Generating genuine multipartite entanglement via XY-interaction and via projective measurements

Mazhar Ali[a)b)†]

[a)] Naturwissenschaftlich-Technische Fakultät, Universität Siegen, Walter-Flex-Straße 3, 57068 Siegen, Germany
[b)] Department of Electrical Engineering, COMSATS Institute of Information Technology, 22060 Abbottabad, Pakistan

(Received 4 December 2013; revised manuscript received 11 August 2014; published online 10 October 2014)

We have studied the generation of multipartite entangled states for the superconducting phase qubits. The experiments performed in this direction have the capacity to generate several specific multipartite entangled states for three and four qubits. Our studies are also important as we have used a computable measure of genuine multipartite entanglement whereas all previous studies analyzed certain probability amplitudes. As a comparison, we have reviewed the generation of multipartite entangled states via von Neumann projective measurements.

Keywords: genuine multipartite entanglement, superconducting qubits, von Neumann measurements

PACS: 03.67.−a, 03.67.Bg **DOI:** 10.1088/1674-1056/23/12/120307

1. Introduction

Quantum entanglement has been recognized as a resource with applications in the emerging field of quantum information and quantum computation.[1,2] The creation and measurement of entangled states is crucial for the various physical implementations of quantum computers.[3,4] Therefore it is of much interest and important to generate entanglement in experiments and also characterize it in theory. The description of quantum entanglement for bipartite quantum systems is relatively simple as any given quantum state is either entangled or separable. However for quantum systems with more than two subsystems, this problem becomes richer and also more difficult. For the simplest multipartite quantum system of three-qubit system, it was shown that three qubits can be entangled in two fundamentally different ways under stochastic local operations and classical communication (SLOCC).[5] In contrast, such inequivalent SLOCC-classes of entangled states for four qubits are already infinite.[6,7]

One of the basic tasks in a quantum computer is the implementation of a set of universal gates, usually a two-qubit gate such as controlled-NOT (CNOT) gate, and single qubit rotations.[8] However, it is also possible that a three-qubit gate like Toffoli gate achieves universality.[9–11] Therefore it is necessary to design experiments with a direct implementation of multi-qubit gates. Recently, such implementations have been successfully performed on the superconducting phase qubits.[12–15] In these experiments, the phase qubits were coupled by connecting them with a capacitor to generate multi-qubit interactions leading to multi-qubit gates rather than designing them from more elementary two-qubit gates. Moreover, it was shown that one can also use two-qubit gates to create $|GHZ\rangle$ states and a more efficient entangling protocol based on a single three-qubit gate to create $|W\rangle$ state.[13] We have extended these studies for three and four qubits and have shown that this experimental setup can offer some unique possibilities which have not been explored till now. We have also proposed an experimental architecture for generating four-qubit $|\chi_4\rangle$ state which might be implemented with a small modification in the original experiment.

Another approach to create SLOCC-inequivalent multipartite entangled states is to apply von Neumann projective measurements on some of the subsystems.[16–19] This way of creating entangled states is due to the property of certain symmetric multipartite entangled states that allow a more flexible preparation of families of SLOCC-inequivalent entangled states by projective measurements on small subsystems. We have reviewed this method to compare the results. We have shown that it is possible to create both GHZ-type and W-type states of three qubits by applying projective measurements on a single qubit of four-qubit states. We have shown that in creating $|\chi_4\rangle$ state, we have used GHZ-type entanglement and W-type entanglement. Although these three states are inequivalent, nevertheless, in the reverse process of projective measurements, we can extract GHZ-type and W-type entanglements from $|\chi_4\rangle$ state. This process of creating inequivalent entangled states has increased our understanding about multipartite entangled states.

This paper is organized as follows. In Section 2, we review the concept of genuine multipartite entanglement and describe the computable measure which detects such type of entanglement. In Section 3, we study the possibility of generating various multipartite entangled states. We re-examine the possibility of generating genuine multipartite entanglement via von Neumann measurements in Section 4. Finally we offer

†Corresponding author. E-mail: mazharaliawan@yahoo.com

© 2014 Chinese Physical Society and IOP Publishing Ltd

some conclusions in Section 5.

2. Genuine multipartite entanglement

In this section we briefly review the concept of genuine entanglement. We consider three qubits as an example to explain the concept. A state is separable with respect to some bipartition, say, $A|BC$, if it can be written as

$$\rho = \sum_k q_k |\phi_A^k\rangle\langle\phi_A^k| \otimes |\psi_{BC}^k\rangle\langle\phi_{BC}^k|, \tag{1}$$

where q_k form a probability distribution. We denote these states as $\rho_{A|BC}^{\text{sep}}$. The two other possibilities are $\rho_{B|AC}^{\text{sep}}$ and $\rho_{C|AB}^{\text{sep}}$. Then a state is called biseparable if it can be written as a mixture of states which are separable with respect to different bipartitions, that is

$$\rho^{\text{bs}} = p_1\rho_{A|BC}^{\text{sep}} + p_2\rho_{B|AC}^{\text{sep}} + p_3\rho_{C|AB}^{\text{sep}}. \tag{2}$$

Any state which is not biseparable is called genuinely entangled. The description of genuine entanglement is quite challenging. Considerable efforts have been devoted to its characterization, quantification, detection, and preparation.[20–28]

Biseparable states can be created by entangling any two of three particles and then one can create a statistical mixture by forgetting to which pairs this operation was applied. To detect genuine entanglement, it is not enough to apply bipartite criterion to every partition, instead one has to show that it cannot be written in form of Eq. (2). As there is no efficient way to search through all possible decompositions, we can consider a superset of the set of separable states which can be characterized more easily than the set of separable states. As a superset of states that are separable with respect to, say, partition $A|BC$, we select the set of states that have a positive partial transpose PPT with respect to partition $A|BC$. A state $\rho = \sum_{ijkl}\rho_{ij,kl}|i\rangle\langle j| \otimes |k\rangle\langle l|$ is PPT if its partially transposed matrix $\rho^{\text{T}_A} = \sum_{ijkl}\rho_{ji,kl}|i\rangle\langle j| \otimes |k\rangle\langle l|$ has no negative eigenvalues. The convex set $\rho_{A|BC}^{\text{sep}}$ is contained in a larger convex set of states which has a positive partial transpose. The benefit of doing so is the easy characterization of PPT set. A well-known fact is that separable states are always PPT.[29–31] We denote the states which are PPT with respect to fixed bipartition by $\rho_{A|BC}^{\text{ppt}}$, $\rho_{B|AC}^{\text{ppt}}$, and $\rho_{C|AB}^{\text{ppt}}$, and ask the question whether a state can be written as

$$\rho^{\text{pptmix}} = p_1\rho_{A|BC}^{\text{ppt}} + p_2\rho_{B|AC}^{\text{ppt}} + p_3\rho_{C|AB}^{\text{ppt}}. \tag{3}$$

Such mixing of PPT states is called PPT-mixture and it can be characterized more easily than biseparable states. This method allows use of semidefinite programming (SDP) and also gives an entanglement monotone. For bipartite systems, this monotone is equivalent to an entanglement measure called negativity.[32] For multipartite systems, this monotone may be called genuine negativity. For multiqubits the value of genuine negativity can be at most $1/2$.[33]

3. Generating entanglement via *XY*-interactions among phase qubits

There are several physical systems such as spins, atoms, or photons, that might be used as quantum information processing devices. Another promising candidate has emerged in recent years as superconducting qubit. Instead of relying on fundamental quantum systems, these devices are engineered circuits that consist of many constituent atoms exhibiting collective quantum behavior. The two key features are superconductivity, which is a collective quantum behavior of many electrons that allows the entire circuit to be treated quantum mechanically, and the Josephson effect, which gives the strong nonlinearity required to make an effective two-level system or qubit.[34] As we study phase qubits in this paper and the circuit diagram for phase qubit is shown in Fig. 2.1 of Ref. [34], we have realized that our studies to generate entanglement are mostly based on mathematical model underlying and a computable measure of genuine entanglement therefore we do not need any detailed descriptions of phase qubits itself. Instead of writing these details here, we refer the readers to see Ref. [34] and references therein.

3.1. Generating entanglement for three qubits

Let us first consider the three qubits case. An arbitrary initial pure state for three qubits can be written as $|\psi(0)\rangle = \sum_{ijk=0}^{1} C_{ijk}(0)|ijk\rangle$, where we have expressed the state in the computational basis $\{|000\rangle, |001\rangle, \ldots, |111\rangle\}$ and $C_{ijk}(0)$ are the initial probability amplitudes. To create $|GHZ_3\rangle = (1/\sqrt{2})(|000\rangle + |111\rangle)$ state, one can design the quantum circuit diagram[13] shown in Fig. 1. This figure utilizes a more natural universal gate called iSWAP gate.[31] The iSWAP gate can be generated by applying the interaction Hamiltonian for the superconducting phase qubits written as[13,35]

$$H_{\text{int}}^{ij} = \frac{\hbar g}{2}\left(\sigma_x^i\sigma_x^j + \sigma_y^i\sigma_y^j\right), \tag{4}$$

for interaction time $t_{\text{iSWAP}} = \pi/(2g)$, where g is the coupling strength and σ_x, σ_y are the Pauli operators on qubits i and j. Alternatively, this coupling Hamiltonian can be written as

$$H_{\text{int}}^{ij} = \hbar g\left(\sigma_+^i\sigma_-^j + \sigma_-^i\sigma_+^j\right), \tag{5}$$

where σ_+ and σ_- are the operators which create and destroy an excitation in each qubit respectively. This form reflects the fact that the interaction leads to excitation swapping back and forth between the two coupled qubits. Under the specified interaction time, qubit states are transformed as $|01\rangle \mapsto -\text{i}|10\rangle$ and $|10\rangle \mapsto -\text{i}|01\rangle$ whereas $|00\rangle$ and $|11\rangle$ invariant. It is simple to figure out that the circuit shown in Fig. 1 produces $|GHZ_3\rangle$ state provided all three initial qubits are in their ground state $|0\rangle$.[13] This means that if we start with all qubits in the ground state, apply rotation $Y(\pi/2)$ achieved by shining a laser on all three qubits, after that we first turn on the interaction for time

$t_{\rm iSWAP}$ between qubits 1 and 2 and then we turn on the interaction between qubits 2 and 3 again for time $t_{\rm iSWAP}$, and finally we apply the rotation $X(-\pi/2)$ again by shining laser, then we get the GHZ state. We note that $|GHZ_3\rangle$ is the maximally entangled state measured by genuine negativity, that is, $E(|GHZ_3\rangle) = 1/2$.

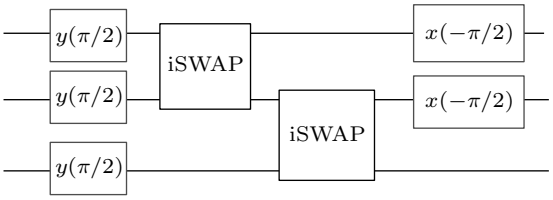


Fig. 1. The circuit diagram for generating $|GHZ_3\rangle$ state using two qubit iSWAP gate, which can be generated directly by capacitive coupling in phase qubits. Single qubit rotations are applied before and after the action of the gate.

Another inequivalent entangled state which has been created in this setup is the $|W_3\rangle = (1/\sqrt{3})(|001\rangle + |010\rangle + |100\rangle)$ state,[13] which does not have the maximum amount of entanglement as measured by genuine negativity, that is $E(|W_3\rangle) \approx 0.4428$. The corresponding circuit diagram for generating $|W_3\rangle$ state is shown in Fig. 2. It can be seen that first the middle qubit is excited by applying a π-pulse to qubit B to excite it with a single photon. The effect is the creation of state $|010\rangle$. After that the interaction among all qubits can be turned on by applying the interaction Hamiltonian

$$H_{\rm int} = H_{\rm int}^{AB} + H_{\rm int}^{AC} + H_{\rm int}^{BC}, \tag{6}$$

for the interaction time $t_W = (4/9)t_{\rm iSWAP}$,[13] which corresponds to $gt_W = (2\pi)/9 \approx 0.7$. This action leaves all qubits in an equal superposition of single excitation. Finally a Pauli Z matrix is applied to correct the phase of qubit B. The Pauli rotations X and Z on the second qubit can be manipulated by focusing a laser before and after turning on interaction for time t_w. The tomographic data[13] yields a very efficient state $|W_3\rangle$ as a result. In the following, we reconfirm this result using computable entanglement monotone E. In addition, we further investigate this setup and suggest that it can offer some more possibilities to create multipartite entangled states.

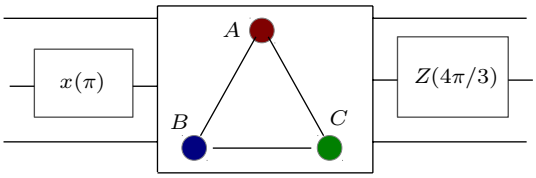


Fig. 2. (color online) The circuit diagram for generating $|W_3\rangle$ state of three qubits using a single entangling step with simultaneous coupling between all qubits. Single qubit rotations are applied on the middle qubit before and after such interaction.

This Hamiltonian (6) leads to two independent sets of coupled equations for probability amplitudes, that is, $\{C_{001}(t), C_{010}(t), C_{100}(t)\}$ and $\{C_{110}(t), C_{101}(t), C_{011}(t)\}$, with $C_{000}(0)$ and $C_{111}(0)$ as invariants. The key observation is that if one has any one of nonzero amplitudes in any set then one can generate either $|W_3\rangle$ state or $|\widetilde{W}_3\rangle = (1/\sqrt{3})(|110\rangle + |101\rangle + |011\rangle)$ state. In Fig. 3 we plot $E(|\psi(t)\rangle)$ against parameter gt with initial condition $C_{001}(0) = 1$ and all other $C_{ijk}(0) = 0$. We observe the creation of $|W_3\rangle$ as predicted[13] at $gt_W \approx 0.7$ and again at $gt_W \approx 1.4$. However, to correct the phase of $|W_3\rangle$ state in the second peak, we need to apply $Z(8\pi/3)$ rotation.

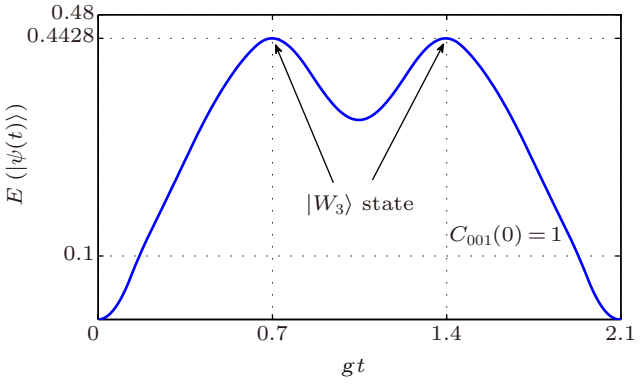


Fig. 3. (color online) Generalized negativity $E(|\psi(t)\rangle)$ is plotted against the parameter gt for initial condition $C_{001}(0) = 1$. It can be seen that $|W_3\rangle$ state is created at $gt \approx 0.7$ and $gt \approx 1.4$.

We investigate the possibility to create the $|G_3\rangle$ states[36] which are defined as

$$|G_N^{\pm}\rangle = (|W_N\rangle \pm |\widetilde{W}_N\rangle)/\sqrt{2}. \tag{7}$$

One interesting feature of $|G_3\rangle$ states is the possibility of transforming them directly into $|GHZ_3\rangle$ state via local filters given by[18]

$$\begin{aligned} f_+ &= \mathcal{H}\left\{\frac{1}{2}\left[\left(\frac{1}{\sqrt{3}}+\mathrm{i}\right)I + \left(\frac{1}{\sqrt{3}}-\mathrm{i}\right)\sigma_z\right]\right\}\mathcal{H}, \\ f_- &= \mathcal{H}\left\{\frac{1}{2}\left[\left(\frac{1}{\sqrt{3}}+\mathrm{i}\right)\sigma_x + \mathrm{i}\left(\frac{1}{\sqrt{3}}-\mathrm{i}\right)\sigma_y\right]\right\}\mathcal{H}, \end{aligned} \tag{8}$$

where σ_i are the Pauli matrices and $\mathcal{H}$ is the Hadamard transformation. As a result, we get

$$\begin{aligned} (f_+ \otimes f_+ \otimes f_+)|G_3^+\rangle &= \frac{1}{3}|GHZ_3\rangle, \\ (f_- \otimes f_- \otimes f_-)|G_3^-\rangle &= \frac{1}{3}|GHZ_3\rangle, \end{aligned} \tag{9}$$

with probability $1/9$. Genuine negativity for $|G_3\rangle$ is $E(|G_3\rangle) \approx 0.3448$. One possibility to create $|G_3\rangle$ seems straight forward from two independent sets of probability amplitudes. By having excitations in each one of this set, the experiment performed for $|W_3\rangle$ state[13] may lead to $|G_3\rangle$ state at $gt \approx 0.7$. However, this is not what we actually observe with initial conditions $C_{001}(0) = C_{011}(0) = 1/\sqrt{2}$. Although at $gt \approx 0.7$ both $|W_3\rangle$ and $|\widetilde{W}_3\rangle$ are formed, nevertheless their respective phases also mix which we were able to correct only in case of $|W_3\rangle$

state creation. Hence the state generated at $gt \approx 0.7$ is

$$|\psi(gt \approx 0.7)\rangle = \frac{1}{\sqrt{2}}(e^{i\phi_1(\phi_2)}|W_3\rangle + e^{i\phi_2(\phi_1)}|\widetilde{W}_3\rangle). \quad (10)$$

Due to this phase dependence, the perfect $|G_3\rangle$ cannot be generated in this method. Another method is to create $|W_3\rangle$ and $|\widetilde{W}_3\rangle$ separately as shown above and then prepare the superposition.

3.2. Generating entanglement for four qubits

Four qubits setup is an extension of three qubits setup which offers some unique possibilities to generate important multipartite entangled states. The interaction Hamiltonian for four-body interactions can be written as

$$H_{\rm int} = H_{\rm int}^{AB} + H_{\rm int}^{AC} + H_{\rm int}^{AD} + H_{\rm int}^{BC} + H_{\rm int}^{BD} + H_{\rm int}^{CD}. \quad (11)$$

As a result of this Hamiltonian the following three sets of coupled differential equations for probability amplitudes emerge, that is, $\{C_{0001}(t),\ C_{0010}(t),\ C_{0100}(t),\ C_{1000}(t)\}$, $\{C_{1110}(t), C_{1101}(t), C_{1011}(t), C_{0111}(t)\}$, and $\{C_{0011}(t), C_{0101}(t), C_{0110}(t), C_{1001}(t),\ C_{1010}(t),\ C_{1100}(t)\}$, whereas $C_{0000}(0)$ and $C_{1111}(0)$ are invariant. We recognize that the first two sets provide the possibility of generating $|W_4\rangle$ and $|\widetilde{W}_4\rangle$ states, respectively, whereas the third set can be utilized to create the singlet state of four qubits as we demonstrate below.

We start with $|GHZ_4\rangle = (1/\sqrt{2})(|0000\rangle + |1111\rangle)$ state with genuine negativity $E(|GHZ_4\rangle) = 1/2$. The circuit diagram in this case is a simple extension of Fig. 1 leading to the desired state. Therefore, we do not repeat the diagram and arguments for the preparation of GHZ state here.

Next we consider the $|W\rangle$ state of four qubits given as $|W_4\rangle = (1/2)(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$, which is not a maximally entangled state as measured by genuine negativity, that is $E(|W_4\rangle) \approx 0.366$. The circuit diagram for the $|W_4\rangle$ is a generalization of Fig. 2. We observed that with initial condition $C_{0001}(0) = 1$, $|W_4\rangle$ state is created at $gt_W = \pi/4$. To correct the phase, we need to apply $Z(\pi)$ rotation, by turning on a laser.

Four qubits $|G_4\rangle = (1/\sqrt{2})(|W_4\rangle + |\widetilde{W}_4\rangle)$ state is interestingly a maximally entangled state as measured by genuine negativity, that is, $E(|G_4\rangle) = 1/2$.

$$|G_4\rangle = \frac{1}{2\sqrt{2}}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle + |0111\rangle + |1011\rangle + |1101\rangle + |1110\rangle). \quad (12)$$

To generate this state, we start with initial condition $C_{0001}(0) = C_{0111}(0) = 1/\sqrt{2}$ and expect $|G_4\rangle$ state to be created at $gt = \pi/4$. However, after correcting the phase, the state we obtain is

$$|\psi_G(gt = \pi/4)\rangle = \frac{1}{2\sqrt{2}}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle + |0111\rangle - |1011\rangle - |1101\rangle + |1110\rangle). \quad (13)$$

This state is also maximally entangled as measurement by genuine negativity, that is, $E(|\psi_G\rangle) = 1/2$.

The question of maximally entangled states for N qubits depends on the entanglement measure being chosen.[37] In fact there are several inequivalent measures of multipartite entanglement, and maximally entangled states differ for different measures. For a class of entanglement measures, based on anti-linear operators and combs,[38,39] there are three different measures of this type for four qubits, resulting in three different maximally entangled states.[37] The first one is the $|GHZ_4\rangle$ state, the second one is the cluster state, and the third one is the state

$$|\chi_4\rangle = \frac{\sqrt{2}|1111\rangle + |0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle}{\sqrt{6}}. \quad (14)$$

Surprisingly, it turns out that all these states are also maximally entangled measured by genuine negativity, that is $E(|\chi_4\rangle) = 1/2$. It has been observed that $|\chi_4\rangle$ state is the symmetric four-qubit state that maximizes certain bipartite entanglement properties.[40] In addition, it was shown[39] that one can generalize this state to five and six qubits, where it is also a maximally entangled state for some comb measure. The generation of this state is still an experimental challenge.[37] We propose a simple modification of current experimental setup which seems quite feasible and would lead to creation of $|\chi_4\rangle$ state. To this end we realize that the state $|\chi_4\rangle$ has a large overlap with $|W_4\rangle$ state with additional component $|1111\rangle$. It is easy to see that $|\chi_4\rangle$ can be written as

$$|\chi_4\rangle = \alpha|1111\rangle + \sqrt{1-\alpha^2}|W_4\rangle, \quad (15)$$

with $\alpha = 1/\sqrt{3}$. It is interesting to note that $|W_4\rangle$ is not maximally entangled, however its entanglement can be maximized by mixing the component $|1111\rangle$. One can also try to maximize entanglement of $|W_3\rangle$ by mixing the component $|111\rangle$. It turns out that one can do that and the resulting state $|\chi_3\rangle = (|001\rangle + |010\rangle + |100\rangle - |111\rangle)/2$ is indeed a maximally entangled state, that is $E(|\chi_3\rangle) = 1/2$, however $|\chi_3\rangle$ is equivalent to $|GHZ_3\rangle$ state,[41] whereas $|\chi_4\rangle$ is inequivalent to $|GHZ_4\rangle$. We propose the circuit diagram for the experiment to generate $|\chi_4\rangle$ in Fig. 4.

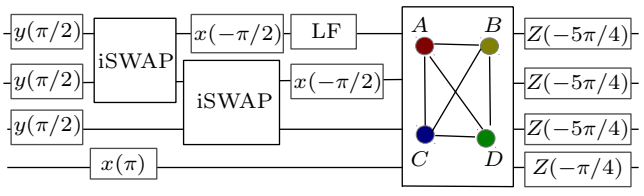


Fig. 4. (color online) Circuit diagram is shown for generating $|\chi_4\rangle$ state. $|GHZ_3\rangle$ state is created in the first three qubits before applying a local filter and four body interactions with fourth qubit in excited state. Phases are corrected by $Z(\gamma)$ rotations at the end.

The idea is to first generate $|GHZ_3\rangle$ state in the first three qubits using the method described in Fig. 1. Once $|GHZ_3\rangle$ state is created then one can apply local filter (LF) on the first

qubit to transform $|GHZ_3\rangle$ state into non-maximally entangled state

$$|\psi_{ABC}\rangle = \sqrt{2/3}|000\rangle + \sqrt{1/3}|111\rangle. \quad (16)$$

The precise form of the filter f which does this job, is given as $f = \mathrm{diag}\{a, 1/a\}$ with $a = (2)^{1/4}$. After that an excitation is created on the fourth qubit and then finally this input state is allowed to interact as four-body interaction for the same amount of time that is required to create $|W_4\rangle$ state, that is, $gt_W = \pi/4$. It is not difficult to see that this procedure is equivalent to produce the initial conditions on probability amplitudes such that $C_{0001}(0) = \sqrt{2/3}$ and $C_{1111}(0) = 1/\sqrt{3}$. In Fig. 5, we have plotted $E(|\psi(t)\rangle)$ against parameter gt with these initial conditions. It can be seen that $|\chi_4\rangle$ state is created as genuine negativity achieves its maximum value at $gt = \pi/4 \approx 0.785$.

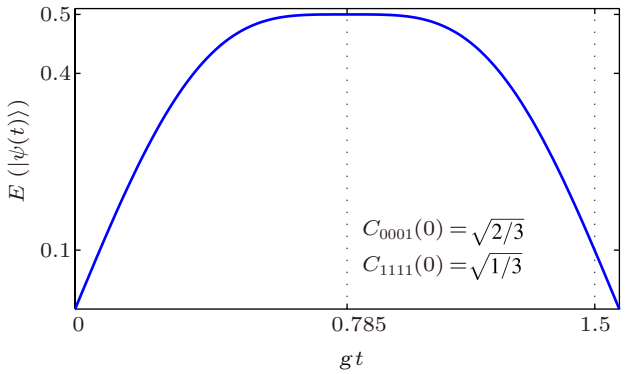


Fig. 5. (color online) $E(|\psi(t)\rangle)$ is plotted against parameter gt with initial conditions $C_{0001}(0) = \sqrt{2/3}$ and $C_{1111}(0) = \sqrt{1/3}$.

To correct phases, we first need to apply $Z(\pi)$ rotation on the fourth qubit as we did in $|W_4\rangle$ case. As $|1111\rangle$ component is in the superposition, so this operation introduces some additional phases and to correct them we need to apply $Z(-5\pi/4)$ rotations on all four qubits. The resultant rotation on the fourth qubit is $Z(\pi - 5\pi/4) = Z(-\pi/4)$.

Our final example is the possibility of creating the multiqubit singlet state. These are pure states $|\Psi\rangle$ which are invariant under simultaneous unitaries on all qubits. Such states only exist for an even number of qubits. For two qubits, singlet state is given as $|\psi^-\rangle = (1/\sqrt{2})(|01\rangle - |10\rangle)$. For four qubits, singlet state[37,42] is given as

$$|\Psi_{S,4}\rangle = \frac{1}{\sqrt{3}}\big[|0011\rangle + |1100\rangle - 1/2(|0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle)\big], \quad (17)$$

with $E(|\Psi_{S,4}\rangle) = 1/2$, which means that this state is also a maximally entangled state according to genuine negativity. The possibility of creating this state appears natural as the third set contains all relevant probability amplitudes for the singlet state. One could work out the corresponding circuit diagram for this case but the essential point is the fact that we need at least one non-zero probability amplitude only in the third set. It follows that the resulting dynamics leads to the singlet state at specific times. Figure 6 shows the evolution of $E(|\psi(t)\rangle)$ against parameter gt with initial condition $C_{0011}(0) = 1$. It can be seen that $|\Psi_{S,4}\rangle$ is created at $gt \approx 0.6$ and at $gt \approx 2.54$. We emphasize at this point that although we have plotted the numerical value of $E(|\psi(t)\rangle)$ for all the cases studied in this paper, the detailed analysis of probability amplitudes confirms that the corresponding states are created at specific instances of time with specific initial conditions.

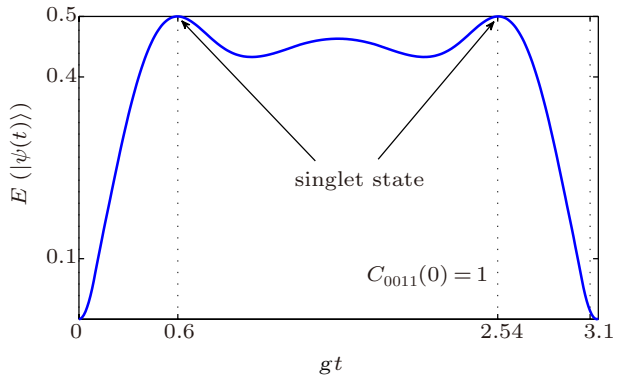


Fig. 6. (color online) $E(|\psi(t)\rangle)$ is plotted against parameter gt with initial conditions $C_{0011}(0) = 1$. The singlet state is generated at $gt \approx 0.6$ and at $gt \approx 2.54$.

4. Generating multipartite entanglement via von Neumann projections

In this section, we review the idea of generating multipartite entanglement via applying von Neumann projective measurements on some of the n-qubit state to get the m-qubit state with $n > m$. We stress here at this point that this idea is not new and extensive work has been done in linear optical quantum computation.[17–19] Our main purpose here in this paper is to compare this method with the previous section in order to show the interesting correspondence between inequivalent genuine multipartite entangled states. Let ρ_n be an initial n-qubit density matrix and ρ_m be the density matrix of m qubits after applying projective measurements on the $n-m$ qubits. Mathematically, one can write this operation as

$$_m = \frac{1}{\mathcal{N}}\mathrm{Tr}_{ij}\big[(I\otimes\cdots M_i\otimes\cdots M_j\otimes\cdots I) \times \rho_n(I\otimes\cdots M_i^\dagger\otimes\cdots M_j^\dagger\otimes\cdots I)\big], \quad (18)$$

where $\mathcal{N}$ is the normalization factor, Tr_{ij} is the partial trace over qubits being measured, and $M_i = V\Pi_i V^\dagger$ is the von Neumann projection operator on ith qubit, where $\Pi_i = |i\rangle\langle i|$ is the projector and $V = tI + \mathrm{i}\boldsymbol{y}\cdot\boldsymbol{\sigma}$ is the unitary matrix, that is, $V \in \mathrm{SU}(2)$, such that $t, y_i \in \mathcal{R}$ and $t^2 + y_1^2 + y_2^2 + y_3^2 = 1$ and $\boldsymbol{\sigma}$ is a vector of Pauli matrices.

In recent experimental work,[17–19] the authors mainly considered this problem for Dicke states of four, five, and six qubits and experimentally demonstrated the possibility of obtaining inequivalent multipartite entangled states by projective

measurements. In this work, we examine several inequivalent multipartite entangled states for four qubits and investigate the possibility of obtaining[1)] inequivalent multipartite entangled states for three qubits. We have chosen only those examples which are not considered before.

Our first example is the cluster state which is one of the two independent graph states for four qubits. The other independent graph state for four qubits is $|GHZ_4\rangle$ state.[37] The cluster state can be written in the computational basis as

$$|CL_4\rangle = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle). \quad (19)$$

Interestingly, it turns out that the cluster state can only be mapped to either one of the $|GHZ\rangle$ states of three qubits, that is

$$|CL_4\rangle \longmapsto \sum_{k=1}^{4} \alpha_k |GHZ^k\rangle_3, \quad (20)$$

where any other three-qubit GHZ state is locally equivalent to $|GHZ_3\rangle$ state. It is known that $|GHZ_3\rangle$ is the only independent graph state for three qubits.[37]

The next example is the four-qubit singlet state $|\Psi_{S,4}\rangle$ which can also be mapped to superposition of $|W_3\rangle$ and $|\widetilde{W}_3\rangle$ state, that is

$$|\Psi_{S,4}\rangle \longmapsto \alpha_6 |W_3\rangle + \beta_6 |\widetilde{W}_3\rangle, \quad (21)$$

which can further be converted into $|G_3\rangle$ state and subsequently into $|GHZ_3\rangle$ state as discussed before.

Our final example is the $|\chi_4\rangle$ state which is the most interesting and illustrative case. We saw in the previous section that we combined GHZ-type entanglement and W-type entanglement to create this state. In the reverse process of projective measurements, we find that $|\chi_4\rangle$ state can be mapped to superposition of these two types of states, that is

$$|\chi_4\rangle \longmapsto \alpha_7 |W_3\rangle + \beta_7 |\widetilde{GHZ}_3\rangle, \quad (22)$$

where one can transform the non-maximally entangled $|\widetilde{GHZ}_3\rangle = \sqrt{1/3}|000\rangle + \sqrt{2/3}|111\rangle$ into $|GHZ_3\rangle$ by local filtering. All these examples have increased our understanding about the structure of genuine multipartite entangled states. We have seen that $|\chi_4\rangle$, $|GHZ_4\rangle$, and $|W_4\rangle$ states are all inequivalent states, however, it is interesting to find that one can create another inequivalent entangled states by combining two different inequivalent states.

5. Conclusions

We have studied the generation of multipartite entangled states for superconducting phase qubits. We have shown that this model has the capacity to generate many important multipartite entangled states. For three qubits, we have reconfirmed the previous studies on creating $|W_3\rangle$ state and have examined the possibility to generate $|G_3\rangle$ states. We have examined the possibility of creating several entangled states for four qubits. In fact, the experimental setup has already been designed for four qubits with studies only on three qubits.[13] All cases studied in this paper may be performed in exactly the same experimental setup designed for generating $|W_3\rangle$ state.[13] We have shown that four-qubit setup is richer than three qubits and offer some unique possibilities to create several important entangled states. Particularly, we have proposed the experimental architecture for generating $|\chi_4\rangle$ state, which seems quite feasible in this setup. Indeed the experimental creation of $|\chi_4\rangle$ state is challenging and such verification would be of much interest and importance for applications of this state in quantum information. We have also re-examined an alternative approach to generate genuine multipartite entangled states via von Neumann projective measurements. In this technique, one can apply projective measurements on one or more qubits of a higher dimensional density matrix before tracing them out to obtain multipartite entangled states in a lower dimensional space. We have only restricted ourselves to four qubits by applying projective measurements only on a single qubit. We have shown that we can generate both GHZ-type and W-type entangled states on a three-qubit space. We have also shown that $|\chi_4\rangle$ state can be mapped to superposition of GHZ-type and W-type entanglements which is intuitive as we actually utilized both these types of entanglement to generate it. Hence this study has increased our understanding of the structure of genuine multipartite entangled states. One of the future avenues is to investigate the robustness of this setup against decoherence.

Acknowledgments

The author would like to thank Andreas Osterloh for discussion, Otfried Gühne for his generous hospitality at the University of Siegen, and Geza Tóth for his kind hospitality at the University of Basque Country, Bilbao, where this work was orally presented. This work has been supported by the EU (Marie Curie CIG 293993/ENFOQI) and the BMBF (Chist-Era Project QUASAR). The author is also grateful to referees for their positive comments which improved the earlier manuscript.

References

[1] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[2] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 *Rev. Mod. Phys.* **81** 865
[3] Ladd T D *et al.* 2010 *Nature* **464** 45
[4] Buluta I *et al.* 2011 *Rep. Prog. Phys.* **74** 104401
[5] Dür W, Vidal G and Cirac J I 2000 *Phys. Rev. A* **62** 062314
[6] Verstraete F, Dehaene J, DeMoor B and Verschelde H 2002 *Phys. Rev. A* **65** 052112

[1)] Here the word "obtaining" means that we get the result after the arrow by applying projective measurements and tracing out the qubit being measured.

[7] Lamata L, Leon J, Salgado D and Solano E 2007 *Phys. Rev. A* **75** 022318
[8] Barenco A, Bennett C H, Cleve R, DiVincenzo D P, Margolus N, Shor P, Sleator T, Smolin J A and Weinfurter H 1995 *Phys. Rev. A* **52** 3457
[9] Shi Y 2003 *Quantum Inf. Comput.* **3** 84
[10] Lanyon B P *et al.* 2008 *Nat. Phys.* **5** 134
[11] Monz T, Kim K, Hänsel Riebe W M, Villar A S, Schindler P, Chwalla M, Hennrich M and Blatt R 2009 *Phys. Rev. Lett.* **102** 040501
[12] Galiautdinov A and Martinis J M 2008 *Phys. Rev. A* **78** 010305
[13] Neeley M, Bialczak R C, Lenander M, Lucero E, Mariantoni M, O'Connell A D, Sank D, Wang H, Weides M, Wenner J, Yin Y, Yamamoto T, Cleland A N and Martinis J M 2010 *Nature* **467** 570
[14] Matsuo S *et al.* 2007 *J. Phys. Soc. Jpn.* **76** 054802
[15] Tsomokos D I *et al.* 2008 *New. J. Phys.* **10** 113020
[16] Ionicioiu R, Popescu A E, Munro W J and Spiller T P 2008 *Phys. Rev. A* **78** 052326
[17] Kiesel N, Schmid C, Toth G, Solano E and Weinfurter H 2007 *Phys. Rev. Lett.* **98** 063604
[18] Wieczorek W, Kiesel N, Schmid C and Weinfurter H 2009 *Phys. Rev. A* **79** 022311
[19] Wieczorek W, Krischek R, Kiesel N, Michelberger P, Toth G and Weinfurter H 2009 *Phys. Rev. Lett.* **103** 020504
[20] Sabín C and García-Alcaine G 2008 *Eur. J. Phys. D* **48** 435
[21] Kay A 2011 *Phys. Rev. A* **83** 020303
[22] Kay A 2010 *J. Phys. A: Math. Theor.* **43** 495301
[23] Gühne O and Seevinck M 2010 *New. J. Phys.* **12** 053002
[24] Gühne O 2011 *Phys. Lett. A* **375** 406
[25] Dür W and Cirac I 2000 *Phys. Rev. A* **61** 042314
[26] Schack R and Caves C M 2000 *J. Mod. Opt* **47** 387
[27] Acín A, Bruß D, Lewenstein M and Sanpera A 2001 *Phys. Rev. Lett.* **87** 040401
[28] Monz T, Schindler P, Barreiro J T, Chwalla M, Nigg D, Coish W A, Harlander M, Hänsel W, Hennrich M and Blatt R 2011 *Phys. Rev. Lett.* **106** 130506
[29] Peres A 1996 *Phys. Rev. Lett.* **77** 1413
[30] Horodecki M, Horodecki P and Horodecki R 1996 *Phys. Lett. A* **223** 1
[31] Alber G *et al.* 2001 *Quantum Information* (Berlin: Springer-Verlag) p. 151
[32] Vidal G and Werner R F 2002 *Phys. Rev. A* **65** 032314
[33] Jungnitsch B, Moroder T and Gühne O 2011 *Phys. Rev. Lett.* **106** 190502
[34] Neeley M G 2010 Ph. D Thesis, University of California at Santa Barbara, USA
[35] Schuch N and Siewert J 2003 *Phys. Rev. A* **67** 032301
[36] Sen(De) A, Sen U and Żukowski M 2003 *Phys. Rev. A* **68** 032309
[37] Gühne O and Tóth G 2009 *Phys. Rep.* **474** 1
[38] Osterloh A and Siewert J 2005 *Phys. Rev. A* **72** 012337
[39] Osterloh A and Siewert J 2006 *Int. J. Quantum Inf.* **4** 531
[40] Kraus B 2003 Ph. D thesis, University of Innsbruck, Austria
[41] Svetlichny G 1987 *Phys. Rev. D* **35** 3066
[42] Weinfurter H and Żukowski M 2001 *Phys. Rev. A* **64** 010102