Some new generating function formulae of the two-variable Hermite polynomials and their application in quantum optics*

Zhan De-Hui(展德会)[a)b)†] and Fan Hong-Yi(范洪义)[a)]

[a)] Department of Material Science and Engineering, University of Science and Technology of China, Hefei 230026, China
[b)] College of Mechanic and Electronic Engineering, Wuyi University, Wuyishan 354300, China

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We derive some new generating function formulae of the two-variable Hermite polynomials, such as $\sum_{n=0}^{\infty}\frac{t^m}{m!}\mathrm{H}_{n,2m}(x)$, $\sum_{n=0}^{\infty}\frac{s^n t^m}{n!m!}\mathrm{H}_{2n,2m}(x,y)$, and $\sum_{n=0}^{\infty}\frac{s^n t^m}{n!m!}\mathrm{H}_{2n+l,2m+k}(x,y)$. We employ the operator Hermite polynomial method and the technique of integration within an ordered product of operators to solve these problems, which will be useful in constructing new optical field states.

Keywords: generating function, two-variable Hermite polynomials, Hermite polynomial method, technique of integral within an ordered product of operators

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1. Introduction

Hermite polynomial $\mathrm{H}_n(x)$, as a special function, is widely used in quantum mechanics and mathematical physics. Physically, $\mathrm{H}_n(x)$ is an eigenfunction of harmonic oscillator. Mathematically, the single-variable Hermite polynomial $\mathrm{H}_n(x)$ is an eigenfunction of fractional Fourier transformation. $\mathrm{H}_n(x)$ can be defined from its generating function

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_n(x)=\mathrm{e}^{2tx-t^2}, \tag{1}$$

it then can be obtained as

$$\mathrm{H}_n(x)=\frac{\partial^n}{\partial t^n}\exp\left(2xt-t^2\right)|_{t=0}.$$

In a preceding paper[1] by combining the operator Hermite polynomial method and the technique of integration within an ordered product (IWOP) of operators, we have derived the generating function of even- and odd-Hermite polynomials for the single-variable case, e.g.,

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{2n}(x)=\frac{1}{\sqrt{1+4t}}\exp\left(\frac{4t}{1+4t}x^2\right), \tag{2}$$

and

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{2n+1}(x)=\frac{2x}{(1+4t)^{3/2}}\exp\left(\frac{4t}{1+4t}x^2\right), \tag{3}$$

as well as

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{2n+l}(x)=\frac{1}{\left(\sqrt{1+4t}\right)^{l+1}}\mathrm{H}_l\left(\sqrt{\frac{1}{4t+1}}x\right)\times\exp\left(\frac{4t}{1+4t}x^2\right). \tag{4}$$

These formulae can be accumulated in mathematical handbooks and have many applications. We have also shown in Ref. [1] that photon-added squeezed state can be expressed by even- and odd-Hermite polynomials. In this work we shall extend Ref. [1] to derive some generating function of even- and odd-Hermite polynomials for the two-variable case, the two-variable Hermite polynomial $\mathrm{H}_{n,m}(x)$[2–6] can also be defined by its generating function

$$\exp\left(tx+\tau y-t\tau\right)=\sum_{n,m=0}^{\infty}\frac{t^m\tau^n}{m!n!}\mathrm{H}_{m,n}(x,y), \tag{5}$$

which is a two-fold summation. We also can obtain the power series expansion of $\mathrm{H}_{m,n}(x,y)$

$$\begin{aligned}\mathrm{H}_{m,n}(x,y)&=\frac{\partial^{n+m}}{\partial t^m\partial\tau^n}\exp\left(tx+\tau y-t\tau\right)|_{t=\tau=0}\\&=\frac{\partial^m}{\partial t^m}\mathrm{e}^{tx}\frac{\partial^n}{\partial\tau^n}\exp\left(\tau\left(y-t\right)\right)|_{t=\tau=0}\\&=\frac{\partial^m}{\partial t^m}\left[\mathrm{e}^{tx}\left(y-t\right)^n\right]|_{t=0}\\&=\sum_{l=0}\binom{m}{l}\frac{\partial^l}{\partial t^l}\left(y-t\right)^n\frac{\partial^{m-l}}{\partial t^{m-l}}\mathrm{e}^{tx}|_{t=0}\\&=\sum_{l=0}^{\min(m,n)}\frac{m!n!(-1)^l}{l!(m-l)!(n-l)!}x^{m-l}y^{n-l}.\end{aligned} \tag{6}$$

The physical meaning of $\mathrm{H}_{m,n}$ has been explored in Ref. [7]. It can describe the quantum mechanical transition amplitude of a forced oscillator from the number state $|m\rangle$ to $|n\rangle$; it can

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†Corresponding author. E-mail: dhzhan@mail.ustc.edu.cn

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also denote the eigenmode in GRIN waveguide.[8] The two-variable Hermite polynomial is also suitable for describing bipartite entangled state representation.[9] From Eq. (5) we see

$$\sum_{n,m=0}\frac{t^n s^m}{n!m!}\mathrm{H}_{n,m}(x,y)=\mathrm{e}^{sy+t(x-s)}=\sum_{n=0}\frac{t^n}{n!}(x-s)^n\,\mathrm{e}^{sy}, \quad (7)$$

so the one-sided summation over $\mathrm{H}_{n,m}(x,y)$ is

$$\sum_{m=0}\frac{s^m}{m!}\mathrm{H}_{n,m}(x,y)=\mathrm{e}^{sy}(x-s)^n. \quad (8)$$

An interesting question naturally arises: what is the summation over even-order subscript, i.e., $\sum_{n=0}^{\infty}\frac{t^m}{m!}\mathrm{H}_{n,2m}(x)$, and $\sum_{n,m=0}^{\infty}\frac{s^n t^m}{n!m!}\mathrm{H}_{2n,2m}(x,y)$? In Sections 2 and 3 we shall employ the operator Hermite polynomial method and the IWOP technique to solve these problems. In Section 4 we shall perform the summation $\sum_{n=0}^{\infty}\frac{s^n t^m}{n!m!}\mathrm{H}_{2n+l,2m+k}(x,y)$. In Section 5 we apply the results to discuss some problems in quantum optics theory.

2. New formula for one-sided summation $\sum_{m=0}^{\infty}\frac{t^m}{m!}\mathrm{H}_{n,2m}(x,y)$

By virtue of the operator Hermite polynomial method we consider $\sum_{n=0}^{\infty}\frac{t^m}{m!}{:}\,\mathrm{H}_{n,2m}\left(a^{\dagger},a\right){:}$. To begin with, we notice

$$\begin{aligned}\mathrm{e}^{ta^{\dagger}}\mathrm{e}^{\tau a}&={:}\exp\left(\tau a+ta^{\dagger}-\tau t\right){:}\\&=\sum_{n,m=0}^{\infty}\frac{t^n\tau^m}{n!m!}{:}\,\mathrm{H}_{n,m}\left(a^{\dagger},a\right){:}\\&=\sum_{n,m=0}^{\infty}\frac{t^n\tau^m}{n!m!}a^{\dagger n}a^m,\end{aligned} \quad (9)$$

which states that the antinormally ordered form of $a^{\dagger n}a^m$ is

$$a^{\dagger n}a^m={:}\,\mathrm{H}_{n,m}\left(a^{\dagger},a\right){:}, \quad (10)$$

so we have

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{t^m}{m!}{:}\,\mathrm{H}_{n,2m}\left(a^{\dagger},a\right){:}&=\sum_{m=0}^{\infty}\frac{t^m}{m!}a^{\dagger n}a^{2m}\\&=\mathrm{e}^{ta^2}\mathrm{e}^{-ta^2}a^{\dagger n}\mathrm{e}^{ta^2}=\mathrm{e}^{ta^2}\left(a^{\dagger}-2ta\right)^n.\end{aligned} \quad (11)$$

We need to convert $\left(a^{\dagger}-2sa\right)^n$ into antinormally ordering, due to

$$\begin{aligned}&\sum_n\frac{\lambda^n}{n!}\left(a^{\dagger}-2sa\right)^n\\&=\mathrm{e}^{\lambda\left(a^{\dagger}-2sa\right)}={:}\,\mathrm{e}^{\lambda\left(a^{\dagger}-2sa\right)}{:}\exp\left[-\frac{\lambda^2}{2}\left[-2sa,a^{\dagger}\right]\right]\\&={:}\,\mathrm{e}^{\lambda\left(a^{\dagger}-2sa\right)}{:}\,\mathrm{e}^{s\lambda^2}\\&={:}\exp\left[-\left(\mathrm{i}\lambda\sqrt{s}\right)^2+2\left(\mathrm{i}\lambda\sqrt{s}\right)\frac{a^{\dagger}-2sa}{2\mathrm{i}\sqrt{s}}\right]{:}\\&=\sum_n\frac{\lambda^n}{n!}\mathrm{i}^n\left(\sqrt{s}\right)^n{:}\,\mathrm{H}_n\left[\frac{a^{\dagger}-2sa}{2\mathrm{i}\sqrt{s}}\right]{:},\end{aligned} \quad (12)$$

we see

$$\left(a^{\dagger}-2sa\right)^n=\mathrm{i}^n\sqrt{s^n}{:}\,\mathrm{H}_n\left(\frac{a^{\dagger}-2at}{2\mathrm{i}\sqrt{s}}\right){:}, \quad (13)$$

then equation (11) becomes

$$\begin{aligned}&\sum_{m=0}^{\infty}\frac{t^m}{m!}{:}\,\mathrm{H}_{n,2m}\left(a^{\dagger},a\right){:}\\&=\mathrm{i}^n\sqrt{t^n}{:}\,\mathrm{e}^{ta^2}\mathrm{H}_n\left(\frac{a^{\dagger}-2at}{2\mathrm{i}\sqrt{t}}\right){:}.\end{aligned} \quad (14)$$

Since both sides of Eq. (14) are in antinormal ordering, and $a^{\dagger}$ and a are permutable within ${:}\,{:}$, we can conclude the following new summation formula about $\mathrm{H}_{n,2m}$

$$\sum_{m=0}^{\infty}\frac{t^m}{m!}\mathrm{H}_{n,2m}(x,y)=\mathrm{i}^n\sqrt{t^n}\mathrm{H}_n\left(\frac{x-2yt}{2\mathrm{i}\sqrt{t}}\right)\mathrm{e}^{ty^2}. \quad (15)$$

Further, by summing over n and using Eq. (1) we obtain

$$\begin{aligned}\sum_{n,m=0}^{\infty}\frac{s^n t^m}{n!m!}\mathrm{H}_{n,2m}(x,y)&=\sum_{n=0}^{\infty}\frac{s^n\mathrm{i}^n\sqrt{t^n}}{n!}\mathrm{H}_n\left(\frac{x-2yt}{2\mathrm{i}\sqrt{t}}\right)\mathrm{e}^{ty^2}\\&=\exp\left[ts^2+s(x-2yt)\right];\end{aligned} \quad (16)$$

this is the generating function for $\mathrm{H}_{n,2m}(x,y)$, which is quite different from Eq. (5).

3. New formula for summation $\sum_{n,m=0}^{\infty}\frac{s^n t^m}{n!m!}\mathrm{H}_{2n,2m}(x,y)$

Next, instead of examining $\sum_{n,m=0}^{\infty}\frac{s^n t^m}{n!m!}\mathrm{H}_{2n,2m}(x,y)$, we consider the operator case. Using Eq. (10) we have

$$\begin{aligned}&\sum_{n,m=0}^{\infty}\frac{s^n t^m}{n!m!}{:}\,\mathrm{H}_{2n,2m}\left(a^{\dagger},a\right){:}\\&=\sum_{n,m=0}^{\infty}\frac{s^n t^m}{n!m!}a^{\dagger 2n}a^{2m}=\mathrm{e}^{sa^{\dagger 2}}\mathrm{e}^{ta^2}.\end{aligned} \quad (17)$$

In order to convert $\mathrm{e}^{sa^{\dagger 2}}\mathrm{e}^{ta^2}$ into antinormal ordering, we employ the operator identity[10]

$$\begin{aligned}\rho(a,a^{\dagger})=&\int\frac{\mathrm{d}^2\beta}{\pi}{:}\,\langle-\beta|\rho(a,a^{\dagger})|\beta\rangle\\&\times\exp\left[|\beta|^2+\beta^* a-\beta a^{\dagger}+a^{\dagger}a\right]{:},\end{aligned} \quad (18)$$

where $|\beta\rangle$ is a coherent state, $|\beta\rangle=\exp\left[-|\beta|^2/2+\beta a^{\dagger}\right]|0\rangle$, thus

$$\begin{aligned}&\sum_{n,m=0}^{\infty}\frac{s^n t^m}{n!m!}{:}\,\mathrm{H}_{2n,2m}\left(a^{\dagger},a\right){:}\\&=\mathrm{e}^{sa^{\dagger 2}}\mathrm{e}^{ta^2}=\int\frac{\mathrm{d}^2\beta}{\pi}{:}\,\langle-\beta|\,\mathrm{e}^{sa^{\dagger 2}}\mathrm{e}^{ta^2}|\beta\rangle\end{aligned}$$

$$\times \exp\left[|\beta|^2+\beta^* a-\beta a^\dagger+a^\dagger a\right]:$$
$$=\int \frac{\mathrm{d}^2\beta}{\pi}:\mathrm{e}^{s\beta^{*2}}\,\mathrm{e}^{t\beta^2}\exp\left[-|\beta|^2+\beta^* a-\beta a^\dagger+a^\dagger a\right]:$$
$$=\frac{1}{\sqrt{1-4st}}:\exp\left[\frac{-aa^\dagger+sa^{\dagger 2}+ta^2}{1-4st}+a^\dagger a\right]:$$
$$=\frac{1}{\sqrt{1-4st}}:\exp\left[\frac{-4staa^\dagger+sa^{\dagger 2}+ta^2}{1-4st}\right]:. \tag{19}$$

Since both sides are in antinormal ordering we can obtain the following new generating function

$$\sum_{n,m=0}^{\infty}\frac{s^n t^m}{n!m!}\mathrm{H}_{2n,2m}(x,y)$$
$$=\frac{1}{\sqrt{1-4st}}\exp\left[\frac{-4stxy+sx^2+ty^2}{1-4st}\right]. \tag{20}$$

4. New formula for summation

$$\sum_{n,m=0}^{\infty}\frac{s^n t^m}{n!m!}\mathrm{H}_{2n+l,2m+k}(x,y)$$

Now we move to consider the more general case $\sum_{n,m=0}^{\infty}\frac{s^n t^m}{n!m!}\mathrm{H}_{2n+l,2m+k}(x,y)$. Instead, we discuss

$$\sum_{n,m=0}^{\infty}\frac{s^n t^m}{n!m!}:\mathrm{H}_{2n+l,2m+k}\left(a^\dagger,a\right):$$
$$=\sum_{n,m=0}^{\infty}\frac{s^n t^m}{n!m!}a^{\dagger 2n+l}a^{2m+k}=a^{\dagger l}\,\mathrm{e}^{sa^{\dagger 2}}\,\mathrm{e}^{ta^2}a^k. \tag{21}$$

With the integration formula[11] we have

$$\int\frac{\mathrm{d}^2 z}{\pi}z^n z^{*m}\exp\left(\zeta|z|^2+\xi z+\eta z^*+fz^2+gz^{*2}\right)$$
$$=A^{-1/2}\left(\mathrm{i}\sqrt{\frac{g}{A}}\right)^n\left(\mathrm{i}\sqrt{\frac{f}{A}}\right)^m$$
$$\times\sum_{k=0}^{\min[n,m]}\left(\frac{\zeta}{\sqrt{fg}}\right)^k\frac{n!m!}{k!(n-k)!(m-k)!}$$
$$\times\mathrm{H}_{n-k}\left(\frac{2g\xi-\zeta\eta}{2\mathrm{i}\sqrt{gA}}\right)\mathrm{H}_{m-k}\left(\frac{2f\eta-\zeta\xi}{2\mathrm{i}\sqrt{fA}}\right)\mathrm{e}^{W}, \tag{22}$$

where

$$A=\zeta^2-4fg,\quad W=\frac{-\zeta\xi\eta+g\xi^2+f\eta^2}{A}.$$

With Eqs. (22) and (18) we put $a^{\dagger l}\,\mathrm{e}^{sa^{\dagger 2}}\,\mathrm{e}^{ta^2}a^k$ into its antinormal ordering

$$\sum_{n,m=0}^{\infty}\frac{s^n t^m}{n!m!}:\mathrm{H}_{2n+l,2m+k}\left(a^\dagger,a\right):$$
$$=a^{\dagger l}\,\mathrm{e}^{sa^{\dagger 2}}\,\mathrm{e}^{ta^2}a^k$$
$$=\int\frac{\mathrm{d}^2\beta}{\pi}:\langle-\beta|a^{\dagger l}\,\mathrm{e}^{sa^{\dagger 2}}\,\mathrm{e}^{ta^2}a^k|\beta\rangle$$
$$\times\exp\left[|\beta|^2+\beta^* a-\beta a^\dagger+a^\dagger a\right]:$$
$$=\int\frac{\mathrm{d}^2\beta}{\pi}\beta^k(-\beta^*)^l$$
$$\times:\exp\left[-|\beta|^2-\beta a^\dagger+\beta^* a+t\beta^2+s\beta^{*2}+a^\dagger a\right]:$$
$$=\frac{(-1)^l}{\sqrt{A'}}\left(\mathrm{i}\sqrt{\frac{s}{A'}}\right)^k\left(\mathrm{i}\sqrt{\frac{t}{A'}}\right)^l$$
$$\times\sum_{m=0}^{\min[k,l]}\left(\frac{-1}{\sqrt{st}}\right)^m\frac{l!k!}{m!(k-m)!(l-m)!}$$
$$\times:\mathrm{H}_{k-m}\left(\frac{-2sa^\dagger+a}{2\mathrm{i}\sqrt{sA'}}\right)\mathrm{H}_{l-m}\left(\frac{2ta-a^\dagger}{2\mathrm{i}\sqrt{tA'}}\right)\mathrm{e}^{W'+a^\dagger a}:, \tag{23}$$

where

$$A'=1-4st,\quad W'=\frac{-aa^\dagger+sa^{\dagger 2}+ta^2}{1-4st}. \tag{24}$$

Thus we obtain the following new generating function formula

$$\sum_{n,m=0}^{\infty}\frac{s^n t^m}{n!m!}\mathrm{H}_{2n+l,2m+k}(x,y)$$
$$=\left(A'\right)^{-1/2}(-1)^l\left(\mathrm{i}\sqrt{\frac{s}{A'}}\right)^k\left(\mathrm{i}\sqrt{\frac{t}{A'}}\right)^l$$
$$\times\sum_{m=0}^{\min[k,l]}\left(\frac{-1}{\sqrt{st}}\right)^m\frac{k!l!}{m!(k-m)!(l-m)!}$$
$$\times\mathrm{H}_{k-m}\left(\frac{x-2sy}{2\mathrm{i}\sqrt{sA'}}\right)\mathrm{H}_{l-m}\left(\frac{2tx-y}{2\mathrm{i}\sqrt{tA'}}\right)$$
$$\times\exp\left(\frac{-4stxy+sy^2+tx^2}{1-4st}\right). \tag{25}$$

Obviously, when $l=k=0$, equation (25) becomes Eq. (20).

5. Application of Eq. (15)

As an application of the formula (15), we consider the one- and two-mode combinatorial squeezed states[12]

$$|\phi\rangle=C\,\mathrm{e}^{sa^{\dagger 2}+ta^\dagger b^\dagger}|00\rangle, \tag{26}$$

where $b^\dagger$ is the second-mode Bose operator, s and t are squeezing parameters respectively for single-mode and two-mode, C is the normalization constant. For quantum controlling purpose we want to produce a non-Gaussian state by subtracting n first-mode photons by repeatedly operating a^n on $|\phi\rangle$. Here we ask what is the state vector of the first-mode n-photon-subtraction of $|\phi\rangle$, i.e., we want to know the explicit form of $a^n|\phi\rangle$. For this aim, using

$$a^n\,\mathrm{e}^{sa^{\dagger 2}+ta^\dagger b^\dagger}=\mathrm{e}^{sa^{\dagger 2}+ta^\dagger b^\dagger}\,\mathrm{e}^{-sa^{\dagger 2}-ta^\dagger b^\dagger}a^n\,\mathrm{e}^{sa^{\dagger 2}+ta^\dagger b^\dagger}$$
$$=\mathrm{e}^{sa^{\dagger 2}+ta^\dagger b^\dagger}\left(a+\left[-sa^{\dagger 2}-ta^\dagger b^\dagger,a\right]\right)^n$$
$$=\mathrm{e}^{sa^{\dagger 2}+ta^\dagger b^\dagger}\left[\left(a+tb^\dagger\right)+2sa^\dagger\right]^n, \tag{27}$$

the normally ordered form of $\left[\left(a+tb^\dagger\right)+2sa^\dagger\right]^n$ can be derived by first noting

$$\mathrm{e}^{\lambda\left(fa+ga^\dagger\right)}=:\mathrm{e}^{\lambda\left(fa+ga^\dagger\right)}:\exp\left(\frac{fg\lambda^2}{2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{\left(-\mathrm{i}\sqrt{\frac{fg}{2}}\lambda\right)^n}{n!} : \mathrm{H}_n\left(\mathrm{i}\frac{fa+ga^\dagger}{\sqrt{2fg}}\right) :$$
$$= \sum_{n=0}^{\infty} \frac{\left[\lambda\left(fa+ga^\dagger\right)\right]^n}{n!}, \tag{28}$$

so

$$\left(fa+ga^\dagger\right)^n = \left(-\mathrm{i}\sqrt{\frac{fg}{2}}\right)^n : \mathrm{H}_n\left(\mathrm{i}\frac{fa+ga^\dagger}{\sqrt{2fg}}\right) : . \tag{29}$$

Then we use the binomial theorem involving Hermite polynomial[13]

$$\sum_{l=0}^{m} \binom{m}{l} y^{m-l} q^l \mathrm{H}_l(x) = q^m \mathrm{H}_m\left(x+\frac{y}{2q}\right) \tag{30}$$

to deduce

$$\left[\left(a+tb^\dagger\right)+2sa^\dagger\right]^n$$
$$= \sum_{l=0}^{n} \binom{n}{l} \left(a+2sa^\dagger\right)^l \left(tb^\dagger\right)^{n-l}$$
$$= \sum_{l=0}^{n} \binom{n}{l} \left(tb^\dagger\right)^{n-l} \left(-\mathrm{i}\sqrt{s}\right)^l : \mathrm{H}_l\left(\mathrm{i}\frac{a+2sa^\dagger}{2\sqrt{s}}\right) :$$
$$= \left(-\mathrm{i}\sqrt{s}\right)^n : \mathrm{H}_n\left(\mathrm{i}\frac{a+2sa^\dagger+tb^\dagger}{2\sqrt{s}}\right) : . \tag{31}$$

Now using (15) we have

$$: (-\mathrm{i})^n \sqrt{s^n} \mathrm{H}_n\left(\mathrm{i}\frac{\left(a+tb^\dagger\right)+2sa^\dagger}{2\sqrt{s}}\right) \mathrm{e}^{sa^{\dagger 2}} :$$
$$= (-1)^n \sum_{m=0}^{\infty} \frac{s^m}{m!} : \mathrm{H}_{n,2m}\left[-\left(a+tb^\dagger\right), a^\dagger\right] :, \tag{32}$$

so

$$a^n \left|\phi(\beta)\right\rangle$$
$$= \mathrm{e}^{sa^{\dagger 2}+ta^\dagger b^\dagger} \left(-\mathrm{i}\sqrt{s}\right)^n : \mathrm{H}_n\left(\mathrm{i}\frac{a+2sa^\dagger+tb^\dagger}{2\sqrt{s}}\right) : |00\rangle$$
$$= (-1)^n \mathrm{e}^{ta^\dagger b^\dagger} \sum_{m=0} \frac{s^m}{m!} : \mathrm{H}_{n,2m}\left[-\left(a+tb^\dagger\right), a^\dagger\right] : |00\rangle$$
$$= (-1)^n \mathrm{e}^{ta^\dagger b^\dagger} \sum_{m=0} \frac{s^m}{m!} \mathrm{H}_{n,2m}\left[-tb^\dagger, a^\dagger\right] |00\rangle$$
$$= (-1)^n \sum_{m=0} \frac{s^m}{m!} \mathrm{H}_{2m,n}\left[a^\dagger, -tb^\dagger\right] \mathrm{e}^{ta^\dagger b^\dagger} |00\rangle, \tag{33}$$

so the first-mode n-photon-subtraction of the one- and two-mode combinatorial squeezed states is equivalent to the summation of the excitation on the two-mode squeezed state $\mathrm{e}^{ta^\dagger b^\dagger}|00\rangle$ by $\mathrm{H}_{2m,n}\left[a^\dagger, -tb^\dagger\right]$, which includes the second-mode excitation, and this exhibits quantum entanglement.

Finally, we point out that using the above results we can propose some new optical fields by calculating their normalization constants.

In summary, by combining the operator Hermite polynomial method and the IWOP technique, we have derived the generating function formulas of even- and odd-Hermite polynomials for a two-variable case. Such formulas can enrich the mathematical handbook and will have potential uses in quantum theory.

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