Approximate solutions of Klein–Gordon equation with improved Manning–Rosen potential in D-dimensions using SUSYQM

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In this paper, we present solutions of the Klein–Gordon equation for the improved Manning–Rosen potential for arbitrary l state in d-dimensions using the supersymmetric shape invariance method. We obtained the energy levels and the corresponding wave functions expressed in terms of Jacobi polynomial in a closed form for arbitrary l state. We also calculate the oscillator strength for the potential.

Keywords: Klein–Gordon equation, improved Manning–Rosen potential, supersymmetric quantum mechanics (SUSYQM)

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1. Introduction

Manning and Rosen[1] proposed a potential model for diatomic molecules of the form,

$$V(r)=\frac{h^2}{8\mu\pi^2b^2}\left[\frac{\beta(\beta-1)\,\mathrm{e}^{-2r/b}}{\left(1-\mathrm{e}^{-2r/b}\right)^2}-\frac{A\,\mathrm{e}^{-r/b}}{1-\mathrm{e}^{-r/b}}\right], \quad (1)$$

where β and A are two dimensionless constants, b is related to the range of the potential model and has the dimension of length. The improved Manning–Rosen potential given by Jia *et al.*[2] can be expressed in the form

$$V_{\mathrm{MR}}(r)=D_{\mathrm{e}}\left(1-\frac{b-1}{\mathrm{e}^{\alpha r}-1}\right)^2,\ \ b=\mathrm{e}^{\alpha r_{\mathrm{e}}}, \quad (2)$$

where $\alpha=1/b$, D_{e} is the dissociation energy, and r_{e} is the equilibrium bond length. Manning–Rosen potential is one of the short range potentials and it has been used to describe the diatomic molecular vibration.[3] This potential has been considered in relativistic and non-relativistic regions.[4,5] The bound state solution of the Klein–Gordon equation with improved Manning–Rosen potential has been studied in Refs. [6] and [7]. Dong and Garcia-Ravelo[8] used the functional analysis approach to investigate the s-wave of the Schrödinger equation with Manning–Rosen potential. There are several quantum mechanical methods which have been used by different authors to obtain the bound state and scattering state solutions of Schrödinger, Klein–Gordon, and Dirac equations with Manning–Rosen potential.[9–11] These methods include Nikiforov–Uvarov,[12] supersymmetry quantum mechanics (SUSYQM),[13,14] exact quantization rule,[15] and asymptotic iteration method (AIM).[16] Among these methods, the SUSYQM method has received great interest. The concept of SUSYQM allows one to determine analytically the eigenfunctions and eigenvalues for a solvable potential model using algebraic operator formulation without solving the Schrödinger-like differential equation by standard series method.

Interestingly, the study of the relativistic wave equation in recent years, particularly the Klein–Gordon equation, has attracted the attention of many authors because the solutions of this equation plays the rule in getting the relativistic effect. The Klein–Gordon equation is the well-known relativistic wave equation that describes spin-zero particles. It is also known that the analytic solutions of the Klein–Gordon equations are only possible in a few cases such as harmonic and Coulomb potentials.[17,18] However, for arbitrary l states ($l\neq 0$), the Klein–Gordon equation does not admit an exact solution. Thus, the Klein–Gordon equation can be solved approximately using different approximation schemes.[19,20] Improved Greene and Aldrich approximation has been used by Chen *et al.*[21] to investigate the bound state solution of the Schrödinger equation with Manning–Rosen potential. Recently, the Klein–Gordon equation in generalized D-dimensions for different potentials is attracting the attention of researchers.[22–24] This multidimensional space analysis of the Klein–Gordon equation has also been investigated for different potentials.[25]

In this paper, we attempt to study the bound state solutions of the Klein–Gordon equation with the improved Manning–Rosen potential using the SUSYQM in D dimen-

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sions. We determine approximately the energy eigenvalues and the eigenfunction by employing the improved Greene–Aldrich approximation scheme.[26] We computed numerically the energy spectrum in different dimensions and presented some figures which show the behavior of the energy of the system versus some parameters of the system.

2. Solutions of Klein–Gordon equation in D dimensions

The Klein–Gordon equation in higher dimensions for spherically symmetric potential reads,[22–25]

$$-\Delta_D \psi_{n,l,m}(r,\Omega_D) = \left\{[E_{n,l} - V(r)]^2 - [m + S(r)]^2\right\} \psi_{n,l.m}(r,\Omega_D), \tag{3}$$

where $E_{n,l}$, m, $V(r)$, and $S(r)$ are the relativistic energy, rest mass, the repulsive vector potential, and the attractive scalar potential respectively, and Δ_D is defined as

$$\Delta_D = \nabla_D^2 = \frac{1}{r^{D-1}} \frac{\partial}{\partial} \left(r^{D-1} \frac{\partial}{\partial r} \right) - \frac{\Lambda_D^2(\Omega_D)}{r^2}. \tag{4}$$

The total wave function in D dimensions is written as,

$$\psi_{n,l,m}(r,\Omega_D) = R_{n,l}(r) \mathrm{Y}_l^m(\Omega_D). \tag{5}$$

The term $\Lambda_D^2(\Omega_D)/r^2$ in Eq. (4) is the generalization of the centrifugal term for the higher dimensional space. The eigenvalues of $\Lambda_D^2(\Omega_D)$ are defined by the relation,

$$\Lambda_D^2(\Omega_D) \mathrm{Y}_l^m(\Omega_D) = l(l + D - 2) \mathrm{Y}_l^m(\Omega_D), \tag{6}$$

where $\mathrm{Y}_l^m(\Omega_D)$, $R_{n,l}$, and l represent the hyperspherical harmonics, the hyper-radial wave function, and the orbital angular momentum quantum number respectively. Now substituting ansatz $R_{n,l}(r) = r^{-(D-1)/2} F_{n,l}(r)$ for the wave function into Eq. (3) yields,

$$\left\{ \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \left(E_{n,l} - V(r)\right)^2 - \left(m + S(r)\right)^2 - \frac{(D + 2l - 1)(D + 2l - 3)}{4r^2} \right\} F_{n,l}(r) = 0. \tag{7}$$

Now considering equal scalar and vector potentials as the improved Manning–Rosen potential $S(r) = V(r)$ in Eq. (7), we obtain the second-order Schrödinger-like equation

$$\left\{ \frac{\mathrm{d}^2}{\mathrm{d}r^2} + E_{n,l}^2 - m^2 - 2(E_{n,l} + m) D_\mathrm{e} \left(1 - \frac{b-1}{\mathrm{e}^{\alpha r} - 1}\right)^2 - \frac{(D + 2l - 1)(D + 2l - 3)}{4r^2} \right\} F_{n,l}(r) = 0. \tag{8}$$

It is well known that equation (8) is exactly solvable when $l = 0$. However, in the presence of the centrifugal term ($l \neq 0$), one can only obtain an approximate solution of Eq. (8). When $\alpha r < 1$, we invoked an improved Greene–Aldrich approximation scheme[4,26] to deal with the centrifugal term as,

$$\frac{1}{r^2} \approx \alpha^2 \left[c_0 + \frac{\mathrm{e}^{\alpha r}}{(\mathrm{e}^{\alpha r} - 1)^2} \right], \tag{9}$$

where $c_0 = 1/12$ is a dimensionless constant obtained from the Taylor expansion of Eq. (9). Substituting Eq. (9) into Eq. (8) yields,

$$-\frac{\mathrm{d}^2 F_{n,l}}{\mathrm{d}r^2} + V_\mathrm{eff}(r) F_{n,l}(r) = \tilde{E}_{n,l} F_{n,l}, \tag{10}$$

where,

$$\begin{aligned} V(r) &= \frac{\chi \mathrm{e}^{2\alpha r} + \rho \mathrm{e}^{\alpha r} + \lambda}{(\mathrm{e}^{\alpha r} - 1)^2}, \\ \chi &= 2(E_{n,J} + m) D_\mathrm{e}, \\ \rho &= -\left(4b(E_{n,l} + m) D_\mathrm{e} + \frac{(D + 2l - 1)(D + 2l - 3)\alpha^2}{4}\right), \\ \lambda &= 2(E_{n,l} + m) D_\mathrm{e} b^2, \end{aligned} \tag{11}$$

$$\tilde{E}_{n,l} = E_{n,l}^2 - m^2 - \frac{\alpha^2 (D + 2l - 1)(D + 2l - 3) c_0}{4}. \tag{12}$$

In the SUSYQM formulation, the ground-state wave function $F_{0,l}(r)$ is given by[27] (see Appendix A)

$$F_{0,l}(r) = \exp\left(-\int W(r)\,\mathrm{d}r\right), \tag{13}$$

in which the integrand is called the superpotential and the Hamiltonian is composed of the raising and lowering operators

$$H_- = \hat{A}^+ \hat{A} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_-(r), \tag{14}$$

$$H_+ = \hat{A}\hat{A}^+ = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_+(r), \tag{15}$$

with

$$\hat{A} = \frac{\mathrm{d}}{\mathrm{d}r} - W(r), \tag{16}$$

$$\hat{A}^+ = -\frac{\mathrm{d}}{\mathrm{d}r} - W(r), \tag{17}$$

$$V_\pm(r) = W^2(r) \mp W'(r). \tag{18}$$

Thus, we have to first solve the associated Riccati equation

$$W^2(r) \mp W'(r) = V_\mathrm{eff}(r) - \tilde{E}_{0,l}, \tag{19}$$

for which we propose a solution of the form

$$W(r) = \frac{-f \mathrm{e}^{\alpha r}}{(\mathrm{e}^{\alpha r} - 1)} + g. \tag{20}$$

Substituting Eqs. (11) and (20) into Eq. (19), we obtain the following relationship

$$f = \frac{\alpha \pm \sqrt{\alpha^2 + 4(\chi + \rho + \lambda)}}{2}, \tag{21}$$

$$g = \left(\frac{f^2 + \lambda - \chi}{2f}\right), \tag{22}$$

$$\tilde{E}_{0,l} = -\left(\frac{f^2+\lambda-\chi}{2f}\right)^2 + \lambda. \quad (23)$$

From Eq. (A2), we can obtain the supersymmetric partner potentials as

$$V_+(r) = \frac{f(f+\alpha)\,\mathrm{e}^{\alpha r}}{(\mathrm{e}^{\alpha r}-1)^2} + \frac{(\chi-\lambda)\,\mathrm{e}^{\alpha r}}{(\mathrm{e}^{\alpha r}-1)} + (g)^2,$$
$$V_-(r) = \frac{f(f-\alpha)\,\mathrm{e}^{\alpha r}}{(\mathrm{e}^{\alpha r}-1)^2} + \frac{(\chi-\lambda)\,\mathrm{e}^{\alpha r}}{(\mathrm{e}^{\alpha r}-1)} + (g)^2. \quad (24)$$

The shape invariancy in this case holds via the mapping

$$a_n = f(a_0) = a_0 + n\alpha = f + n\alpha, \quad (25)$$

and therefore the residuals are

$$R(a_1) = \left(\frac{(a_0)^2+\lambda-\chi}{2a_0}\right)^2 - \left(\frac{(a_1)^2+\lambda-\chi}{2a_1}\right)^2,$$
$$R(a_2) = \left(\frac{(a_1)^2+\lambda-\chi}{2a_1}\right)^2 - \left(\frac{(a_2)^2+\lambda-\chi}{2a_2}\right)^2,$$
$$\cdots$$
$$R(a_n) = \left(\frac{(a_{n-1})^2+\lambda-\chi}{2a_{n-1}}\right)^2 - \left(\frac{(a_n)^2+\lambda-\chi}{2a_n}\right)^2. \quad (26)$$

The energy eigenvalues can be obtained as follows:

$$\tilde{E}_{n,l} = \tilde{E}^-_{n,l} + \tilde{E}_{0,l}, \quad (27)$$

where,

$$\tilde{E}^-_{n,l} = \sum_{k=1}^{n} R(a_k) = \left(\frac{(a_0)^2+\lambda-\chi}{2a_0}\right)^2 - \left(\frac{(a_n)^2+\lambda-\chi}{2a_n}\right)^2. \quad (28)$$

Finally, the energy relation is written as

$$E^2_{n,l} - m^2 - \frac{\alpha^2(D+2l-1)(D+2l-3)c_0}{4}$$
$$= 2(E_{n,l}+m)D_e b^2 - \left(\frac{(a_n)^2+2(E_{n,l}+m)D_e\left(b^2-1\right)}{2a_n}\right)^2. \quad (29)$$

In order to find the corresponding wave function, we define a new variable $y = \mathrm{e}^{-\alpha r}$ and substituting it into Eq. (10), we obtain the following second-order differential equation,

$$\frac{\mathrm{d}^2F_{n,l}(y)}{\mathrm{d}y^2} - \frac{1-y}{y(1-y)}\frac{\mathrm{d}F_{n,l}(y)}{\mathrm{d}y} + \frac{1}{y^2(1-y)^2}\left\{-\gamma_1 y^2 + \gamma_2 y - \gamma_3\right\}F_{n,l}(y) = 0, \quad (30)$$

where we define the parameters γ_1, γ_2, and γ_3 as follows:

$$\gamma_1 = \frac{1}{\alpha^2}\left(\lambda - \tilde{E}_{n,l}\right), \quad (31)$$
$$\gamma_1 = -\frac{1}{\alpha^2}\left(\rho + 2\tilde{E}_{n,l}\right), \quad (32)$$
$$\gamma_3 = \frac{1}{\alpha^2}\left(\chi - \tilde{E}_{n,l}\right). \quad (33)$$

By using Eqs. (31)–(33) and Eq. (30), the wave function $F_{n,l}(r)$ can be determined as,

$$F_{n,l}(r) = N\left(\mathrm{e}^{-\alpha r}\right)^{\sqrt{\gamma_3}}\left(1-\mathrm{e}^{-\alpha r}\right)^{1/2+\sqrt{1/4+\gamma_1-\gamma_2+\gamma_3}} \times P_n^{\left(2\sqrt{\gamma_3},2\sqrt{1/4+\gamma_1-\gamma_2+\gamma_3}\right)}\left(1-2\mathrm{e}^{-\alpha r}\right), \quad (34)$$

where N is the normalization constant.

In Table 1, we have reported some numerical results for different states and different dimensions. We see that we have obtained degenerated states when l increases to $l+1$ and D reduces two units, i.e., to $D-2$. In Fig. 1 we have plotted the wave function of the system for $|0,0\rangle$, $|1,0\rangle$, and $|2,0\rangle$ states. The behavior of the energy of the system under the improved Manning–Rosen potential versus D_e, r_e, and α are indicated in Figs. 2–4, respectively. It is clear that as D_e and $r_e(\alpha)$ increase (s), the energy of the system has decreasing (increasing) behavior.

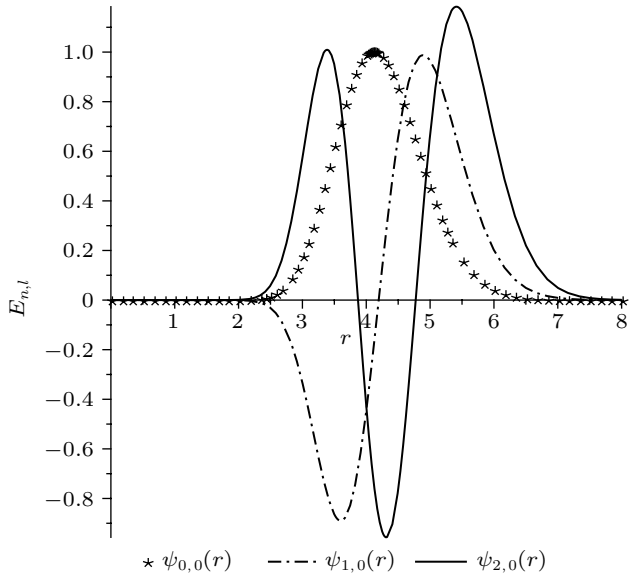


Fig. 1. The wave function of the system for different states $m = 2$, $D_e = 7$, $r_e = 4$, $\alpha = 0.1$, $D = 3$.

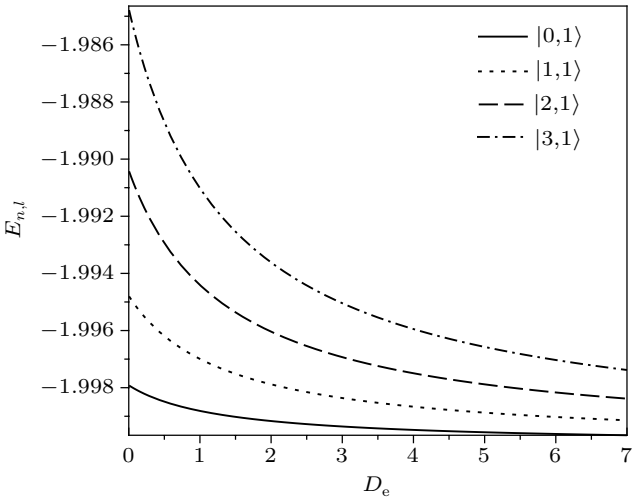


Fig. 2. Behavior of the energy of the system versus D_e and $m = 2, r_e = 4$, $\alpha = 0.1$.

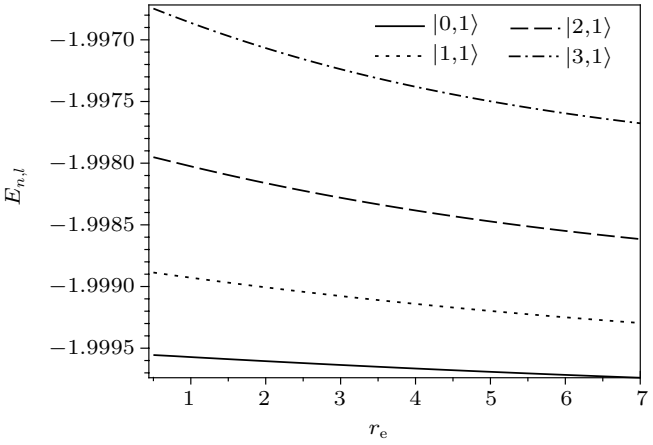


Fig. 3. Behavior of the energy of the system versus r_e and $m = 2, D_e = 7$, $\alpha = 0.1$.

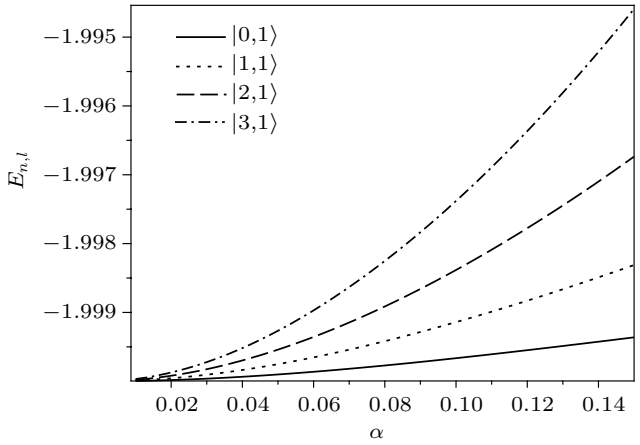


Fig. 4. Behavior of the energy of the system versus α and $m = 2$, $D_e = 7$, $r_e = 4$.

Table 1. Energy of the system for different states and dimensions and $m = 2$, $D_e = 7$, $r_e = 4$, $\alpha = 0.1$.

	l	$E^D_{n,l}\,(D=1)$	$E^D_{n,l}\,(D=2)$	$E^D_{n,l}\,(D=3)$	$E^D_{n,l}\,(D=4)$
$n=0$	0	−1.999896703	−1.999953926	−1.999896703	−1.999797423
		2.526144766	2.523004775	2.526144766	2.535534054
	1	−1.999896703	−1.999797423	−1.999664773	−1.999499031
		2.526144766	2.535534054	2.551081928	2.572641543
	2	−1.999664773	−1.999499031	−1.999300253	−1.999068456
		2.551081928	2.572641543	2.600015995	2.632965995
	3	−1.999300253	−1.999068456	−1.998803646	−1.998505826
		2.600015995	2.632965995	2.671218502	2.714475676
	4	−1.998803646	−1.998505826	−1.998174995	−1.997811152
		2.671218502	2.714475676	2.762423536	2.814739926
$n=1$	0	−1.999560934	−1.999665772	−1.999560934	−1.999370769
		3.391169862	3.388851062	3.391169862	3.398112092
	1	−1.999560934	−1.999370769	−1.999140892	−1.998876167
		3.391169862	3.398112092	3.409635614	3.425671446
	2	−1.999140892	−1.998876167	−1.998577689	−1.998245833
		3.409635614	3.425671446	3.446125555	3.470881207
		3.446125555	3.470881207	3.499801755	3.532733716
	3	−1.998577689	−1.998245833	−1.997880756	−1.997482537
		3.446125555	3.470881207	3.499801755	3.532733716
	4	−1.997880756	−1.997482537	−1.997051218	−1.996586820
		3.499801755	3.532733716	3.569509984	3.609953044
$n=2$	0	−1.998963163	−1.999089518	−1.998963163	−1.998701785
		4.111614867	4.109748739	4.111614867	4.117204973
	1	−1.998963163	−1.998701785	−1.998383442	−1.998024415
		4.111614867	4.117204973	4.126494371	4.139442443
	2	−1.998383442	−1.998024415	−1.997629093	−1.997199078
		4.126494371	4.139442443	4.155993398	4.176077285
	3	−1.997629093	−1.997199078	−1.996735078	−1.996237452
		4.155993398	4.176077285	4.199611228	4.226500826
	4	−1.996735078	−1.996237452	−1.995706400	−1.995142039
		4.199611228	4.226500826	4.256641666	4.289920902

3. The oscillator strength for the modified Manning–Rosen potential

In quantum mechanics it is known that the absorption of light brings about a transition from one quantum state to another. The resulting transition probability density plays an important role in many physical systems such as stellar to subatomic physics.[28] Importantly, the idea of oscillator strength gives information on the fine structure and selection rules of the optical absorption in physics and chemistry.[29] In the transition from a lower state ψ_i to an upper state ψ_j, we have the following form for the absorption oscillator strength:

$$f^l_{ij} = \frac{2}{3}\frac{m}{\hbar^2}\left(E_j - E_i\right)\left|\left\langle \psi_j\right| r\left|\right\rangle \psi_i\right\rangle\Big|^2, \tag{35}$$

where f^l_{ij} is the length strength. In Figs. 5–7, the variation of the length strength versus D_e, r_e, and α is shown. It is obvious that as these parameters increase, the value of the length

strength decreases.

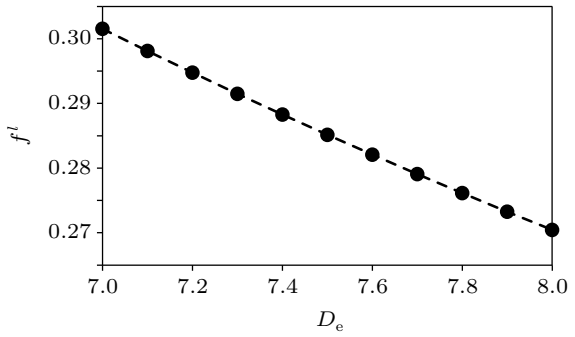


Fig. 5. Variation of the length strength versus D_e for $m = 2$, $r_e = 4$, $\alpha = 0.1$, $D = 3$.

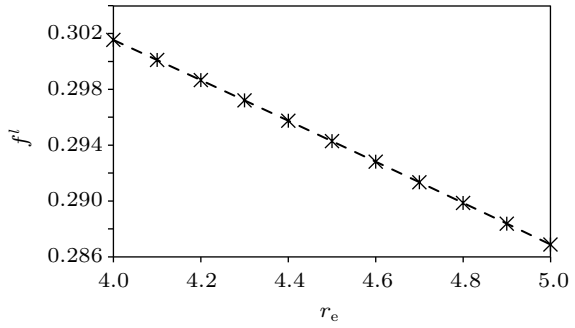


Fig. 6. Variation of the length strength versus r_e for $m = 2$, $D_e = 7$, $r_e = 4$, $\alpha = 0.1$, $D = 3$.

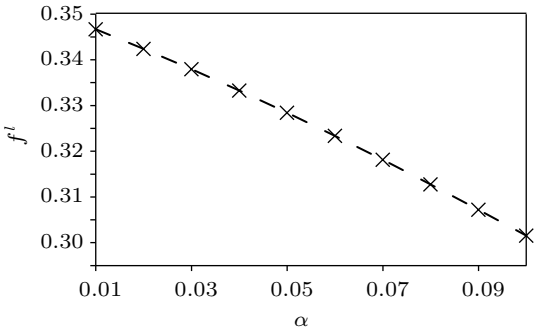


Fig. 7. Variation of the length strength versus α for $m = 2$, $D_e = 7$, $r_e = 4$, $D = 3$.

4. Conclusions

In this paper, we have calculated approximately the energy eigenvalues for any l state and the corresponding eigenfunctions in D dimensions for improved Manning–Rosen potential recently suggested in Refs. [4] and [5] using the SUSYQM formulation within the framework of the relativistic Klein–Gordon equation. In obtaining the approximate solution, we apply the improved Greene–Aldrich approximation scheme to the centrifugal term. Finally, we can say that the results of our work are interesting for both theoretical and experimental physicists.

Acknowledgments

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Appendix A: Supersymmetry quantum mechanics

In the SUSYQM we normally deal with the partner Hamiltonians

$$H_{\pm} = \frac{p^2}{2m} + V_{\pm}(x), \tag{A1}$$

where

$$V_{\pm}(x) = \Phi^2(x) \pm \Phi'(x). \tag{A2}$$

In the case of good SUSY, i.e., $E_0 = 0$, the ground state of the system is obtained via

$$\phi_0^-(x) = C\mathrm{e}^{-U}, \tag{A3}$$

where C is a normalization constant and

$$U(x) = \int_{x_0}^{x} \mathrm{d}z \Phi(z). \tag{A4}$$

Next, if the shape-invariant condition

$$V_+(a_0,x) = V_-(a_1,x) + R(a_1), \tag{A5}$$

where a_1 is a new set of parameters uniquely determined from the old set a_0 via the mapping $F : a_0 \mapsto a_1 = F(a_0)$ and $R(a_1)$ does not include x, the higher state solutions are obtained via

$$E_n = \sum_{s=1}^{n} R(a_s), \tag{A6a}$$

$$\phi_n^-(a_0,x) = \prod_{s=0}^{n-1} \left(\frac{A^+(a_s)}{[E_n - E_s]^{1/2}} \right) \phi_0^-(a_n,x), \tag{A6b}$$

$$\phi_0^-(a_n,x) = C\exp\left\{ -\int_0^x \mathrm{d}z \Phi(a_n,z) \right\}, \tag{A6c}$$

where

$$A_s^+ = -\frac{\partial}{\partial x} + \Phi(a_s,x). \tag{A7}$$

Therefore, this condition determines the spectrum of the bound states of the Hamiltonian

$$H_s = -\frac{\partial^2}{\partial x^2} + V_-(a_s,x) + E_s, \tag{A8}$$

and the energy eigenfunctions of

$$H_s \phi_{n-s}^-(a_s,x) = E_n \phi_{n-s}^-(a_s,x), \quad n \geq s \tag{A9}$$

are related via[27–29]

$$\phi_{n-s}^-(a_s,x) = \frac{A^+}{[E_n - E_s]^{1/2}} \phi_{n-(s+1)}^-(a_{s+1},x). \tag{A10}$$

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