A generalized Padé approximation method of solving homoclinic and heteroclinic orbits of strongly nonlinear autonomous oscillators*

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An intrinsic extension of Padé approximation method, called the generalized Padé approximation method, is proposed based on the classic Padé approximation theorem. According to the proposed method, the numerator and denominator of Padé approximant are extended from polynomial functions to a series composed of any kind of function, which means that the generalized Padé approximant is not limited to some forms, but can be constructed in different forms in solving different problems. Thus, many existing modifications of Padé approximation method can be considered to be the special cases of the proposed method. For solving homoclinic and heteroclinic orbits of strongly nonlinear autonomous oscillators, two novel kinds of generalized Padé approximants are constructed. Then, some examples are given to show the validity of the present method. To show the accuracy of the method, all solutions obtained in this paper are compared with those of the Runge–Kutta method.

Keywords: generalized Padé approximation method, homoclinic and heteroclinic orbits, strongly nonlinear oscillators

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1. Introduction

In the study of chaotic behavior of dynamical system and traveling wave solutions in partial differential equations, homoclinic and heteroclinic orbits play a great role.[1–8] Thus, many methods have been proposed in researching homoclinic and heteroclinic orbits of nonlinear dynamics, such as the Melnikov method,[9] hyperbolic perturbation method,[10] hyperbolic Lindstedt–Poincaré method,[11] perturbation-incremental method,[12] generalized hyperbolic perturbation method,[13] and so on. Besides the perturbation-based method, the rational approximation method can also be used for solving homoclinic and heteroclinic orbits such as Padé approximation. Emaci *et al.*[14] constructed the solution of nonlinear Schrödinger equation based on the classical Padé approximant. Martin and Baker[15] extended the classical Padé approximant and constructed a kind of quasi-Padé approximant which contained the powers and exponential function. Then Mikhlin[16] used the quasi-Padé approximant to construct the homoclinic orbits of two-dimensional (2D) and three-dimensional (3D) dynamic systems. Manucharyan and Mikhlin[17] constructed a novel kind of quasi-Padé approximant which just contained the exponential functions for investigating the homoclinic and heteroclinic orbits of non-autonomous Duffing equation and the Van der Pol–Duffing equation. Under the enlightenment of Ref. [17], Zhang *et al.*[18] and Feng *et al.*[19] constructed their quasi-Padé approximants in solving homoclinic and heteroclinic orbits of Duffing equation. Padé approximation can also be combined with other methods such as Adomian decomposition method.[20]

Based on previous studies, an intrinsic extension of Padé approximation method, called the generalized Padé approximation method, is proposed in this paper. According to the proposed method, the numerator and denominator of Padé approximant are extended from polynomial functions to a series composed of any kind of function, which means that the generalized Padé approximant is not limited to some forms, but can be constructed in different forms in solving different problems. It unifies many existing modifications of Padé approximation method theoretically, which means that the Padé and quasi-Padé approximants constructed in Refs. [14]–[18] can be considered to be the special cases of the proposed method. In solving homoclinic and heteroclinic orbits of strongly nonlinear autonomous oscillators, the hyperbolic functions are adopted in constructing two novel kinds of generalized Padé approximants. Then, some examples are given to show that the present method is not limited to some systems but can be applied widespread. To show the accuracy of the method, all solutions obtained in this paper are compared with those of the Runge–Kutta method. The rest of this paper is organized as follows. In Section 2 the definition of generalized Padé approximation is given. In Section 3, the procedures of solving homoclinic and heteroclinic orbits of a strongly nonlinear autonomous oscillator based on the present method is established. In Section 4, several examples are calculated. The conclusions are drawn in Section 5 finally.

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2. Generalized Padé approximation method

The Padé approximation technique consists in approximating a truncated series by a rational function. In general, the latter function has the advantage in extending the range of validity of the initial polynomial. Padé approximation has thus far been successfully applied to statistical mechanics[21] and more recently to other physical problems.[22,23] To express the definition of Padé approximation, the following signs are introduced:

$$H_m = \left\{ P : P(z) = \sum_{i=0}^{m} a_i z^i,\ z, a_i \in C \right\}, \tag{1}$$

$$R(m,n) = \{ R = P/Q : P \in H_m,\ Q \in H_n \backslash \{0\} \}, \tag{2}$$

where C is the complex number and $H_n\backslash\{0\}$ indicates the elimination of the elements which are equal to zero from H_n. Based on these, the classical definition of Padé approximation can be expressed below.

Definition 1 (given by Baker) Assume that function $f(z)$ can be expressed by the following power series

$$f(z) = \sum_{i=0}^{\infty} a_i z^i, \tag{3}$$

if a rational function $P_L(z)/Q_M(z) \in R(L,\ M)$ ($P_L(z)$ and $Q_M(z)$ relatively-prime) exists such that

$$f(z) - P_L(z)/Q_M(z) = O(z^{L+M+1}), \tag{4}$$

$$Q_M(0) = 1, \tag{5}$$

then $P_L(z)/Q_M(z)$ is called a Padé approximant of $f(z)$ in $R(L,\ M)$, denoted as $[L/M]_f(z)$, briefly as $[L/M]_f$ or $[L/M]$.

Emaci *et al.*[14] constructed the following classical Padé approximant:

$$[L/M] = \frac{\sum_{i=0}^{L} \alpha_i t^i}{1 + \sum_{i=1}^{M} \beta_i t^i}. \tag{6}$$

Martin and Baker[15] constructed a kind of quasi-Padé approximant which contained the powers and exponential function, and it can be expressed as

$$QPA_n = \mathrm{e}^{-t} \frac{\sum_{i=0}^{n} \alpha_i t^i}{1 + \sum_{i=1}^{n} \beta_i t^i}. \tag{7}$$

Manucharyan and Mikhlin[17] constructed the following kind of quasi-Padé approximant which contained only the exponential functions

$$QPA_n = \mathrm{e}^{k\pm t} \frac{\sum_{i=0}^{n} \alpha_i \mathrm{e}^{\mathrm{i}k\pm t}}{1 + \sum_{i=1}^{n} \beta_i \mathrm{e}^{\mathrm{i}k\pm t}}. \tag{8}$$

Based on Eq. (8), Zhang *et al.*[18] constructed their quasi-Padé approximant as

$$QPA_n = \mathrm{e}^{-\sqrt{-c_1}t} \frac{\sum_{i=0}^{n} \alpha_i \mathrm{e}^{\mathrm{i}\sqrt{-c_1}t}}{1 + \sum_{i=1}^{n} \beta_i \mathrm{e}^{\mathrm{i}\sqrt{-c_1}t}}. \tag{9}$$

The modifications of Padé approximation method mentioned above were just the modifying of the form of Padé approximant, rather than the extending of the definition of Padé approximant. Based on the previous work, a generalized Padé approximation method is proposed in this paper which unifies the above-mentioned modifications of Padé approximation method theoretically. To express the definition of generalized Padé approximation, we redefine Eqs. (1) and (2) as

$$\hat{H}_m = \left\{ P : P(z) = \sum_{i=0}^{m} a_i g_i(z)^i,\ g_i :\ C \to C, a_i \in C \right\}, \tag{10}$$

$$G(m,n) = \{ R = P/Q : P \in \hat{H}_m, Q \in \hat{H}_n \backslash \{0\} \}, \tag{11}$$

where C is a complex number, $\hat{H}_n\backslash\{0\}$ indicates the elimination of the elements which are equal to zero from $\hat{H}_n$. Based on these, the definition of generalized Padé approximation can be expressed below.

Definition 2 Assume that function $f(z)$ can be expressed by the following power series:

$$f(z) = \sum_{i=0}^{\infty} a_i z^i. \tag{12}$$

If a rational function $P_L(z)/Q_M(z) \in G(L,\ M)$ exists such that

$$f(z) - P_L(z)/Q_M(z) = O(z^{L+M+1}), \tag{13}$$

then $P_L(z)/Q_M(z)$ is called a generalized Padé approximant of $f(z)$ in $G(L,\ M)$, denoted as $GPA[L/M]_f(z)$ or briefly as $GPA[L/M]$.

Remark 1 Compared with Definition 1, Definition 2 extends $P_L(z)$ and $Q_M(z)$ from polynomial functions to a series composed of any kind of function. If choosing $P_L(z)$ as $\sum_{i=0}^{L} \alpha_i t^i$ and $Q_M(z)$ as $1 + \sum_{i=1}^{M} \beta_i t^i$, Definition 2 reduces to Definition 1. Accordingly, Padé and quasi-Padé approximants expressed by Eqs. (6)–(9) can be considered to be special cases of the proposed method. If some traits of the function we want to approximate can be known, such as initial point, limitation, etc., we can construct a generalized Padé approximant by choosing a suitable function via Definition 2, and use it to approximate the target more efficiently.

In the next section, we will present the procedures of solving homoclinic and heteroclinic orbits of strongly nonlinear autonomous oscillators via the present method. Meanwhile, based on Definition 2, two new kinds of generalized Padé approximants will be constructed by using hyperbolic functions.

3. Homoclinic and heteroclinic orbits of strongly nonlinear autonomous oscillators

Consider the following form of nonlinear autonomous oscillator:

$$\ddot{x}+g(x)=0, \tag{14}$$

where $g(x)$ is a nonlinear function. When

$$g(x)=c_1x+c_nx^n\ (n=2,\ 3),$$

the exact solution of Eq. (14) can be expressed by elliptic function and hyperbolic function. When $g(x)$ is another high order polynomial function or rational function, solving Eq. (14) is still a very worthwhile piece of work. In this section, we utilize the generalized Padé approximation method to solve the homoclinic and heteroclinic orbits of Eq. (14) when $g(x)$ is an arbitrary polynomial function and rational function.

Assume that the solution of Eq. (14) can be expressed by the following power series:

$$x(t)=\sum_{i=0}^{\infty}a_it^i, \tag{15}$$

where a_0 and a_1 are the initial values which indicate that

$$x(0)=a_0,\ \ \dot{x}(0)=a_1. \tag{16}$$

Without loss of generality, we assume the saddle point of Eq. (14) to be $H_i(h_i,0)$ $(i=1,\ 2,\ \ldots)$. For homoclinic orbits, $a_1=0$, a_0 can be determined by

$$\int_{a_0}^{h_i}g(x)\,\mathrm{d}x=0, \tag{17}$$

where h_i is the abscissa of saddle point. For heteroclinic orbit, the initial values $a_0=0$ and a_1 which can be determined by

$$\frac{1}{2}a_1^2+k(0)=k(h_j), \tag{18}$$

where

$$k(x)=\int g(x)\,\mathrm{d}x,$$

h_j is the abscissa of any saddle point which is linked by the heteroclinic orbit. When $g(x)$ is a polynomial function and rational function, the parameters $a_i\,(i=2,3,\ldots)$ can always be obtained by substituting Eq. (15) into Eq. (14). The homoclinic solution should satisfy the following condition:

$$x(\pm\infty)=h_i. \tag{19}$$

The heteroclinic solution should satisfy

$$x(+\infty)=h_i,\ \ x(-\infty)=h_j,\ \ i\neq j. \tag{20}$$

Obviously, power series (15) does not meet the conditions. Thus, we should find another expression to replace the obtained series solution in searching for the homoclinic and heteroclinic orbits. Based on Definition 2 and the traits of homoclinic and heteroclinic orbits, we construct the following approximant by using hyperbolic functions:

(i) for homoclinic orbit

$$GPA[L/M]=\frac{\sum\limits_{i=0}^{L}\alpha_i\mathrm{sech}^it}{1+\sum\limits_{i=1}^{M}\beta_i\mathrm{sech}^it}; \tag{21}$$

(ii) for heteroclinic orbit

$$GPA[L/M]=\frac{\sum\limits_{i=0}^{L}\alpha_i\mathrm{Tanh}^it}{1+\sum\limits_{i=1}^{M}\beta_i\mathrm{Tanh}^it}, \tag{22}$$

where α_i and β_i are unknown parameters to be determined later. From Definition 2, we know that if choosing the suitable values of α_i and β_i such that

$$\sum_{i=0}^{N}a_it^i-\frac{\sum\limits_{i=0}^{L}\alpha_i\mathrm{sech}^it}{1+\sum\limits_{i=1}^{M}\beta_i\mathrm{sech}^it}=O(t^{L+M+1}) \tag{23}$$

and

$$\sum_{i=0}^{N}a_it^i-\frac{\sum\limits_{i=0}^{L}\alpha_i\tanh^it}{1+\sum\limits_{i=1}^{M}\beta_i\tanh^it}=O(t^{L+M+1}), \tag{24}$$

then equations (21) and (22) can be called the generalized Padé approximants of the power series (15). Expanding the Taylor series of the right parts of Eqs. (21) and (22) around $t=0$ and equating the same orders of the power series (15), we can determine the unknown parameters α_i and β_i by solving a set of nonlinear algebraic equations. The accuracies of approximants (21) and (22) depend on the parameters L and M, each of which is called the approximation order. One can choose the appropriate values of L and M according to the different accuracy levels required. The higher the required accuracy, the larger the values of chosen L and M will be. The advantages of the present method can be summarized as follows. 1) The present method is not limited in certain systems but can be applied widespread. 2) The Taylor series expansions of our approximants (21) and (22) are simpler than other approximants,[15–18] which means that the calculation amount of the present method is less than others' in calculating the same order approximation. 3) The accuracy of the solutions obtained in this paper is higher than that from the classic Padé approximation method especially around the saddle point.

4. Applications of the present method

4.1. Duffing equation with cubic and quintic nonlinearities

Duffing equations can be utilized in describing many kinds of nonlinear oscillatory systems in the areas of physics, mechanics, and engineering. They are well known in nonlinear dynamics and have been used to consider various types of nonlinearities. However, because of the complexity of the Duffing equation with cubic and quintic nonlinearities,[24–26] it has not been studied as extensively as the Duffing equation with only cubic nonlinearity. In this section, we use the present method to solve the homoclinic and heteroclinic orbits of the cubic–quintic Duffing equation.

Consider the following unforced, undamped cubic–quintic Duffing equation:

$$\ddot{x}+c_1x+c_3x^3+c_5x^5=0. \tag{25}$$

If choosing the suitable values of c_i, equation (25) can have five real fix points. Substituting the power series (15) into Eq. (25), the coefficients a_i can be determined as

$$\begin{aligned}
a_2 &= -\frac{1}{2}a_0[c_1+a_0^2(c_3+a_0^2c_5)],\\
a_3 &= -\frac{1}{6}a_1[c_1+a_0^2(3c_3+5a_0^2c_5)],\\
a_4 &= \frac{1}{24}(a_0(c_1^2-6a_1^2c_3)+8a_0^7c_3c_5+5a_0^9c_5^2\\
&\quad +2a_0^3(2c_1c_3-10a_1^2c_5)+3a_0^5(c_3^2+2c_1c_5)),\\
a_5 &= \frac{1}{120}a_1[c_1^2-6a_1^2c_3+27a_0^4c_3^2-60a_0^2a_1^2c_5\\
&\quad +108a_0^6c_3c_5+85a_0^8c_5^2+2a_0^2c_1(12c_3+35a_0^2c_5)],\\
&\quad \cdots,
\end{aligned} \tag{26}$$

where a_0 and a_1 are initial values which can be determined by Eqs. (17) and (18). Expanding Eqs. (21) and (22) into the Taylor series around $t=0$ and equating the same order of the power series (15), we can determine the unknown parameters α_i and β_i through solving a set of nonlinear algebraic equations. Then substituting the values of α_i and β_i into Eqs. (21) and (22), the approximate analytical solutions of Eq. (25) can be obtained. Let $c_1=2$, $c_3=-10$, and $c_5=9$, then the oscillator (25) is a symmetric system which has five fix points, i.e.,

i) $H_1(-0.9216983907,\ 0)$, center;
ii) $H_2(-0.5114520384,\ 0)$, saddle point;
iii) $H_3(0,\ 0)$, center;
iv) $H_4(0.5114520385,\ 0)$, saddle point;
v) $H_5(0.9216983907,\ 0)$, center.

Thus, there are two homoclinic orbits around H_1 and H_5, and a pair of heteroclinic orbits linked the saddle points H_2 and H_4. In calculating the homoclinic orbits, we let $L=M=4$ in Eq. (21). In calculating the heteroclinic orbits, we let $L=M=7$ in Eq. (22). Through the procedure of generalized Padé approximation method, the homoclinic around H_1 can be obtained as

$$\begin{aligned}
&GPA[4/4]\\
&=(-0.511452+0.543138\mathrm{sech}(t)-0.0398480\mathrm{sech}(t)^2\\
&\quad -0.0307003\mathrm{sech}(t)^3-0.0193938\mathrm{sech}(t)^4)\\
&\quad /(1-1.15465\mathrm{sech}(t)-0.290139\mathrm{sech}(t)^2\\
&\quad +0.663979\mathrm{sech}(t)^3-0.164709\mathrm{sech}(t)^4);
\end{aligned} \tag{27}$$

the homoclinic around H_5 can be obtained as

$$\begin{aligned}
&GPA[4/4]\\
&=(0.511452-0.690798\mathrm{sech}(t)+0.340373\mathrm{sech}(t)^2\\
&\quad -0.163161\mathrm{sech}(t)^3+0.0579433\mathrm{sech}(t)^4)\\
&\quad /(1-1.46123\mathrm{sech}(t)+0.430319\mathrm{sech}(t)^2\\
&\quad +0.103981\mathrm{sech}(t)^3-0.0208819\mathrm{sech}(t)^4);
\end{aligned} \tag{28}$$

the heteroclinic orbits which link the saddle points H_2 and H_4 can be determined as

$$\begin{aligned}
&GPA[7/7]\\
&=\pm(0.484494\tanh(t)-0.522761\tanh(t)^3\\
&\quad +0.105844\tanh(t)^5+0.00615708\tanh(t)^7)\\
&\quad /(1-1.07898\tanh(t)^2+0.201095\tanh(t)^4\\
&\quad +0.02340627\tanh(t)^6).
\end{aligned} \tag{29}$$

The phase portraits of the above solutions are shown in Fig. 1. To illustrate the accuracy of the present method, the numerical solution via the Runge–Kutta method is also shown in Fig. 1. From this figure, we can find that the solutions obtained by the present method are very close to the numerical ones. Meanwhile, the lower order approximate solutions obtained via the same approximants are shown in Fig. 2. In this figure, the homoclinic orbits are calculated when $L=M=3$. The heteroclinic orbits are calculated when $L=M=5$. From Fig. 2, we can find that the accuracy with using the lower approximation order is not sufficient.

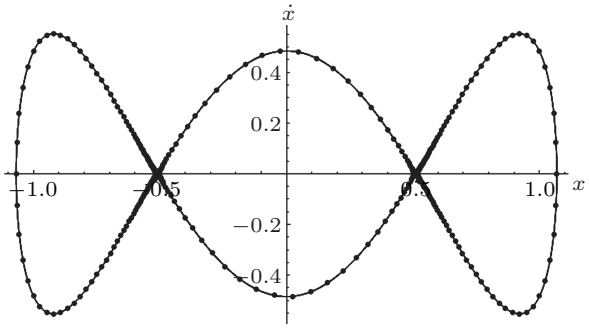


Fig. 1. Homoclinic and heteroclinic orbits of oscillator (25), where the solid line denotes the orbit obtained by the Runge–Kutta method and the dotted line refers to the orbit obtained by the present method.

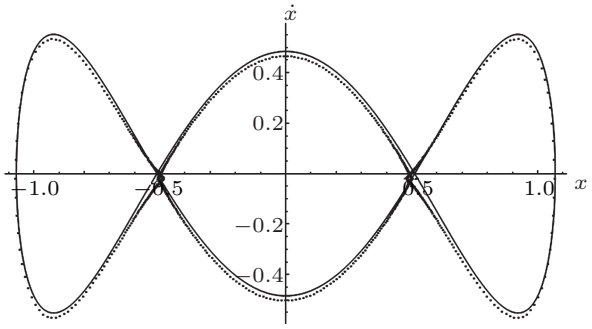


Fig.2. Homoclinic and heteroclinic orbits of oscillator (25), where the solid line denotes the orbit obtained by the Runge–Kutta method and the dotted line refers to the orbit obtained by the present method with lower approximation order.

4.2. Duffing equation with Φ^6 potential

Consider the following Duffing equation with Φ^6 potential

$$\ddot{x}+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5=0. \tag{30}$$

This equation can be considered as a conservative system for the following parametrically and externally forced Φ^6 oscillator[27] and the Φ^6-van der Pol oscillator

$$\begin{aligned}&\ddot{x}+(c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5)(1+\eta\sin 2\omega t)\\&\quad+(1-\mu\dot{x}^2)x=f\cos\omega t,\end{aligned} \tag{31}$$

$$\begin{aligned}&\ddot{x}-(\mu_0-\mu_1x^2)\dot{x}+(c_1x+c_2x^2+c_3x^3\\&\quad+c_4x^4+c_5x^5)=0.\end{aligned} \tag{32}$$

Thus, the investigations of oscillator (30) are meaningful. Substituting the power series (15) into Eq. (30), the coefficients a_i can be determined as

$$\begin{aligned}a_2 &= -\frac{1}{2}a_0\{c_1+a_0[c_2+a_0(c_3+a_0(c_4+a_0c_5))]\};\\ a_3 &= -\frac{1}{6}a_1\{c_1+a_0[2c_2+a_0(3c_3+4a_0c_4+5a_0^2c_5)]\};\\ a_4 &= \frac{1}{24}[-2a_1^2c_2+a_0(c_1^2-6a_1^2c_3)+3a_0^2(c_1c_2-4a_1^2c_4)\\&\quad+5a_0^4(c_2c_3+c_1c_4)+9a_0^8c_4c_5+5a_0^9c_5^2\\&\quad+2a_0^3(c_2^2+2c_1c_3-10a_1^2c_5)\\&\quad+3a_0^5(c_3^2+2c_2c_4+2c_1c_5)+7a_0^6(c_3c_4+c_2c_5)\\&\quad+4a_0^7(c_4^2+2c_3c_5)];\\ a_5 &= \frac{1}{120}a_1\{c_1^2-6a_1^2c_3+36a_0^3c_2c_3-24a_0a_1^2c_4\\&\quad+a_0^4(27c_3^2+58c_2c_4)+136a_0^7c_4c_5+85a_0^8c_5^2\\&\quad+10a_0^2(c_2^2-6a_1^2c_5)+a_0^5(78c_3c_4+86c_2c_5)\\&\quad+4a_0^6(13c_4^2+27c_3c_5)+2a_0c_1(5c_2+a_0(12c_3\\&\quad+22a_0c_4+35a_0^2c_5))\};\\&\quad\cdots,\end{aligned} \tag{33}$$

where a_0 and a_1 are initial values which can be determined by Eqs. (17) and (18). Through the steps presented in the previous subsection, the approximate analytical solutions of Eq. (30) can be obtained. Let $c_1=3.2639$, $c_2=0.271331$, $c_3=-4.11882$, $c_4=-0.416409$, and $c_5=1$, then equation (30) has five fix points, i.e.,

i) $H_1(-1.4835888240,\ 0)$, center;

ii) $H_2(-1.1000016335,\ 0)$, saddle point;

iii) $H_3(0,\ 0)$,center;

iv) $H_4(1.0000003835,\ 0)$, saddle point;

v) $H_5(1.9999990740,\ 0)$, center.

In calculating the homoclinic orbits, we let $L=M=6$ in Eq. (21). In calculating the heteroclinic orbits, we let $L=M=7$ in Eq. (22). Through the procedure of generalized Padé approximation method, the homoclinic around H_1 can be obtained as

$$\begin{aligned}&GPA[6/6]\\&=(-1.11321-1.57075\mathrm{sech}(t)^2+0.419233\mathrm{sech}(t)^4\\&\quad+0.0324173\mathrm{sech}(t)^6)/(1+0.966579\mathrm{sech}(t)^2\\&\quad-0.677887\mathrm{sech}(t)^4+0.0701468\mathrm{sech}(t)^6);\end{aligned} \tag{34}$$

the homoclinic around H_5 can be obtained as

$$\begin{aligned}&GPA[6/6]\\&=(0.983641-1.94166\mathrm{sech}(t)^2+1.22177\mathrm{sech}(t)^4\\&\quad-0.239343\mathrm{sech}(t)^6)/(1-2.28326\mathrm{sech}(t)^2\\&\quad+1.72161\mathrm{sech}(t)^4-0.427928\mathrm{sech}(t)^6);\end{aligned} \tag{35}$$

the heteroclinic orbits linking the saddle points H_2 and H_4 can be determined as

$$\begin{aligned}&GPA[7/7]\\&=(1.24585\tanh(t)+0.802112\tanh(t)^2-0.718015\tanh(t)^3\\&\quad-0.252246\tanh(t)^4-0.102913\tanh(t)^5\\&\quad-0.138857\tanh(t)^6+0.0199502\tanh(t)^7)\\&\quad/(1+0.643826\tanh(t)-0.365674\tanh(t)^2\\&\quad-0.0386767\tanh(t)^3-0.205941\tanh(t)^4\\&\quad-0.175999\tanh(t)^5+0.0133468\tanh(t)^6\\&\quad-0.0179511\tanh(t)^7);\end{aligned} \tag{36}$$

$$\begin{aligned}&GPA[7/7]\\&=(1.24585\tanh(t)+0.802112\tanh(t)^2+0.718015\tanh(t)^3\\&\quad-0.252246\tanh(t)^4+0.102913\tanh(t)^5\\&\quad-0.138857\tanh(t)^6-0.0199502\tanh(t)^7)\\&\quad/(1-0.643826\tanh(t)-0.365674\tanh(t)^2\\&\quad+0.0386767\tanh(t)^3-0.205941\tanh(t)^4\\&\quad+0.175999\tanh(t)^5+0.0133468\tanh(t)^6\\&\quad+0.0179511\tanh(t)^7).\end{aligned} \tag{37}$$

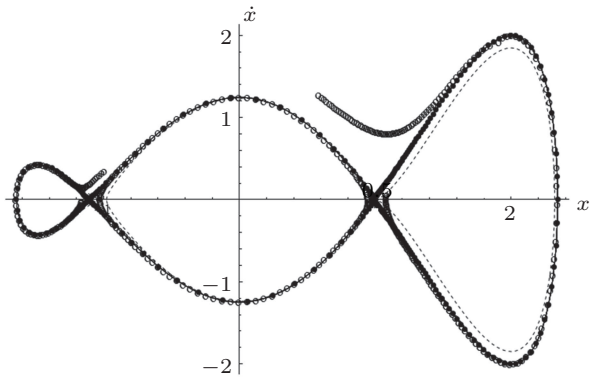


Fig. 3. Homoclinic and heteroclinic orbits of oscillator (30), where the solid line denotes the orbit obtained by the Runge–Kutta method and the dotted line refers to the orbit obtained by the present method, the dashed line represents the orbit obtained by Feng *et al*.,[27] and the empty circle line is the orbit obtained via approximant (7).

The phase portraits of the above solutions are shown in Fig. 3. To show the accuracy of the present method, the numerical solutions via the Runge–Kutta method and the solutions obtained by Feng *et al*.[27] via classical Padé approximant are shown in Fig. 3. Meanwhile, the solutions of this equation via approximant (7) with the same approximation orders are also shown in Fig. 3. From this figure, we can find that the solutions obtained by the present method are very close to the numerical ones, and the accuracies of our solutions are higher than those of the solutions obtained via approximant (7) and the solutions given by Feng *et al.*

4.3. Generalized Duffing-harmonic oscillator

Consider the following generalized Duffing-harmonic oscillator:

$$\ddot{x}+\frac{\lambda x+\mu x^3}{1+\nu x^2}=0. \tag{38}$$

To make comparisons of accuracy and efficiency among approximate methods, some popular nonlinear models are taken as benchmark examples. One of them is the Duffing-harmonic oscillator. In Ref. [28] the periodic solutions of Eq. (38) are investigated by using nonlinear time transformation method. When $\lambda=0$, equation (38) is solved by a modified harmonic balance method.[29] Hu[30] applied a modified iteration procedure to Eq. (38). By combining Newton's method with the harmonic balance method, Lim[31] presented a new method of obtaining the analytical solutions of Duffing-harmonic oscillator. Öziş and Yıldırım used the energy balance method presented by He to determine the frequency–amplitude relation of the Duffing-harmonic oscillator.[32] Beléndez *et al*.[33] reinvestigated the explicit approximate solution to the Duffing-harmonic oscillator by a cubication method. However, the analytical homoclinic and heteroclinic solutions of Eq. (38) are seldom studied. To show the widespread application of our method, we will solve Eq. (38) based on the generalized Padé approximation method. Substituting power series (15) into Eq. (38), the coefficients a_i can be determined as

$$\begin{aligned}
a_2 &= \frac{-\lambda a_0-\mu a_0^3}{2\left(1+\nu a_0^2\right)},\\
a_3 &= \frac{-\lambda a_1-3\mu a_0^2a_1+\lambda\nu a_0^2a_1-\mu\nu a_0^4a_1}{6\left(1+\nu a_0^2\right)^2},\\
a_4 &= \frac{a_0\left\{\lambda^2+3\mu^2a_0^4+\mu^2\nu a_0^6-6\left(\mu-\lambda\nu\right)a_1^2+a_0^2\left(\lambda\left(4\mu-\lambda\nu\right)+2\nu\left[\mu-\lambda\nu\right]a_1^2\right)\right\}}{24\left(1+\nu a_0^2\right)^3},\\
a_5 &= \{a_1[\lambda^2+4\lambda\mu\nu^2a_0^6+\mu^2\nu^2a_0^8-6(\mu-\lambda\nu)a_1^2-4a_0^2[\lambda(-6\mu+5\lambda\nu)+9\nu(-\mu+\lambda\nu)a_1^2]\\
&\quad+a_0^4(27\mu^2-28\lambda\mu\nu+7\lambda^2\nu^2+6\nu^2(-\mu+\lambda\nu)a_1^2)]\}/120(1+\nu a_0^2)^4,\\
&\quad\cdots.
\end{aligned} \tag{39}$$

When $\lambda=-1$, $\mu=5$, and $\nu=5$, equation (38) has two homoclinic orbits. Letting $N=M=3$ in Eq. (21) and through the procedure of generalized Padé approximation method, the homoclinic orbits of Eq. (38) can be obtained as

$$GPA[3/3]=\pm\frac{1.26143\mathrm{sech}(t)+6.18703\mathrm{sech}(t)^2+17.0891\mathrm{sech}(t)^3}{1+4.91221\mathrm{sech}(t)+14.368\mathrm{sech}(t)^2+14.3322\mathrm{sech}(t)^3}. \tag{40}$$

When $\lambda=3$, $\mu=-5$, and $\nu=2$, equation (38) has a pair of heteroclinic orbits. Letting $N=M=8$ in Eq. (22) and through the procedure of generalized Padé approximation method, the heteroclinic orbits of Eq. (38) can be obtained as

$$\begin{aligned}
&GPA[8/8]\\
&=\pm(0.81747\tanh(t)+1.15089\tanh(t)^3\\
&\quad-0.264601\tanh(t)^5-0.356103\tanh(t)^7)\\
&\quad/(1+1.57453\tanh(t)^2-0.203802\tanh(t)^4
\end{aligned}$$

$$-0.579723\tanh(t)^6 - 0.0256511\tanh(t)^8). \quad (41)$$

The phase portraits of the above solutions are shown in Figs. 4 and 5. To show the accuracy of the present method, the numerical solutions via the Runge–Kutta method are shown in Figs. 4 and 5. Meanwhile, the solutions of this equation via approximants (6) and (7) with the same approximation orders are also shown in Figs. 4 and 5. From the comparisons, we can find that the solutions obtained by the present method are very close to the numerical ones, and the accuracies of our solutions are higher than those of the solutions obtained via approximants (6) and (7).

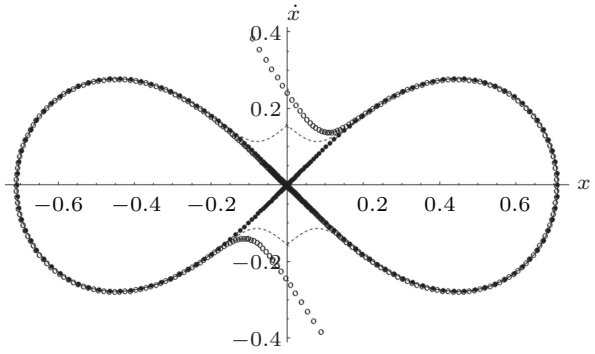


Fig. 4. Homoclinic orbits of oscillator (38) when $\lambda = -1$, $\mu = 5$, and $\nu = 5$, where the solid line denotes the orbit obtained by the Runge–Kutta method, the dotted line represents the orbit obtained by the present method, the dashed line refers to the orbit obtained via approximant (6); the empty circles indicate the orbit obtained via approximant (7).

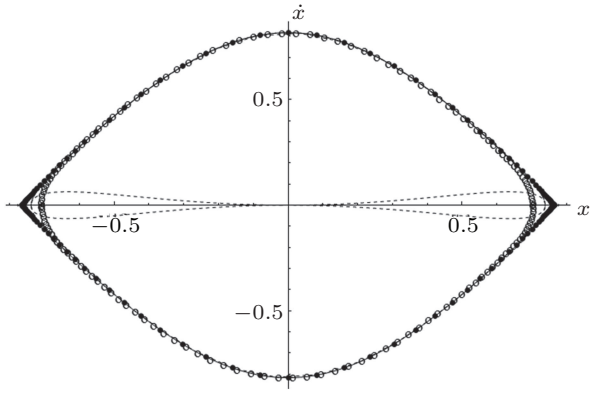


Fig. 5. Heteroclinic orbits of oscillator (38) when $\lambda = 3$, $\mu = -5$, and $\nu = 2$, where the solid line denotes the orbit obtained by the Runge–Kutta method, the dotted line represents the orbit obtained by the present method, the dashed line refers to the orbit obtained via approximant (6), the empty circles indicate the orbit obtained via approximant (7).

5. Conclusions

As a kind of rational approximation method, Padé approximation has been used widespread in many areas. In this paper, we extend the traditional definition of Padé approximation method and propose a generalized Padé approximation method. According to the proposed method, any kind of function, if necessary, can be introduced in constructing generalized Padé approximant. Many existing modifications of Padé approximation method therefore can be considered to be the special cases of the proposed method. From the examples in Section 4, we can find that the proposed method is not limited in some systems but can be applied widespread, many other oscillators can also be solved by this method. Meanwhile, the solutions obtained by the present method are very close to the numerical ones, and their accuracies are higher than those of the solutions obtained by classical Padé approximation method. Accordingly, the proposed method can be considered to be a feasible and high-accuracy method which provides a new ideal tool in the use of Padé approximation.

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