t)+1| - |x_3(t)-1|)\Big)^{\mathrm{T}}$$

a nonlinear function. The positive constant is chosen as $L = 1$. When $[\Delta C(t), \Delta B(t)] = EH(t)[D_1, D_2]$ with $E = \mathrm{diag}\{0.1, 0.05, 0.3\}$, $H(t) = (\sin(t))I$, $D_1 = -0.6I$, and $D_2 = 0.2I$. The chaotic attractor of system (17) is shown in Fig. 6. In this example, we consider the case where $t_i - t_{i-1} = \hat{t}_j - \hat{t}_{j-1} = 0.1$, $i = 1, 2, \ldots, N_1$, $j = 1, 2, \ldots, N_2$.

Meanwhile, the time sequence satisfies that $t_1, \hat{t}, t_2, \hat{t}_2, \ldots, t_i, \hat{t}_i, \ldots$. Let $b^{\mathrm{desy}} = 0.7$ and $d^{\mathrm{sy}} = -0.5$, figure 7 shows the time-varying impulse sequence. The topology of Example 2 is the same as that shown in Fig. 1 when each node has the same dynamical behavior. By Theorem 2, one can derive that the uncertain complex dynamic networks (3) are globally exponentially cluster-synchronized if

$$\Delta t < -\frac{\ln(1+b_*^{\mathrm{desy}})^2(1+d_*^{sy})^2}{\omega} = 0.038.$$

Thus the impulsive interval can be chosen to be $\Delta t = 0.03$. Figures 8–10 represent the trajectories of the synchronization errors in communities I, II, and III, respectively.

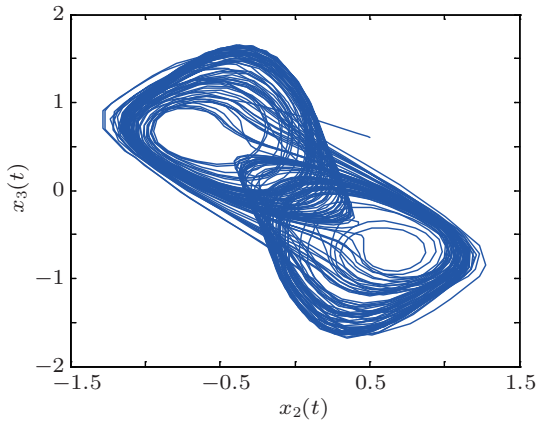


Fig. 6. (color online) Chaotic attractor of the isolated dynamical system (18) with uncertain parameters.

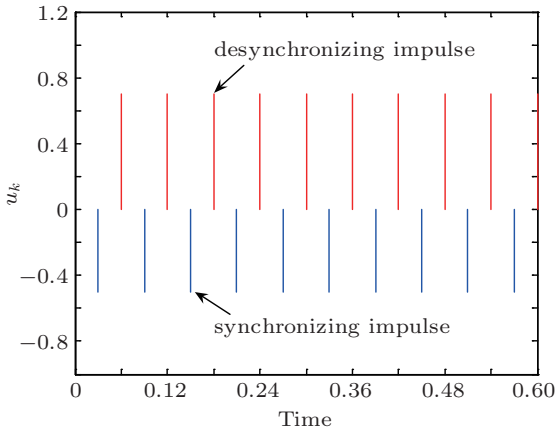


Fig. 7. (color online) Time-varying impulse sequence with $b^{\mathrm{desy}} = 0.7$, $d^{\mathrm{sy}} = -0.5$, and $\Delta t = 0.03$.

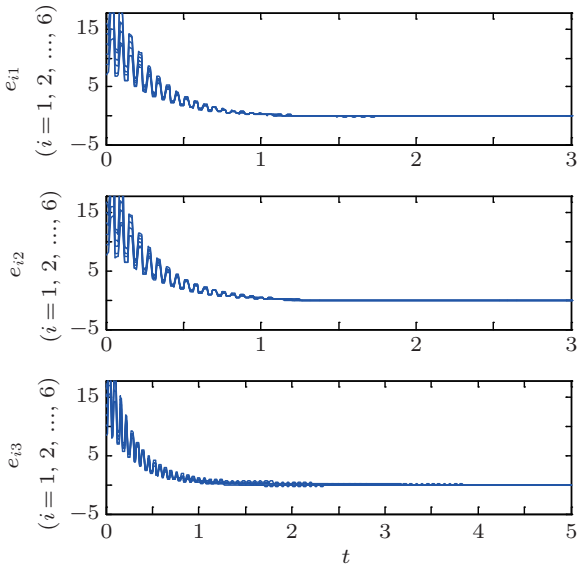


Fig. 8. (color online) Synchronization errors in community I.

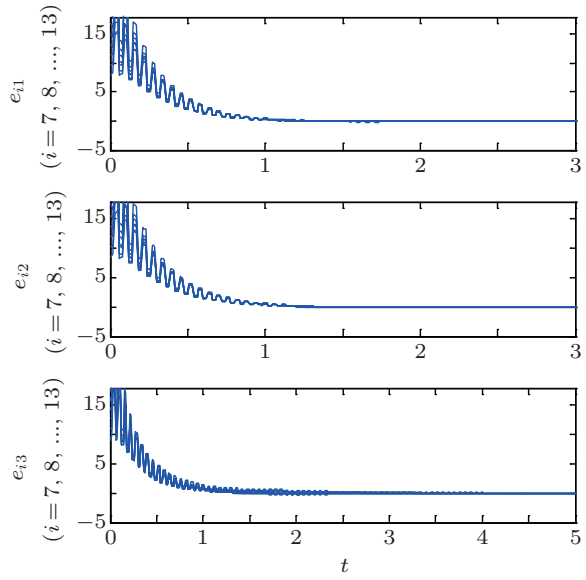


Fig. 9. (color online) Synchronization errors in community II.

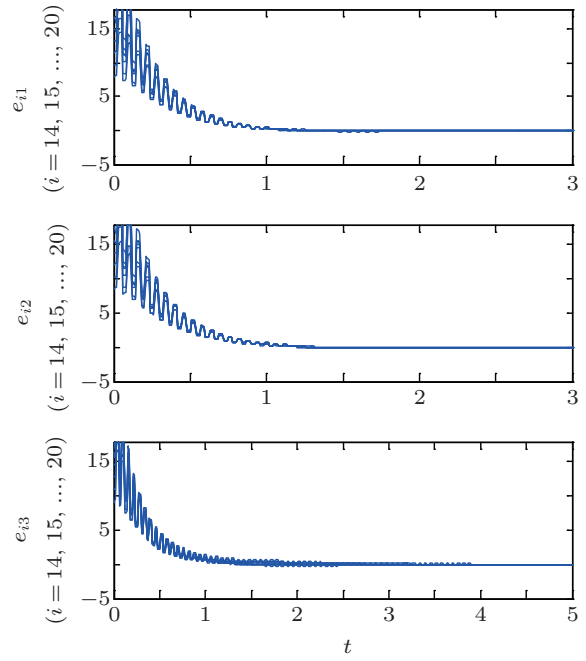


Fig. 10. (color online) Synchronization errors in community III.

5. Conclusion

In this paper, a detailed analysis is presented for the cluster synchronization of uncertain complex dynamical networks with impulse disturbance. In terms of control energy consumption, we consider feedback control and impulse control. Some globally exponential cluster-synchronization criteria for the uncertain complex networks with impulse disturbance are established by utilizing the Lyapunov method and Lipschitz condition. The results show that the feedback control or synchronizing impulse control remains valid for cluster synchronization even if there are some desynchronizing impulses in complex networks. Two examples and numerical simulations illustrate the effectiveness of the theoretical results.

Recently, multi-agent dynamical networks have received considerable attention due to their broad engineering applications.[31,32] Motivated by these models and studies, we will focus on the cluster consensus of multi-agent networks with desynchronizing impulse in our future work.

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