Wave functions of a new kind of nonlinear single-mode squeezed state*

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We explore the theoretical possibility of extending the usual squeezed state to those produced by nonlinear single-mode squeezing operators. We derive the wave functions of $\exp[-(ig/2)(\sqrt{1-X^2}P+P\sqrt{1-X^2})]|0\rangle$ in the coordinate representation. A new operator's disentangling formula is derived as a by-product.

Keywords: wave functions, nonlinear single-mode squeezed states, coordinate representation, disentangling formula

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1. Introduction

Squeezed state theory has been an important topic not only in quantum optics but also in quantum communication because it can describe a light field that has less quantum fluctuation in one quadrature than the coherent state.[1–3] Theoretically, the single-mode squeezed state is produced by operating the squeezing operator

$$S=\exp\left[\frac{\lambda}{2}\left(a^2-a^{\dagger 2}\right)\right], \tag{1}$$

on the vacuum state $|0\rangle$, where $|0\rangle$ is annihilated by the photon annihilation operator, $a|0\rangle=0$, $\left[a,a^{\dagger}\right]=1$. In Ref. [4] we have shown that the squeezing operator possesses a natural representation in the coordinate representation

$$\exp\left[\frac{\lambda}{2}\left(a^2-a^{\dagger 2}\right)\right]=\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sqrt{\mu}}\left|x/\mu\right\rangle\langle x|,\quad \mu=\mathrm{e}^{\lambda}, \tag{2}$$

where $|x\rangle$ is the coordinate eigenstate which in Fock space is expressed as

$$\begin{aligned}&|x\rangle=\pi^{-1/4}\exp\left[-\frac{1}{2}x^2+\sqrt{2}xa^{\dagger}-\frac{a^{\dagger 2}}{2}\right]|0\rangle,\\ &X|x\rangle=x|x\rangle,\end{aligned} \tag{3}$$

the integration in Eq. (2) can be performed by using the technique of integration within normally ordered product (IWOP) of operators and $|0\rangle\langle 0|=:\mathrm{e}^{-a^{\dagger}a}:$[5–7]

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sqrt{\mu}}\left|x/\mu\right\rangle\langle x|\\ &=\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sqrt{\pi\mu}}:\exp\left[-\frac{x^2}{2}\left(1+\frac{1}{\mu^2}\right)\right.\\ &\quad\left.+\sqrt{2}x\left(\frac{a^{\dagger}}{\mu}+a\right)-\frac{a^{\dagger 2}+a^2}{2}-a^{\dagger}a\right]:\\ &=\mathrm{sech}^{1/2}\lambda:\exp\left[-\frac{a^{\dagger 2}+a^2}{2}\tanh\lambda+a^{\dagger}a\left(\mathrm{sech}\,\lambda-1\right)\right]:\\ &=\exp\left(-\frac{a^{\dagger 2}}{2}\tanh\lambda\right)\exp\left[\left(a^{\dagger}a+\frac{1}{2}\right)\ln\mathrm{sech}\,\lambda\right]\\ &\quad\times\exp\left(\frac{a^2}{2}\tanh\lambda\right)\\ &=\exp\left[\frac{\lambda}{2}\left(a^2-a^{\dagger 2}\right)\right].\end{aligned} \tag{4}$$

Using

$$X=\frac{1}{\sqrt{2}}\left(a+a^{\dagger}\right),\quad P=\frac{1}{\sqrt{2}i}\left(a-a^{\dagger}\right),\quad [X,P]=\mathrm{i}, \tag{5}$$

we see

$$\exp\left[\frac{\lambda}{2}\left(a^2-a^{\dagger 2}\right)\right]=\exp\left[\mathrm{i}\lambda\left(PX+\frac{\mathrm{i}}{2}\right)\right]. \tag{6}$$

In this work we let $X\to\sqrt{1-X^2}$ (a nonlinear factor) to see if $\exp[-\mathrm{i}gP\sqrt{1-X^2}]$, whose exponential part includes the nonlinear term $P\sqrt{1-X^2}$, plays the role of squeezing. Our motivation arises from the observation that a squeezed state is experimentally produced in some nonlinear media. We hope to know the wave function of $\exp[-\mathrm{i}gP\sqrt{1-X^2}]|0\rangle$. For this aim we shall first deduce a new operator identity and then examine the wave function of $\exp[\mathrm{i}g\sqrt{1-X^2}P]|0\rangle$. The sharp difference between these two wavefunctions warns us that we should be very cautious to treat nonlinear operators in quantum mechanics.

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2. Wave function of $\exp[ig\sqrt{1-X^2}P]|0\rangle$ in the coordinate representation

From

$$\exp\left[g\frac{\mathrm{d}}{\mathrm{d}x}\right]f(x)=f(x+g), \tag{7}$$

and setting

$$x=\mathrm{e}^{y},\ \ \frac{\mathrm{d}}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}y}=\mathrm{e}^{-y}\frac{\mathrm{d}}{\mathrm{d}y},$$

we have

$$\begin{aligned}\exp\left[gx\frac{\mathrm{d}}{\mathrm{d}x}\right]f(x) &= \exp\left[g\frac{\mathrm{d}}{\mathrm{d}y}\right]f(\mathrm{e}^{y})\\ &= f\left(\mathrm{e}^{y+g}\right)\\ &= f(\mathrm{e}^{g}x).\end{aligned} \tag{8}$$

On the other hand, in Eq. (7) by letting

$$x=\sin y,\ \ \frac{\mathrm{d}}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}y}=\frac{1}{\cos y}\frac{\mathrm{d}}{\mathrm{d}y}=\frac{1}{\sqrt{1-x^2}}\frac{\mathrm{d}}{\mathrm{d}y},$$

we see

$$\begin{aligned}&\exp\left[g\sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x}\right]f(x)\\ &=\exp\left[g\frac{\mathrm{d}}{\mathrm{d}y}\right]f(\sin y)\\ &=f[\sin(y+g)]\\ &=f[\sin y\cos g+\cos y\sin g]\\ &=f\left[x\cos g+\sqrt{1-x^2}\sin g\right].\end{aligned} \tag{9}$$

Due to

$$P|x\rangle=\frac{\mathrm{i}d}{\mathrm{d}x}|x\rangle,$$

so equation (9) implies

$$\begin{aligned}&\exp\left[-\mathrm{i}gP\sqrt{1-X^2}\right]|x\rangle\\ &=\exp\left[g\sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x}\right]|x\rangle\\ &=\left|x\cos g+\sqrt{1-x^2}\sin g\right\rangle.\end{aligned} \tag{10}$$

Taking its Hermite conjugate we obtain

$$\langle x|\exp\left[\mathrm{i}g\sqrt{1-X^2}P\right]=\left\langle x\cos g+\sqrt{1-x^2}\sin g\right|, \tag{11}$$

which exhibits nonlinearity. Through the form of Eq. (11) we can see the squeezing factor explicitly. Using Eq. (11) we obtain the wave function of the state $\exp[\mathrm{i}g\sqrt{1-X^2}P]|0\rangle$

$$\begin{aligned}&\langle x|\exp\left[\mathrm{i}g\sqrt{1-X^2}P\right]|0\rangle\\ &=\left\langle x\cos g+\sqrt{1-x^2}\sin g\middle|0\right\rangle,\end{aligned} \tag{12}$$

using Eq. (3) and $a|0\rangle=0$, we have

$$\begin{aligned}&\left\langle x\cos g+\sqrt{1-x^2}\sin g\right|0\rangle\\ &=\pi^{-1/4}\langle 0|\exp\left[-\frac{1}{2}\left(x\cos g+\sqrt{1-x^2}\sin g\right)^2\right.\\ &\quad\left.+\sqrt{2}\left(x\cos g+\sqrt{1-x^2}\sin g\right)a-\frac{a^2}{2}\right]|0\rangle\\ &=\pi^{-1/4}\langle 0|\exp\left[-\frac{1}{2}\left(x\cos g+\sqrt{1-x^2}\sin g\right)^2\right]|0\rangle\\ &=\pi^{-1/4}\exp\left[\frac{-1}{2}\left(x^2\cos 2g\right.\right.\\ &\quad\left.\left.+x\sqrt{1-x^2}\sin 2g+\sin^2 g\right)\right],\end{aligned} \tag{13}$$

then we obtain

$$\begin{aligned}&\langle x|\exp\left[\mathrm{i}g\sqrt{1-X^2}P\right]|0\rangle\\ &=\pi^{-1/4}\exp\left[\frac{-1}{2}\left(x^2\cos 2g\right.\right.\\ &\quad\left.\left.+x\sqrt{1-x^2}\sin 2g+\sin^2 g\right)\right].\end{aligned} \tag{14}$$

3. A new disentangling formula

In Eqs. (9) and (10) we have discussed the expression $\exp[-\mathrm{i}gP\sqrt{1-X^2}]|x\rangle$, it is natural to ask the action of $\exp[-\mathrm{i}g\sqrt{1-X^2}P]$ on $|x\rangle$, in fact, using formula

$$[f(X),P]=\mathrm{i}\frac{\partial f(X)}{\partial X},\ \ (\hbar=1),$$

we have

$$\left[\sqrt{1-X^2},P\right]=-\mathrm{i}\frac{X}{\sqrt{1-X^2}},$$

then

$$\begin{aligned}&\exp\left[-\mathrm{i}g\sqrt{1-X^2}P\right]|x\rangle\\ &=\exp\left[-\mathrm{i}g\left(P\sqrt{1-X^2}+\left[\sqrt{1-X^2},P\right]\right)\right]|x\rangle\\ &=\exp\left[-\mathrm{i}gP\sqrt{1-X^2}-g\frac{X}{\sqrt{1-X^2}}\right]|x\rangle\\ &=\exp\left[g\sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x}-g\frac{x}{\sqrt{1-x^2}}\right]|x\rangle,\end{aligned} \tag{15}$$

thus we need to disentangle

$$\exp\left[g\sqrt{1-x^2}(\mathrm{d}/\mathrm{d}x)-gx/\sqrt{1-x^2}\right].$$

For this aim, we propose a new disentangling formula

$$\begin{aligned}&\exp\left[\lambda\frac{\mathrm{d}}{\mathrm{d}y}+\lambda f(y)\right]\\ &=\exp\left[\int_0^{\lambda}f(y+t)\,\mathrm{d}t\right]\exp\left[\lambda\frac{\mathrm{d}}{\mathrm{d}y}\right].\end{aligned} \tag{16}$$

or equivalently

$$\begin{aligned}&\exp[\mathrm{i}\lambda P+\lambda f(X)]\\ &=\exp\left[\int_0^{\lambda}f(X+t)\,\mathrm{d}t\right]\exp[\mathrm{i}\lambda P].\end{aligned} \tag{17}$$

We prove Eq. (16) as follows:

Defining

$$\begin{aligned} O(\lambda) &= \exp[\lambda A]\exp[\lambda B] \\ &= \exp\left[\lambda\frac{\mathrm{d}}{\mathrm{d}y}+\lambda f(y)\right]\exp\left[-\lambda\frac{\mathrm{d}}{\mathrm{d}y}\right], \end{aligned} \tag{18}$$

where

$$\begin{aligned} &A=\frac{\mathrm{d}}{\mathrm{d}y}+f(y),\ \ B=-\frac{\mathrm{d}}{\mathrm{d}y}, \\ &\left[\frac{\mathrm{d}}{\mathrm{d}y},f(y)\right]=f'(y), \end{aligned} \tag{19}$$

we see

$$\begin{aligned} &[A,B]=f'(y), \\ &[A,[A,B]]=f''(y), \\ &\underbrace{[A,[\ldots,[A}_{n},B]\ldots]=f^{(n)}(y). \end{aligned} \tag{20}$$

It then follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda}O(\lambda) &= AO(\lambda)+\exp[\lambda A]B\exp[\lambda B] \\ &= AO(\lambda)+\exp[\lambda A]B\exp[-\lambda A]O(\lambda) \\ &= (A+B)O(\lambda)+\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\underbrace{[A,[\ldots,[A}_{n},B]\ldots]O(\lambda) \\ &= f(y)O(\lambda)+\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\underbrace{[A,[\ldots,[A}_{n},B]\ldots]O(\lambda) \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}f^{(n)}(y)O(\lambda) \\ &= f(y+\lambda)O(\lambda), \end{aligned} \tag{21}$$

thus

$$\begin{aligned} O(\lambda) &= \exp\left[\int_0^{\lambda}f(y+t)\,\mathrm{d}t\right]O(0) \\ &= \exp\left[\int_0^{\lambda}f(y+t)\,\mathrm{d}t\right]. \end{aligned} \tag{22}$$

Comparing Eq. (22) with Eq. (18) we obtain Eq. (16), QED.

Now we consider the following exponential operator

$$O(g,h)=\exp\left\{-\mathrm{i}gP\sqrt{1-X^2}-gh\frac{X}{\sqrt{1-X^2}}\right\}, \tag{23}$$

where h is a real parameter, since

$$O(g,h=1)=\exp[-\mathrm{i}g\sqrt{1-X^2}P].$$

Operating $O(g,h)$ on $|x\rangle$ yields

$$O(g,h)|x\rangle=\exp\left[g\sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x}-gh\frac{x}{\sqrt{1-x^2}}\right]|x\rangle. \tag{24}$$

Letting

$$x=\sin y,\ \ \frac{\mathrm{d}}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}y}=\frac{1}{\cos y}\frac{\mathrm{d}}{\mathrm{d}y}=\frac{1}{\sqrt{1-x^2}}\frac{\mathrm{d}}{\mathrm{d}y},$$

we have

$$g\sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x}-gh\frac{x}{\sqrt{1-x^2}}=g\frac{\mathrm{d}}{\mathrm{d}y}+g(-h\tan y). \tag{25}$$

Using Eq. (16) we see

$$\begin{aligned} &O(g,h)|x\rangle \\ &=\exp\left[g\frac{\mathrm{d}}{\mathrm{d}y}+g(-h\tan y)\right]|\sin y\rangle \\ &=\exp\left[-h\int_0^g\tan(y+t)\,\mathrm{d}t\right]\exp\left[g\frac{\mathrm{d}}{\mathrm{d}y}\right]|\sin y\rangle \\ &=\left[\frac{\cos(y+g)}{\cos y}\right]^h|\sin(y+g)\rangle \\ &=\left[\cos g-\frac{x}{\sqrt{1-x^2}}\sin g\right]^h\left|x\cos g+\sqrt{1-x^2}\sin g\right\rangle. \end{aligned} \tag{26}$$

Then we employ Eq. (10) to derive

$$\begin{aligned} O(g,h)|x\rangle &= \left[\cos g-\frac{x}{\sqrt{1-x^2}}\sin g\right]^h \\ &\quad\times\exp\left\{-\mathrm{i}gP\sqrt{1-X^2}\right\}|x\rangle, \end{aligned} \tag{27}$$

thus

$$\begin{aligned} O(g,h) &= \exp\left\{-\mathrm{i}gP\sqrt{1-X^2}\right\} \\ &\quad\times\left[\cos g-\frac{X}{\sqrt{1-X^2}}\sin g\right]^h. \end{aligned} \tag{28}$$

4. Wave function of $\exp\left[\mathrm{i}gP\sqrt{1-X^2}\right]|0\rangle$ in the coordinate representation

It then follows from Eqs. (15), (23), and Eq. (28) that

$$\begin{aligned} &\exp\left[-\mathrm{i}g\sqrt{1-X^2}P\right] \\ &=\exp\left[-\mathrm{i}gP\sqrt{1-X^2}-g\frac{X}{\sqrt{1-X^2}}\right] \\ &=O(g,h=1) \\ &=\exp\left\{-\mathrm{i}gP\sqrt{1-X^2}\right\}\left[\cos g-\frac{X}{\sqrt{1-X^2}}\sin g\right] \end{aligned} \tag{29}$$

and

$$\begin{aligned} &\exp\left[-\mathrm{i}g\sqrt{1-X^2}P\right]|x\rangle \\ &=\left[\cos g-\frac{x}{\sqrt{1-x^2}}\sin g\right]\exp\left\{-\mathrm{i}gP\sqrt{1-X^2}\right\}|x\rangle. \end{aligned} \tag{30}$$

Its Hermite conjugate gives the wave function of $\exp\left[\mathrm{i}gP\sqrt{1-X^2}\right]|0\rangle$

$$\begin{aligned} &\langle x|\exp\left[\mathrm{i}gP\sqrt{1-X^2}\right]|0\rangle \\ &=\left[\cos g-\frac{x}{\sqrt{1-x^2}}\sin g\right]\langle x|\exp\left\{\mathrm{i}g\sqrt{1-X^2}P\right\}|0\rangle \\ &=\pi^{-1/4}\left[\cos g-\frac{x}{\sqrt{1-x^2}}\sin g\right] \end{aligned}$$

$$\times \exp\left[\frac{-1}{2}\left(x^2\cos 2g + x\sqrt{1-x^2}\sin 2g + \sin^2 g\right)\right]. \quad (31)$$

Comparing Eq. (31) with Eq. (14) we see a big difference. This teaches us that one should be very cautious when dealing with nonlinear operators in quantum mechanics; actually, treating nonlinear operators Hermitian or unitarity is a challenge to theoreticians.

5. Wave function of $\exp[-ig(\sqrt{1-X^2}P + P\sqrt{1-X^2})/2]\,|0\rangle$

Based on the above results Eq. (23) and Eq. (28) we have

$$\begin{aligned}
&\exp\left[-\frac{ig}{2}\left(\sqrt{1-X^2}P + P\sqrt{1-X^2}\right)\right] \\
&= \exp\left[-igP\sqrt{1-X^2} - \frac{g}{2}\frac{X}{\sqrt{1-X^2}}\right] \\
&= O(g,1/2) \\
&= \exp\left\{-igP\sqrt{1-X^2}\right\}\left[\cos g - \frac{X}{\sqrt{1-X^2}}\sin g\right]^{1/2}. \quad (32)
\end{aligned}$$

Then we use Eq. (10) to know

$$\begin{aligned}
&\langle 0|\exp\left[-\frac{ig}{2}\left(\sqrt{1-X^2}P + P\sqrt{1-X^2}\right)\right]|x\rangle \\
&= \left[\cos g - \frac{x}{\sqrt{1-x^2}}\sin g\right]^{1/2}\left\langle 0\left|x\cos g + \sqrt{1-x^2}\sin g\right.\right\rangle \\
&= \pi^{-1/4}\left[\cos g - \frac{x}{\sqrt{1-x^2}}\sin g\right]^{1/2} \\
&\quad\times \exp\left[\frac{-1}{2}\left(x^2\cos 2g + x\sqrt{1-x^2}\sin 2g + \sin^2 g\right)\right]. \quad (33)
\end{aligned}$$

In summary, for the first time we derive the wave function of the nonlinear squeezed state

$$\langle 0|\exp\left[-\frac{ig}{2}\left(\sqrt{1-X^2}P + P\sqrt{1-X^2}\right)\right]|x\rangle.$$

A new operator's disentangling formula has been derived as a by-product.

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