Relativistic effect of pseudospin symmetry and tensor coupling on the Mie-type potential via Laplace transformation method

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(Received 25 February 2014; revised manuscript received 22 April 2014; published online 10 October 2014)

A relativistic Mie-type potential for spin-1/2 particles is studied. The Dirac Hamiltonian contains a scalar $S(r)$ and a vector $V(r)$ Mie-type potential in the radial coordinates, as well as a tensor potential $U(r)$ in the form of Coulomb potential. In the pseudospin (p-spin) symmetry setting $\Sigma = C_{\rm ps}$ and $\Delta = V(r)$, an analytical solution for exact bound states of the corresponding Dirac equation is found. The eigenenergies and normalized wave functions are presented and particular cases are discussed with any arbitrary spin–orbit coupling number κ. Special attention is devoted to the case $\Sigma = 0$ for which p-spin symmetry is exact. The Laplace transform approach (LTA) is used in our calculations. Some numerical results are obtained and compared with those of other methods.

Keywords: Dirac equation, Mie-type potential, Laplace transform approach, tensor interaction, p-spin symmetry

PACS: 03.65.Pm, 03.65.Ge, 02.30.Gp **DOI:** 10.1088/1674-1056/23/12/120304

1. Introduction

The exact bound-state solutions of the Dirac equation with physically significant potentials play a major role in quantum mechanics. Over the decades, the exact analytical solutions of the Dirac equation having spin and p-spin symmetry have been extensively studied, for example, see Refs. [1]–[14]. Nevertheless, solving this equation is still a challenging problem even though it started to be derived more than 80 years ago. Further, the Dirac equation is very useful to investigate the relativistic effects.[15] In the nuclear level, the Dirac equation is used to describe the behavior of nucleons in the nucleus and also in solving problems of high-energy physics and chemistry.[16] For this reason, it has been used extensively in relativistic heavy ion collisions, heavy ion spectroscopy, laser–matter interaction (for more reviews, see Ref. [15] and the references therein), condensed matter physics,[17] atomic and molecular physics, and chemistry.

The Mie-type potential (Kratzer potential) is an exactly solvable potential model[18] and used to study the diatomic molecules.[19] It is also used to determine the molecular structure and recently has received much attention.[20] Various methods are used to solve exactly the Schrödinger equation (SE) in different dimensions $N = 1,2,3$ for a system bound by the Mie-type potentials, such as exact quantization rule,[21] polynomial solution,[22] $1/N$ expansion method,[23] path integral,[24] ladder operators,[25] Laplace transformation approach,[26] wave function ansatz method,[27] asymptotic iteration method,[28] smooth transformation,[29] Lie algebraic method,[30] etc. Recently, the relativistic bound state of spin-1/2 particle in the presence of Mie-type potential has been found under spin and p-spin symmetry[31] by means of the asymptotic iteration method.

In this work, we attempt to solve the Dirac equation for the Mie-type potential[4,32] including a tensor coupling Coulomb potential in the framework of the Laplace transform approach (LTA).[28] The Mie-type potential takes the form[21,22]

$$V_{\rm Mie}(r) = \frac{a}{r^2} - \frac{b}{r} + c, \tag{1}$$

and we consider the tensor interaction potential in the Coulomb form as

$$U(r) = -\frac{T}{r}, \quad T = \frac{Z_{\rm a} Z_{\rm b} e^2}{4\pi\varepsilon_0}, \quad r \geq R_{\rm c}, \tag{2}$$

where $R_{\rm c}$ is the Coulomb radius, $Z_{\rm a}$ and $Z_{\rm b}$ denote the charges of the projectile particle a and the target nucleus b, respectively.[33,34] Tensor potentials are introduced into the Dirac equation with the substitution $\boldsymbol{p} \to \boldsymbol{p} - {\rm i}m\omega\beta \cdot \hat{r}U(r)$.[35–37] In this way, a spin–orbit coupling term is added to the Dirac Hamiltonian.

The rest of this paper is organized as follows. In Section 2 we present the general Dirac equation with scalar and vector potentials in the Mie-type form and a tensor potential in the Coulomb form. We then obtain and discuss the energy eigenvalues and normalized wave functions of this equation for $\Delta = V_{\rm Mie}(r)$ and $\Sigma = C_{\rm ps} =$ constant by means of LTA. In Section 3 we look into some particular cases like Kratzer–Fues and modified Kratzer potentials and solution in the absence of the tensor potential. Some numerical results are also

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displayed. Finally, the conclusion is given in Section 4.

2. Solution of the Dirac equation including tensor coupling

In a relativistic description, nuclei are characterized by a strong attractive scalar potential $S(r)$, strong repulsive vector potential $V(r)$, and a tensor potential $U(r)$. The Dirac equation for a nucleon in ($\hbar = c = 1$ units) reads

$$[\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta(M+S(r))-\mathrm{i}\beta\boldsymbol{\alpha}\cdot\hat{r}U(r)]\,\psi(\boldsymbol{r})=[E-V(r)]\,\psi(\boldsymbol{r}),\tag{3}$$

where E is the relativistic energy, M is the fermion mass, $\boldsymbol{p}=-\mathrm{i}\boldsymbol{\nabla}$ is the momentum operator and $\boldsymbol{\alpha}$ and β are 4×4 matrices which, in the usual representations, take the forms of

$$\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix},\quad \beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix},\tag{4}$$

with I being the 2×2 unitary matrix, and $\boldsymbol{\sigma}$ the three-vector spin matrices

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.\tag{5}$$

The total angular momentum operator $\boldsymbol{J}$ and spin–orbit $K=(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$, where $\boldsymbol{L}$ is the orbital angular momentum, of the spherical nucleons commute with the Dirac Hamiltonian. The eigenvalues of spin–orbit coupling operator are $\kappa=(j+1/2)>0$ and $\kappa=-(j+1/2)<0$ for unaligned spin $j=l-1/2$ and the aligned spin $j=l+1/2$, respectively. $(H^2,\ K,\ J^2,\ J_z)$ can be taken as a complete set of the conservative quantities. Thus, the spinor wave functions are classified according to their angular momentum j, spin–orbit quantum number κ, and the radial quantum number n, and can be written as follows:

$$\psi_{n\kappa}(\boldsymbol{r})=\begin{pmatrix}f_{n\kappa}(\boldsymbol{r})\\ g_{n\kappa}(\boldsymbol{r})\end{pmatrix}=\frac{1}{r}\begin{pmatrix}F_{n\kappa}(r)\mathrm{Y}^{l}_{jm}(\theta,\varphi)\\ \mathrm{i}G_{n\kappa}(r)\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\varphi)\end{pmatrix},\tag{6}$$

where $f_{n\kappa}(\boldsymbol{r})$ is the upper (large) component and $g_{n\kappa}(\boldsymbol{r})$ is the lower (small) component of the Dirac spinors; $\mathrm{Y}^{l}_{jm}(\theta,\phi)$ and $\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi)$ are spin and p-spin spherical harmonics, respectively; m is the projection of the angular momentum on the z axis. Substituting Eq. (6) into Eq. (3) and using the following relations:[38]

$$(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B})=\boldsymbol{A}\cdot\boldsymbol{B}+\mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{B}),\tag{7a}$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{P})=\boldsymbol{\sigma}\cdot\hat{r}\left(\hat{r}\cdot\boldsymbol{P}+\mathrm{i}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{L}}{r}\right),\tag{7b}$$

further, making use of the following properties

$$\boldsymbol{\sigma}\cdot\boldsymbol{L}\begin{Bmatrix}\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi)\\ \mathrm{Y}^{l}_{jm}(\theta,\phi)\end{Bmatrix}=\begin{cases}(\kappa-1)\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi)\\ -(\kappa-1)\mathrm{Y}^{l}_{jm}(\theta,\phi)\end{cases},\tag{8a}$$

$$\boldsymbol{\sigma}\cdot\hat{r}\begin{Bmatrix}\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi)\\ \mathrm{Y}^{l}_{jm}(\theta,\phi)\end{Bmatrix}=\begin{cases}-\mathrm{Y}^{l}_{jm}(\theta,\phi),\\ -\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi),\end{cases}\tag{8b}$$

one obtains two coupled differential equations for upper and lower radial wave functions $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ as

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\right)F_{n\kappa}(r)=(M+E_{n\kappa}-\Delta(r))\,G_{n\kappa}(r),\tag{9a}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\right)G_{n\kappa}(r)=(M-E_{n\kappa}+\Sigma(r))\,F_{n\kappa}(r),\tag{9b}$$

with

$$\Delta(r)=V(r)-S(r),\tag{10a}$$

$$\Sigma(r)=V(r)+S(r).\tag{10b}$$

Eliminating $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ from Eqs. (9), we can obtain the following two Schrödinger-like differential equations for the upper and lower radial spinor components:

$$\begin{aligned}&\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa+1)}{r^2}+\frac{2\kappa}{r}U(r)-\frac{\mathrm{d}U(r)}{\mathrm{d}r}-U^2(r)\right]F_{n\kappa}(r)\\&+\frac{\dfrac{\mathrm{d}\Delta(r)}{\mathrm{d}r}}{M+E_{n\kappa}-\Delta(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\right)F_{n\kappa}(r)\\&=[(M+E_{n\kappa}-\Delta(r))\,(M-E_{n\kappa}+\Sigma(r))]\,F_{n\kappa}(r),\end{aligned}\tag{11a}$$

and

$$\begin{aligned}&\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa-1)}{r^2}+\frac{2\kappa}{r}U(r)+\frac{\mathrm{d}U(r)}{\mathrm{d}r}-U^2(r)\right]G_{n\kappa}(r)\\&+\frac{\dfrac{\mathrm{d}\Sigma(r)}{\mathrm{d}r}}{M-E_{n\kappa}+\Sigma(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\right)G_{n\kappa}(r)\\&=[(M+E_{n\kappa}-\Delta(r))\,(M-E_{n\kappa}+\Sigma(r))]\,G_{n\kappa}(r),\end{aligned}\tag{11b}$$

respectively, where $\kappa(\kappa-1)=\tilde{l}(\tilde{l}+1)$ and $\kappa(\kappa+1)=l(l+1)$. These radial wave functions are required to satisfy the necessary boundary conditions. The spin–orbit quantum number κ is related to the quantum numbers for spin symmetry l and p-spin symmetry $\tilde{l}$ as

$$\kappa=\begin{cases}-(l+1)=-(j+1/2),\ \ (\mathrm{s}_{1/2},\ \mathrm{p}_{3/2},\ \ \text{etc.}),\\ \qquad j=l+1/2,\ \ \text{aligned spin}\,(\kappa<0),\\ +l=+(j+1/2),\ \ (\mathrm{p}_{1/2},\ \mathrm{d}_{3/2},\ \ \text{etc.}),\\ \qquad j=l-1/2,\ \ \text{unaligned spin}\,(\kappa>0),\end{cases}\tag{12}$$

and the quasi-degenerate doublet structure can be expressed in terms of a p-spin angular momentum $\tilde{s}=1/2$ and pseudo-orbital angular momentum $\tilde{l}$, which is defined as

$$\kappa=\begin{cases}-\tilde{l}=-(j+1/2),\ \ (\mathrm{s}_{1/2},\ \mathrm{p}_{3/2},\ \ \text{etc.}),\\ \qquad j=\tilde{l}-1/2,\ \ \text{aligned p-spin}\,(\kappa<0),\\ +(\tilde{l}+1)=+(j+1/2),\ \ (\mathrm{d}_{3/2},\ \mathrm{f}_{5/2},\ \ \text{etc.}),\\ \qquad j=\tilde{l}+1/2,\ \ \text{unaligned p-spin}\,(\kappa>0),\end{cases}\tag{13}$$

where $\kappa=\pm1,\pm2,\ldots$. For example, $(1\mathrm{s}_{1/2},0\mathrm{d}_{3/2})$ and $(1\mathrm{p}_{3/2},0\mathrm{f}_{5/2})$ can be considered as p-spin doublets. In this work, we will deal with the so-called p-spin symmetric case which has a remarkable application in the study of deformation, super-deformation, identical bands, and magnetic moment in the nuclear structure.[39–41] The pseudospin symme-

try occurs when $\mathrm{d}\Sigma(r)/\mathrm{d}r = 0$ or equivalently $\Sigma(r) = C_{\rm ps} =$ const.[42–45] Equation (11b) cannot be solved analytically because of

$$\frac{\mathrm{d}\Sigma(r)/\mathrm{d}r}{M - E_{n\kappa} + \Sigma(r)} \left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r} + U(r) \right)$$

term.[46,47]

Hence, we can take the potentials as

$$\Delta(r) = \frac{a}{r^2} - \frac{b}{r} + c, \quad \Sigma(r) = C_{\rm ps}, \quad U(r) = -\frac{T}{r}, \tag{14}$$

to recast Eq. (11a) in the form of

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{(\kappa+T)(\kappa+T-1)}{r^2} - \left(E_{n\kappa} - M - C_{\rm ps}\right)\left(\frac{a}{r^2} - \frac{b}{r} + c\right) - \left[(M + E_{n\kappa})\left(M - E_{n\kappa} + C_{\rm ps}\right)\right] G_{n\kappa}(r) = 0. \tag{15}$$

Identifying the following parameters:

$$\begin{aligned} \lambda^2 &= (\kappa+T)(\kappa+T-1) + a(E_{n\kappa} - M - C_{\rm ps}), \\ \delta^2 &= -b(E_{n\kappa} - M - C_{\rm ps}), \\ \varepsilon^2 &= -(E_{n\kappa} - M - C_{\rm ps})(E_{n\kappa} + M - c), \end{aligned} \tag{16}$$

one can rewrite Eq. (15) as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \varepsilon^2 - \frac{\delta^2}{r} - \frac{\lambda^2}{r^2} \right) G_{n\kappa}(r) = 0. \tag{17}$$

Defining the new function $G_{n\kappa}(r) = \sqrt{r}\varphi(r)$, equation (17) turns into

$$\left[r^2 \frac{\mathrm{d}^2}{\mathrm{d}r^2} + r\frac{\mathrm{d}}{\mathrm{d}r} - \left(\frac{\gamma^2}{r^2} + \frac{\delta^2}{r} + \varepsilon^2 \right) r^2 \right] \varphi(r) = 0, \tag{18}$$

where $\gamma^2 = \lambda^2 + 1/4$, and further setting $\varphi(r) = r^\tau \chi(r)$, where τ is a constant, equation (18) becomes

$$\left\{ r^2 \frac{\mathrm{d}^2}{\mathrm{d}r^2} + (2\tau+1) r \frac{\mathrm{d}}{\mathrm{d}r} - \left(\gamma^2 + \delta^2 r + \varepsilon^2 r^2 - \tau^2\right) \right\} \chi(r) = 0. \tag{19}$$

Now, to obtain a finite solution for the wave function at $r \to \infty$, we must take $\tau = -\gamma$ in the above equation (19) and then we can obtain

$$\left\{ r \frac{\mathrm{d}^2}{\mathrm{d}r^2} - (2\gamma - 1) \frac{\mathrm{d}}{\mathrm{d}r} - \left(\delta^2 + \varepsilon^2 r\right) \right\} \chi(r) = 0. \tag{20}$$

Here, we will use the LTA to solve Eq. (20). It is an integral transform and comprehensively useful in physics and engineering.[48] Recently, LTA has been used by many authors to solve the Schrödinger equations for different potential models[26,49–52] and it serves as a powerful algebraic treatment in solving the second-order differential equations. The advantage of LTA is that it converts the second-order equation into a simpler form whose solutions may be obtained easily.[49] Applying the Laplace transform defined by[53]

$$L\{\chi(r)\} = f(t) = \int_0^\infty \mathrm{d}r \, \mathrm{e}^{-tr} \chi(r), \tag{21}$$

to Eq. (20), we can obtain a first-order differential equation

$$\left(t^2 - \varepsilon^2\right) \frac{\mathrm{d}f(t)}{\mathrm{d}t} + \left[(2\gamma+1)t + \delta^2\right] f(t) = 0, \tag{22}$$

whose solution is simply given by

$$f(t) = N (t+\varepsilon)^{-(2\gamma+1)} \left(\frac{t-\varepsilon}{t+\varepsilon} \right)^{-(\delta^2/2\varepsilon) - (2\gamma+1)/2}, \tag{23}$$

where N is a normalization constant. The wave functions must be single-valued and we require that

$$-\frac{\delta^2}{2\varepsilon} - \frac{(2\gamma+1)}{2} = n, \quad n = 0, 1, 2, \ldots. \tag{24}$$

To obtain the energy eigenvalue of the radial part, we insert the values of the parameters given in Eq. (16) into Eq. (24). Taking this requirement into account, one can expand Eq. (23) into the series:

$$f(t) = N' \sum_{k=0}^{n} \frac{(-1)^k n! (2\varepsilon)^k (t+\varepsilon)^{-(2\gamma+1+k)}}{(n-k)!k!}, \tag{25}$$

where N' is a constant. Further, using the inverse Laplace transformation[54] in the above equation, we immediately obtain

$$\chi(r) = N'' r^{2\gamma} \mathrm{e}^{-\varepsilon r} \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!k!} \frac{\Gamma(2\gamma+1)}{\Gamma(2\gamma+1+k)} (2\varepsilon r)^k. \tag{26}$$

Finally, using $\varphi(r) = r^{-\gamma}\chi(r)$, we obtain

$$\varphi(r) = N''' r^{\gamma} \mathrm{e}^{-\varepsilon r} \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!k!} \frac{\Gamma(2\gamma+1)}{\Gamma(2\gamma+1+k)} (2\varepsilon r)^k, \tag{27}$$

where N''' is a constant. On the other hand, the confluent hyper-geometric functions are defined in a series expansion as[54]

$${}_1\mathrm{F}_1(-n, \sigma, s) = \sum_{m=0}^{n} \frac{(-1)^m n!}{(n-m)!m!} \frac{\Gamma(\sigma)}{\Gamma(\sigma+m)} s^m. \tag{28}$$

Comparing Eq. (28) with Eq. (27), we can finally obtain the lower radial part of the wave function as

$$G_{n\kappa}(r) = N r^{\gamma+1/2} \mathrm{e}^{-\varepsilon r} {}_1\mathrm{F}_1\left(-n, 2\gamma+1, 2\varepsilon r\right), \tag{29}$$

where N is a normalization constant, and γ and ε are defined as

$$\begin{aligned} \gamma &= \sqrt{(\kappa + T - 1/2) + a(E_{n\kappa} - M - C_{\rm ps})}, \\ \varepsilon &= \sqrt{(M - E_{n\kappa} + C_{\rm ps})(E_{n\kappa} + M - c)}. \end{aligned}$$

By using the normalization condition $\int_0^\infty |G_{n\kappa}(r)|^2 \, \mathrm{d}r = 1$, and the relation between the Laguerre polynomials and confluent hyper-geometric functions as[54]

$$\mathrm{L}_n^p(x) = \frac{\Gamma(n+p+1)}{n!\Gamma(p+1)} {}_1\mathrm{F}_1(-n, p+1, x),$$

the normalization constant in Eq. (29) is written as

$$N = \Gamma(2\gamma+1) \sqrt{\frac{n!\,(2n+2\gamma+1)}{\Gamma(n+2\gamma+1)}}, \tag{30}$$

where we have used[45]

$$\int_0^\infty x^q \mathrm{e}^{-x} \mathrm{L}_n^q(x) \mathrm{L}_{n'}^q(x) \mathrm{d}x = \frac{\Gamma(q+n+1)}{n!} \delta_{nn'}. \quad (31)$$

On the other hand, the upper spinor component of the wave function can be calculated from Eq. (9b) as

$$F_{n\kappa}(r) = \sqrt{\frac{n!\,(2n+2\gamma+1)}{\Gamma(n+2\gamma+1)}} \frac{\Gamma(2\gamma+1)}{(M-E_{n\kappa}+C_{\mathrm{ps}})} \times \left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa+T}{r}\right) r^{\gamma+1/2} \mathrm{e}^{-\varepsilon r} {}_1\mathrm{F}_1\left(-n, 2\gamma+1, 2\varepsilon r\right), \quad (32)$$

where $E_{n\kappa} \neq M$ when $C_{\mathrm{ps}} = 0$, which means that only negative energy spectrum is permissible for a normalizable and well-defined wave function.[55] Inserting the parameters in Eq. (16) into Eq. (24), one obtains the energy eigenvalues of the radial part as follows:

$$\frac{b(E_{n\kappa}-M-C_{\mathrm{ps}})}{\sqrt{(M+E_{n\kappa}-c)(M-E_{n\kappa}+C_{\mathrm{ps}})}} = 1+2n+2\sqrt{(\kappa+T-1/2)^2+a(E_{n\kappa}-M-C_{\mathrm{ps}})}, \quad n=0,1,2,\ldots, \quad (33)$$

which is identical to Eq. (40) in Ref. [31]. In the exact p-spin limit and in the absence of tensor interaction, we have the energy equation

$$\frac{b(E_{n\kappa}-M)}{\sqrt{M^2-E_{n\kappa}^2+c(E_{n\kappa}-M)}} = 2\left(\sqrt{(\kappa-1/2)^2+a(E_{n\kappa}-M)}+n+1/2\right), \quad n=0,1,2,\ldots \quad (34)$$

and the wave function is given by

$$G_{n\kappa}(r) = \Gamma(2\gamma+1)\sqrt{\frac{n!\,(2n+2\gamma+1)}{\Gamma(n+2\gamma+1)}} r^{\gamma+1/2} \times \mathrm{e}^{-\varepsilon r} {}_1\mathrm{F}_1\left(-n, 2\gamma+1, 2\varepsilon r\right) \quad (35)$$

with

$$\gamma = \sqrt{(\kappa-1/2)^2+a(E_{n\kappa}-M)}, \quad \varepsilon = \sqrt{M^2-E_{n\kappa}^2+c(E_{n\kappa}-M)}. \quad (36)$$

3. Results and discussion

In this section, we study some special cases of our solution given in Section 2 and also calculate numerical results for the eigenvalues of these cases. Further, we compare these results with the ones found by other methods.

3.1. Coulomb potential ($a = c = 0$)

The eigensolutions of the Coulomb potential, in the presence of exact p-spin symmetry, can be found via Eqs. (34) and (35) to be

$$b(E_{n\kappa}-M) = 2(\kappa+T+n)\sqrt{M^2-E_{n\kappa}^2}, \quad n=0,1,2,\ldots, \quad (37)$$

and

$$G_{n\kappa}(r) = \Gamma(2\kappa+2T)\sqrt{\frac{n!\,(2n+2\kappa+2T)}{\Gamma(n+2\kappa+2T)}} r^{\kappa+T} \mathrm{e}^{-\sqrt{M^2-E_{n\kappa}^2}r} \times {}_1\mathrm{F}_1\left(-n, 2\kappa+2T+1, 2\sqrt{M^2-E_{n\kappa}^2}r\right), \quad |E_{n\kappa}| < M. \quad (38)$$

On the other hand, in view of exact spin symmetry ($E_{n\kappa} \to -E_{n\kappa}$, $V(r) \to -V(r)$, $\kappa \to \kappa+1$, $G_{n,\kappa}(r) \to F_{n,\kappa}(r)$), they take the following forms:[33,56–60]

$$b(E_{n\kappa}+M) = 2(\kappa+T+n+)\sqrt{M^2-E_{n\kappa}^2}, \quad n=0,1,2,\ldots, \quad (39)$$

and

$$F_{n\kappa}(r) = \Gamma(2\kappa+2T+2)\sqrt{\frac{n!\,(2n+2\kappa+2T+2)}{\Gamma(n+2\kappa+2T+2)}} r^{\kappa+T+1} \times \mathrm{e}^{-\sqrt{M^2-E_{n\kappa}^2}r} {}_1\mathrm{F}_1\left(-n, 2\kappa+2T+3, 2\sqrt{M^2-E_{n\kappa}^2}r\right), \quad |E_{n\kappa}| < M. \quad (40)$$

In the non-relativistic limit ($E_{n\kappa}+M \to 2\mu$, $E_{n\kappa}-M \to E_{nl}$, $T=0$, $F_{n,\kappa}(r) \to R_{n,l}(r)$, $\kappa \to l$), we have[33,56–60]

$$E_{nl} = -\frac{\mu b^2}{2(n+l+1)^2}, \quad n=0,1,2,\ldots,\ l=0,1,2,\ldots, \quad (41)$$

and

$$R_{nl}(r) = \Gamma(2l+2)\sqrt{\frac{n!\,(2n+2l+2)}{\Gamma(n+2l+2)}} r^{l+1} \mathrm{e}^{-\sqrt{-2\mu E_{nl}}r} \times {}_1\mathrm{F}_1\left(-n, 2l+3, 2\sqrt{-2\mu E_{nl}}r\right), \quad E_{nl} < 0. \quad (42)$$

3.2. Kratzer–Fues potential

The Kratzer–Fues potential can be given as a simple example from the Mie-type potential by setting $a = D_{\mathrm{e}} r_{\mathrm{e}}^2$, $b = 2D_{\mathrm{e}} r_{\mathrm{e}}$ and $c = 0$[31,32]

$$V_{\mathrm{KF}}(r) = -D_{\mathrm{e}}\left(\frac{2r_{\mathrm{e}}}{r} - \frac{r_{\mathrm{e}}^2}{r^2}\right), \quad (43)$$

where D_{e} is the dissociation energy and r_{e} is the equilibrium inter-nuclear length.

The energy equation (33) becomes

$$\frac{2D_{\mathrm{e}}r_{\mathrm{e}}(E_{n\kappa}-M-C_{\mathrm{ps}})}{\sqrt{(M+E_{n\kappa}-c)(M-E_{n\kappa}+C_{\mathrm{ps}})}} - 2\sqrt{(\kappa+T-1/2)^2+D_{\mathrm{e}}r_{\mathrm{e}}^2(E_{n\kappa}-M-C_{\mathrm{ps}})} = 2n+1, \quad n=0,1,2,\ldots \quad (44)$$

and the wave function is the same as Eq. (29) with

$$\gamma = \sqrt{(\kappa+T-1/2)^2+D_{\mathrm{e}}r_{\mathrm{e}}^2(E_{n\kappa}-M-C_{\mathrm{ps}})}, \quad \varepsilon = \sqrt{M^2-E_{n\kappa}^2+C_{\mathrm{ps}}(E_{n\kappa}+M)}. \quad (45)$$

The numerical results for this case are given in Table 1 and compared with the results in Refs. [31], and [32].

Table 1. Bound state energies of the Kratzer–Fues potential for several values of n and κ with $M = 5$ fm^{-1}, $T = [0, 10, 15]$, $D_e = 1.25$ fm^{-1}, $r_e = 0.35$ fm, and $C_{ps} = 0$.

$\tilde{l}$	$n, \kappa < 0$	(l, j)	$E_{n,\kappa<0}$ Ref. [31]	$E_{n,\kappa<0}$ Ref. [32]	$E_{n,\kappa<0}$ Present work		
			for $T = 0$	for $T = 10$	$T = 0$	$T = 10$	$T = 15$
1	1, -1	$1s_{1/2}$	−4.672305	−4.980546	−4.67231	−4.98055	−4.99144
2	1, −2	$1p_{3/2}$	−4.860421	−4.975880	−4.86042	−4.97588	−4.99016
3	1, −3	$1d_{5/2}$	−4.916782	−4.969281	−4.91678	−4.96928	−4.98857
4	1, −4	$1f_{7/2}$	−4.943953	−4.959486	−4.94395	−4.95949	−4.98656
1	2, −1	$2s_{1/2}$	−4.833547	−4.983944	−4.83355	−4.98394	−4.99248
2	2, −2	$2p_{3/2}$	−4.913595	−4.980499	−4.91319	−4.9805	−4.99143
3	2, −3	$2d_{5/2}$	−4.942941	−4.975793	−4.94294	−4.97579	−4.99015
4	2, −4	$2f_{7/2}$	−4.959104	−4.969109	−4.9591	−4.96911	−4.98856
$\tilde{l}$	$n-1, \kappa > 0$	$(l+2, j+1)$			$E_{n-1,\kappa>0}$		
1	0, 2	$0d_{3/2}$	−4.672305	−4.988570	−4.67231	−4.98857	−4.99407
2	0, 3	$0f_{5/2}$	−4.860421	−4.990158	−4.86042	−4.99016	−4.99468
3	0, 4	$0g_{7/2}$	−4.916782	−4.991436	−4.91678	−4.99144	−4.9952
4	0, 5	$0h_{9/2}$	−4.983077	−4.992479	−4.94395	−4.99248	−4.99565
1	1, 2	$1d_{3/2}$	−4.833547	−4.990150	−4.83355	−4.99015	−4.99467
2	1, 3	$1f_{5/2}$	−4.913595	−4.991430	−4.91319	−4.99143	−4.9952
3	1, 4	$1g_{7/2}$	−4.942941	−4.992475	−4.94294	−4.99248	−4.99564
4	1, 5	$1h_{9/2}$	−4.959104	−4.993340	−4.9591	−4.99334	−4.99603

3.3. Modified Kratzer potential

The other example is the modified Kratzer potential obtained by sitting $a = D_e r_e^2$, $b = 2D_e r_e$, and $c = D_e$ [31,32] which is expressed as

$$V_{MK}(r) = D_e \left(\frac{r - r_e}{r} \right)^2 . \tag{46}$$

The energy equation is

$$\frac{2D_e r_e (E_{n\kappa} - M - C_{ps})}{\sqrt{(M + E_{n\kappa} - D_e)(M - E_{n\kappa} + C_{ps})}} - 2\sqrt{(\kappa + \alpha - 1/2)^2 + D_e r_e^2 (E_{n\kappa} - M - C_{ps})} = 2n + 1, \quad n = 0, 1, 2, \ldots \tag{47}$$

and the wave function is the same as Eq. (29) with

$$\gamma = \sqrt{(\kappa + \alpha - 1/2) + D_e r_e^2 (E_{n\kappa} - M - C_{ps})},$$
$$\varepsilon = \sqrt{(M - E_{n\kappa} + C_{ps})(M + E_{n\kappa} - D_e)}. \tag{48}$$

We use the parameter values $M = 5$ fm^{-1} (1 fm $= 10^{-15}$ m), $T = 0, 10, 15$, $C_{ps} = 0$, $r_e = [0.35, \ldots, 0.85]$ fm, and $D_e = [1.25, \ldots, 4.25]$ fm^{-1} in our calculations. The numerical results are displayed in Table 1 and compared with the results in Refs. [31] and [32].

In Table 1, we see that the energies of bound states such as $(1s_{1/2}, 0d_{3/2})$, $(1p_{3/2}, 0f_{5/2})$, $(1d_{5/2}, 0g_{7/2})$, $(1f_{7/2}, 0h_{9/2})$, ... (where each pair is considered as a p-spin doublet) in the absence of the tensor potential are degenerate but in the presence of the tensor potential, the degeneracies are removed. In checking Table 1, we find the fact that the presence of tensor potential leads to non-degenerate states and cancelling tensor provides degenerate states. This is the main reason for introducing the tensor term in solving the Mie-type potential in Dirac equation. Also, in Figs. 1 and 2, we show the variations of bound energy with D_e and r_e parameters under the exact p-spin symmetry ($C_{ps} = 0$) when $T = 0$ respectively. In Figs. 1 and 2, it is obvious that as D_e and r_e parameters increase, the bound state energy eigenvalues of the Kratzer–Fues potential increase for several states.

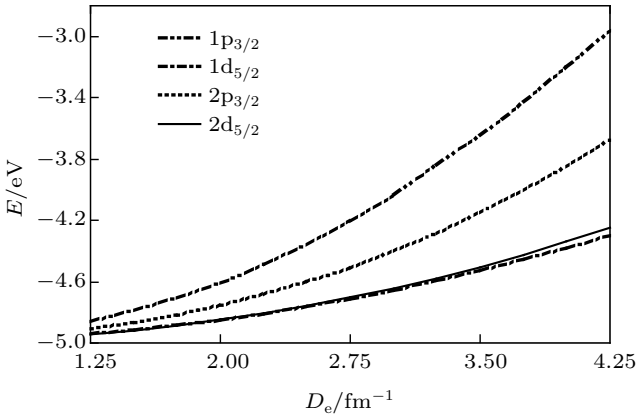


Fig. 1. Variations of several energy states with parameter D_e in the Kratzer–Fues potential case when $M = 5$ fm^{-1} and $r_e = 0.35$ fm.

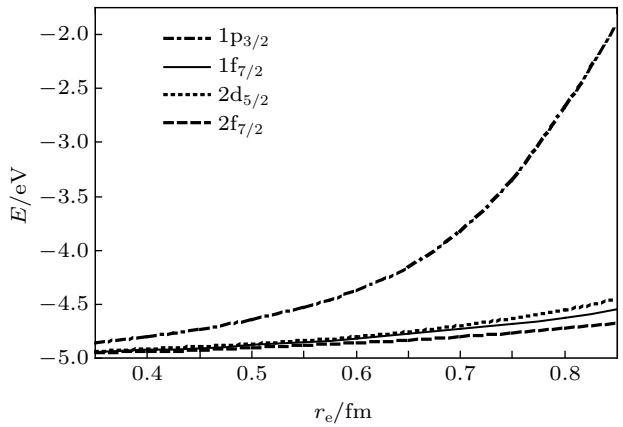


Fig. 2. Variations of several energy states with parameter r_e in the Kratzer–Fues potential case when $M = 5$ fm^{-1} and $D_e = 1.25$ fm^{-1}.

4. Conclusions

In this paper, the Dirac equation is solved exactly for the Mie-type potential in the presence of the tensor interaction by using the LTA in view of the p-spin symmetry. The bound state energy equation and the corresponding normalized eigenfunctions of the Dirac equation are obtained in closed form. Presented in Table 1 are some numerical results in the presence of the tensor interaction for the Kratzer–Fues potential. Further, figures 1 and 2 show the variations of energy with D_e and r_e parameters for the Kratzer–Fues potential considering various states. It is noticed that the tensor interaction removes the degeneracy between two states in spin doublets. Some particular cases of interest are studied and some numerical results are also obtained. Our results are found to be identical with those obtained by other methods.

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