Non-probabilistic solutions of imprecisely defined fractional-order diffusion equations

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The fractional diffusion equation is one of the most important partial differential equations (PDEs) to model problems in mathematical physics. These PDEs are more practical when those are combined with uncertainties. Accordingly, this paper investigates the numerical solution of a non-probabilistic viz. fuzzy fractional-order diffusion equation subjected to various external forces. A fuzzy diffusion equation having fractional order $0 < \alpha \leq 1$ with fuzzy initial condition is taken into consideration. Fuzziness appearing in the initial conditions is modelled through convex normalized triangular and Gaussian fuzzy numbers. A new computational technique is proposed based on double parametric form of fuzzy numbers to handle the fuzzy fractional diffusion equation. Using the single parametric form of fuzzy numbers, the original fuzzy diffusion equation is converted first into an interval-based fuzzy differential equation. Next, this equation is transformed into crisp form by using the proposed double parametric form of fuzzy numbers. Finally, the same is solved by Adomian decomposition method (ADM) symbolically to obtain the uncertain bounds of the solution. Computed results are depicted in terms of plots. Results obtained by the proposed method are compared with the existing results in special cases.

Keywords: double parametric form of fuzzy number, fuzzy fractional diffusion equation, ADM

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1. Introduction

In recent years, fractional differential equations have become an important research area mainly due to their wide range of applications in physics, chemistry, biology, engineering, fluid mechanics, viscoelasticity, economics, etc. In particular, fractional diffusion equations are found to be important to model various problems in mathematical physics. The time-fractional diffusion equation differs from the standard diffusion equation when the first-order time derivative is replaced by a fractional derivative, making the problem global in time. The basic process in the diffusion phenomenon is the flow of fluid from a region of higher density to one of lower density. Several studies on fractional calculus have been reported in the last few decades. Excellent books related to this have also been written by different authors representing the scopes and various aspects of fractional calculus such as those in Refs. [1]–[6]. These books also give an extensive review of the fractional derivative and its applications which may help the readers to understand the basic concepts of fractional calculus with its applications in mathematical physics and beyond. Many authors have developed various methods to solve fractional differential and integral equations of physical systems. Different numerical methods and applications of fractional differential equations are also introduced in Refs. [7]–[21].

Fractional diffusion equations have also been analyzed by a few authors.[22–36] In general, the system parameters and variables involved in the diffusion processes are considered to be crisp or defined exactly. Other authors[37–50] have also investigated some application problems with crisp variables and parameters. But in actual practice, rather than the particular value, only uncertain or vague estimates about the variables and parameters are known, because those are found in general by some observations and experiments. So, to handle these uncertainties and vagueness, one may use fuzzy parameters and variables in the governing differential equations.

Both uncertainty and fractional differential equations play a vital role in solving real life problems. Very recently a few authors studied fractional differential equations with uncertainty. Some recent contributions to the theory of fuzzy differential equations and fuzzy fractional differential equations may be seen in Refs. [51]– [74]. The concept of the fuzzy fractional differential equation was introduced recently by Agrawal *et al.*[63] Arshad and Lupulescu[64] proved some results on the existence and uniqueness of solutions of fuzzy fractional differential equations based on the concept of fuzzy differential equations of fractional order introduced by Agrawal.[63] Arshad and Lupulescu[65] investigated the fractional differential equation with the fuzzy initial condition. Jeong[66] discussed the existence and uniqueness results for fuzzy fractional differential equations with infinite delay. The boundary value problem for fuzzy fractional differential equations with finite delay has been solved by Wang and Liu.[67]

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They established the existence of solution by contraction mapping principle. For Riemann–Liouville H-differentiability, Allahviranloo *et al.*[68] studied the explicit solutions of fuzzy fractional differential equations or uncertain fractional differential equations (UFDEs). The variational iteration method was used by Khodadadi and Celik[69] for solving the fuzzy fractional differential equations. Mohammed *et al.*[70] used the differential transform method for solving the fuzzy fractional initial value problems. Salahshour *et al.*[71] developed Riemann–Liouville differentiability by using Hukuhara difference called Riemann–Liouville H-differentiability and solved fuzzy fractional differential equations by Laplace transforms. Recently, Salah *et al.*[72] used the homotopy analysis transform method for solving the fuzzy fractional heat equation. In Ref. [72], the authors have solved the fuzzy fractional differential equation twice, i.e., for lower and upper bounds separately to obtain the solution bound. Also Ahmadian *et al.*[73] implemented Jacobi operational matrix for solving a fuzzy fractional differential equation. Behera and Chakraverty[74] successfully used homotopy perturbation method to find the uncertain impulse response of the imprecisely defined fractional-order mechanical system.

Recently, Adomian decomposition method (ADM) is found to be a powerful tool for solving various linear and nonlinear, ordinary and partial differential equations. The ADM was first developed by Adomian[75,76] and then many authors used this method to solve various scientific and engineering problems. The solution is considered as the sum of infinite series, which converges rapidly to accurate solutions. Convergence of ADM is proved in Refs. [77]–[80]. Wazwaz[81] has done a reliable modification of ADM. Very recently ADM has been applied to a wide class of problems.[82–86]

In the present analysis, ADM is used for numerically solving the uncertain fractional-order diffusion equation subjected to various external forces where the initial condition is defined uncertainly. Uncertainty in the initial condition is defined in terms of triangular and Gaussian fuzzy numbers. A literature review revealed that fuzzy fractional differential equations are always converted into two crisp fractional differential equations in general to obtain the solution bound. But in this paper, a new approach is proposed based on the double parametric form of the fuzzy number as defined in Definition 4. Here, the fractional fuzzy differential equation has been converted into a single crisp parametric fractional differential equation. Finally the corresponding single crisp parametric fractional differential equation may be solved by any known numerical method symbolically to obtain the solution in the double parametric form. Then substituting the parametric values one may obtain the final solution bounds.

The rest of this paper is organized as follows. In Section 2, some basic preliminaries related to the present investigation are given. The proposed technique is combined with ADM in Section 3 to solve the fuzzy fractional diffusion equation. Next, the uncertain solutions for different types of forces are presented followed by numerical results and discussion. Finally conclusions are drawn in Section 4.

2. Preliminaries

In this section, we present some notations, definitions, and preliminaries which are used further in this paper.[87–90]

Definition 1 Fuzzy number

A fuzzy number $\tilde{U}$ is a convex-normalized fuzzy set $\tilde{U}$ of the real line R such that

$$\{\mu_{\tilde{U}}(x) : R \to [0,1], \forall x \in R\},$$

where, $\mu_{\tilde{U}}$ is called the membership function of the fuzzy set and it is piecewise continuous.

Definition 2 Triangular fuzzy number

A triangular fuzzy number $\tilde{U}$ is a convex-normalized fuzzy set $\tilde{U}$ of the real line R such that 1) there exists exactly one $x_0 \in R$ with $\mu_{\tilde{U}}(x_0) = 1$ (x_0 is called the mean value of $\tilde{U}$), where $\mu_{\tilde{U}}$ is called the membership function of the fuzzy set; 2) $\mu_{\tilde{U}}(x)$ is piecewise continuous.

Consider an arbitrary triangular fuzzy number $\tilde{U} = (a,b,c)$. The membership function $\mu_{\tilde{U}}$ of $\tilde{U}$ will be defined as follows:

$$\mu_{\tilde{U}}(x) = \begin{cases} 0, & x \le a, \\ \dfrac{x-a}{b-a}, & a \le x \le b, \\ \dfrac{c-x}{c-b}, & b \le x \le c, \\ 0, & x \ge c. \end{cases}$$

The triangular fuzzy number $\tilde{U} = (a,b,c)$ can be represented with an ordered pair of functions through r-cut approach or in single parametric form viz. $[\underline{u}(r), \bar{u}(r)] = [(b-a)r + a, -(c-b)r + c]$ where, $r \in [0,1]$.

Definition 3 Gaussian fuzzy number

Now define an arbitrary asymmetrical Gaussian fuzzy number, $U = (\delta, \sigma_{\rm l}, \sigma_{\rm r})$. The membership function $\mu_{\tilde{U}}$ of $\tilde{U}$ will be as follows:

$$\mu_U(x) = \begin{cases} \exp[-(x-\delta)^2/2\sigma_{\rm l}^2] & \text{for } x \le \delta, \\ \exp[-(x-\delta)^2/2\sigma_{\rm r}^2] & \text{for } x \ge \delta, \end{cases} \quad \forall x \in R,$$

where the modal value is denoted as δ, $\sigma_{\rm l}$ and $\sigma_{\rm r}$ denote the left-hand and right-hand spreads (fuzziness) corresponding to the Gaussian distribution. For symmetric Gaussian fuzzy number, the left-hand and right-hand spreads are equal, i.e., $\sigma_{\rm l} = \sigma_{\rm r} = \sigma$. So the symmetric Gaussian fuzzy number may be

written as $U=(\delta,\sigma,\sigma)$ and corresponding membership function may be defined as $\mu_{\tilde{U}}(x)=\exp\{-\beta(x-\delta)^2\}$, $\forall x\in R$, where, $\beta=1/2\sigma^2$. The symmetric Gaussian fuzzy number in the single parametric form can be represented as

$$\tilde{U}=[\underline{u}(r),\bar{u}(r)]=\left[\delta-\sqrt{-\frac{(\log_e r)}{\beta}},\delta+\sqrt{-\frac{(\log_e r)}{\beta}}\right],$$

where $r\in[0,1]$.

Definition 4 Double parametric form of fuzzy number

Using the single parametric form, we have $\tilde{U}=[\underline{u}(r),\bar{u}(r)]$. Now this may be written as a crisp number using double parametric form $\tilde{U}(r,\beta)=\beta(\bar{u}(r)-\underline{u}(r))+\underline{u}(r)$, where r and $\beta\in[0,1]$.

Definition 5 Fuzzy arithmetic

For any two arbitrary fuzzy numbers, i.e., $\tilde{x}=[\underline{x}(r),\bar{x}(r)]$ and $\tilde{y}=[\underline{y}(r),\bar{y}(r)]$ and scalar k, the fuzzy arithmetic is defined as follows:

(i) $\tilde{x}=\tilde{y}$ if and only if $\underline{x}(r)=\underline{y}(r)$ and $\bar{x}(r)=\bar{y}(r)$,

(ii) $\tilde{x}+\tilde{y}=[\underline{x}(r)+\underline{y}(r),\bar{x}(r)+\bar{y}(r)]$,

(iii) $\tilde{x}\times\tilde{y}=\begin{bmatrix}\min\left(\underline{x}(r)\times\underline{y}(r),\underline{x}(r)\times\bar{y}(r),\right.\\ \left.\bar{x}(r)\times\underline{y}(r),\bar{x}(r)\times\bar{y}(r)\right),\\ \max\left(\underline{x}(r)\times\underline{y}(r),\underline{x}(r)\times\bar{y}(r),\right.\\ \left.\bar{x}(r)\times\underline{y}(r),\bar{x}(r)\times\bar{y}(r)\right),\end{bmatrix}$,

(iv) $k\tilde{x}(r)=\begin{cases}[k\bar{x}(r),k\underline{x}(r)], & k<0,\\ [k\underline{x}(r),k\bar{x}(r)], & k\geq 0.\end{cases}$

3. Double parametric-based fuzzy fractional diffusion equation

First we convert the fuzzy fractional diffusion equation into an interval-based fuzzy fractional diffusion equation using a single parametric form of fuzzy numbers. Then the interval-based differential equation is reduced to a crisp differential equation by using the double parametric form of fuzzy numbers. Next, we used ADM to obtain the solution in the double parametric form.

Consider the fuzzy-fractional diffusion equation

$$\frac{\partial^\alpha\tilde{v}(x,t)}{\partial t^\alpha}=K\frac{\partial^2\tilde{v}(x,t)}{\partial x^2}-\frac{\partial}{\partial x}(F(x)\tilde{v}(x,t)),\quad 0<\alpha\leq 1,\quad t>0,\quad x>0 \tag{1}$$

with fuzzy initial condition

$$\tilde{v}(x,0)=\tilde{f}(x),$$

where $\partial^\alpha/\partial t^\alpha$ is the Caputo derivative of order α, $\tilde{v}(x,t)$ represents the uncertain probability density function, K is the constant dependent on the temperature and $F(x)$ is the external force. In the present analysis we consider $F(x)=-g(x)$.

Now equation (1) may be written as

$$\frac{\partial^\alpha\tilde{v}(x,t)}{\partial t^\alpha}=K\frac{\partial^2\tilde{v}(x,t)}{\partial x^2}+\frac{\partial}{\partial x}(g(x)\tilde{v}(x,t)). \tag{2}$$

Here, the initial condition has been taken as fuzzy with an idea that the condition may actually be uncertain viz. it may be due to the error in observation or experiment, etc., where we may take the error or uncertainty in terms of fuzzy membership functions. As such this will make the governing differential equation uncertain and the corresponding outcome or the output (result) will be in uncertain form. In this way we may have the actual essence of the uncertainty. So, we need to have efficient methods to handle these problems.

As per the single parametric form we may write the above fuzzy fractional differential equation (Eq. (2)) as

$$\left[\frac{\partial^\alpha\underline{v}(x,t;r)}{\partial t^\alpha},\frac{\partial^\alpha\bar{v}(x,t;r)}{\partial t^\alpha}\right]=K\left[\frac{\partial^2\underline{v}(x,t;r)}{\partial x^2},\frac{\partial^2\bar{v}(x,t;r)}{\partial x^2}\right]+g(x)\left[\frac{\partial\underline{v}(x,t;r)}{\partial x},\frac{\partial\bar{v}(x,t;r)}{\partial x}\right] \tag{3}$$

subjected to fuzzy-initial condition

$$[\underline{v}(x,0;r),\bar{v}(x,0;r)]=[\underline{f}(x;r),\bar{f}(x;r)].$$

Next, using the double parametric form (as discussed in Definition 4), equation (3) can be expressed as

$$\begin{aligned}&\left\{\beta\left(\frac{\partial^\alpha\bar{v}(x,t;r)}{\partial t^\alpha}-\frac{\partial^\alpha\underline{v}(x,t;r)}{\partial t^\alpha}\right)+\frac{\partial^\alpha\underline{v}(x,t;r)}{\partial t^\alpha}\right\}\\&=\left\{\beta\left(\frac{\partial^2\bar{v}(x,t;r)}{\partial x^2}-\frac{\partial^2\underline{v}(x,t;r)}{\partial x^2}\right)+\frac{\partial^2\underline{v}(x,t;r)}{\partial x^2}\right\}\\&\quad+g(x)\left\{\beta\left(\frac{\partial\bar{v}(x,t;r)}{\partial x}-\frac{\partial\underline{v}(x,t;r)}{\partial x}\right)+\frac{\partial\underline{v}(x,t;r)}{\partial x}\right\}\end{aligned} \tag{4}$$

subjected to the fuzzy initial conditions

$$\begin{aligned}&\{\beta(\bar{v}(x,0;r)-\underline{v}(x,0;r))+\underline{v}(x,0;r)\}\\&=\beta\left(\bar{f}(x;r)-\underline{f}(x;r)\right)+\underline{f}(x;r),\end{aligned}$$

where, $\beta\in[0,1]$. Denote

$$\begin{aligned}&\left\{\beta\left(\frac{\partial^\alpha\bar{v}(x,t;r)}{\partial t^\alpha}-\frac{\partial^\alpha\underline{v}(x,t;r)}{\partial t^\alpha}\right)+\frac{\partial^\alpha\underline{v}(x,t;r)}{\partial t^\alpha}\right\}\\&=\frac{\partial^\alpha\tilde{v}(x,t;r,\beta)}{\partial t^\alpha},\\&\left\{\beta\left(\frac{\partial^2\bar{v}(x,t;r)}{\partial x^2}-\frac{\partial^2\underline{v}(x,t;r)}{\partial x^2}\right)+\frac{\partial^2\underline{v}(x,t;r)}{\partial x^2}\right\}\\&=\frac{\partial^2\tilde{v}(x,t;r,\beta)}{\partial x^2},\\&\left\{\beta\left(\frac{\partial\bar{v}(x,t;r)}{\partial x}-\frac{\partial\underline{v}(x,t;\alpha)}{\partial x}\right)+\frac{\partial\underline{v}(x,t;r)}{\partial x}\right\}\\&=\frac{\partial\tilde{v}(x,t;r,\beta)}{\partial x},\\&\{\beta(\bar{v}(0;r)-\underline{v}(0;r))+\underline{v}(0;r)\}=\tilde{v}(0;r,\beta)\end{aligned}$$

and

$$\beta\left(\bar{f}(x;r)-\underline{f}(x;r)\right)+\underline{f}(x;r)=\tilde{f}(x;r;\beta).$$

Substituting these into Eq. (4) we obtain

$$\frac{\partial^\alpha \tilde{v}(x,t;r,\beta)}{\partial t^\alpha} = K\frac{\partial^2 \tilde{v}(x,t;r,\beta)}{\partial x^2} + \frac{\partial}{\partial x}(g(x)\tilde{v}(x,t;r,\beta)) \quad (5)$$

with initial condition

$$\tilde{v}(x,0;r,\beta) = \tilde{f}(x;r;\beta).$$

Solving the corresponding crisp differential equation (Eq. (5)) one may have the solution as $\tilde{v}(x,t;r,\beta)$. To obtain the lower and upper bounds of the solution in single parametric form we let $\beta = 0$ and 1 respectively which may be represented as

$$\tilde{v}(x,t;r,0) = \underline{v}(x,t,r) \text{ and } \tilde{v}(x,t,r,1) = \bar{v}(x,t,r).$$

3.1. Solution by ADM[75,76] using proposed methodology

We have used ADM to solve Eq. (5) and consider Eq. (5) as

$$L_t^\alpha \tilde{v}(x,t;r,\beta) = KL_{xx}\tilde{v}(x,t;r,\beta) + L_x(g(x)\tilde{v}(x,t;r,\beta)), \quad (6)$$

where, $L_t^\alpha \equiv \partial^\alpha/\partial t^\alpha$, $L_{xx} \equiv \partial^2/\partial x^2$, and $L_x \equiv \partial/\partial x$.

Applying the operator $L_t^{-\alpha}$ (which is the inverse operator of L_t^α) on both sides of Eq. (6) we have

$$\begin{aligned} L_t^{-\alpha}L_t^\alpha \tilde{v}(x,t;r,\beta) &= L_t^{-\alpha}\big(KL_{xx}\tilde{v}(x,t;r,\beta) \\ &\quad + L_x(g(x)\tilde{v}(x,t;r,\beta))\big). \end{aligned} \quad (7)$$

Using $L_t^{-\alpha}L_t^\alpha \tilde{v}(x,t;r,\beta) = v(x,0;r,\beta)$ equation (7) becomes

$$\begin{aligned} \tilde{v}(x,t;r,\beta) &= v(x,0;r,\beta) + L_t^{-\alpha}\big(KL_{xx}\tilde{v}(x,t;r,\beta) \\ &\quad + L_x(g(x)\tilde{v}(x,t;r,\beta))\big). \end{aligned} \quad (8)$$

According to ADM,[75,76] we assume an infinite series solution for unknown function $v(x,t;r,\beta)$ to be

$$v(x,t;r,\beta) = \sum_{n=0}^{\infty} v_n(x,t;r,\beta) \quad (9)$$

with $v_0(x,t;r,\beta) = v(x,0;r,\beta)$ and the components $v_n(x,t;r,\beta)$, which are usually determined by $(n>0)$

$$v_n(x,t;r,\beta) = L_t^{-\alpha}\left(\frac{\partial^2 v_{n-1}(x,t;r,\beta)}{\partial x^2} + \frac{\partial\,(g(x)v_{n-1}(x,t;r,\beta))}{\partial x}\right).$$

Substituting these terms into Eq. (9), one may obtain the approximate solution of Eq. (5) as follows:

$$\begin{aligned} v(x,t;r,\beta) &= v_0(x,t;r,\beta) + v_1(x,t;r,\beta) + v_2(x,t;r,\beta) \\ &\quad + v_3(x,t;r,\beta) + \cdots. \end{aligned} \quad (10)$$

3.2. Solution bounds for different external forces

In this subsection we consider different external forces to find the uncertain bounds for fuzzy fractional differential equations using ADM based on the proposed technique.

Case 1 We consider $K = 1$ and $g(x) = x$ with fuzzy initial condition in the parametric form as $\tilde{v}(x,0;r) = [0.2r + 0.8, 1.2 - 0.2r]$. Hence equation (2) will become

$$\frac{\partial^\alpha \tilde{v}(x,t)}{\partial t^\alpha} = K\frac{\partial^2 \tilde{v}(x,t)}{\partial x^2} + \frac{\partial}{\partial x}(x\tilde{v}(x,t)), \quad (11)$$

such that the fuzzy initial condition $\tilde{v}(x,0;r) = [0.2r + 0.8, 1.2 - 0.2r]$.

The double parametric form of Eq. (5) with the fuzzy initial condition becomes

$$\frac{\partial^\alpha \tilde{v}(x,t;r,\beta)}{\partial t^\alpha} = K\frac{\partial^2 \tilde{v}(x,t;r,\beta)}{\partial x^2} + \frac{\partial}{\partial x}(x\tilde{v}(x,t,r,\beta))$$

with the initial condition

$$\tilde{v}(0;r,\beta) = \beta(0.4 - 0.4r) + (0.2r + 0.8).$$

By using the ADM we have

$$\tilde{v}_0(x,t;r,\beta) = \{\beta(0.4-0.4r)+(0.2r+0.8)\}, \quad (12)$$

$$\tilde{v}_1(x,t;r,\beta) = \{\beta(0.4-0.4r)+(0.2r+0.8)\}\frac{t^\alpha}{\Gamma(\alpha+1)}, \quad (13)$$

$$\tilde{v}_2(x,t;r,\beta) = \{\beta(0.4-0.4r)+(0.2r+0.8)\}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \quad (14)$$

$$\tilde{v}_3(x,t;r,\beta) = \{\beta(0.4-0.4r)+(0.2r+0.8)\}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \quad (15)$$

$$\tilde{v}_4(x,t;r,\beta) = \{\beta(0.4-0.4r)+(0.2r+0.8)\}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}. \quad (16)$$

In a similar manner, higher order approximation may be obtained as discussed above. Therefore, the solution can be written in general form as

$$\tilde{v}(x,t;r,\beta) = \{\beta(0.4 - 0.4r) + (0.2r + 0.8)\}\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \quad (17)$$

To obtain the solution bounds in the single parametric form we may let $\beta = 0$ and 1 to obtain the lower and upper bounds of the solution respectively as

$$\underline{v}(x,t;r,0) = \underline{v}(x,t;r) = (0.2r + 0.8)\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \quad (18)$$

and

$$\bar{v}(x,t;r,1) = \bar{v}(t;r,1) = (1.2 - 0.2r)\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \quad (19)$$

One may note that in the special case where $r = 1$ the results (crisp) obtained by the proposed method are exactly the same as that of the solution obtained by the method of Godal *et al.*[30]

Case 2 Now we consider the same force and constant K as those in Case 1 with a different fuzzy initial condition $\tilde{v}(x,0;r) = x[0.2r + 0.8, 1.2 - 0.2r]$.

Again, by using the procedure discussed previously, we have the general solution in the double parametric form as

$$\tilde{v}(x,t;r,\beta) = x\{\beta(0.4 - 0.4r) + (0.2r + 0.8)\}\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \quad (20)$$

Putting $\beta = 0$ and 1 into $\tilde{v}(x,t;r,\beta)$ we have the lower and upper bounds of the fuzzy solutions respectively as

$$\underline{v}(x,t;r,0) = \underline{v}(x,t;r) = x(0.2r+0.8)\sum_{n=0}^{\infty}\frac{2^n t^{n\alpha}}{\Gamma(n\alpha+1)} \quad (21)$$

and

$$\bar{v}(x,t;r,1) = \bar{v}(x,t;r) = x(1.2-0.2r)\sum_{n=0}^{\infty}\frac{2^n t^{n\alpha}}{\Gamma(n\alpha+1)}. \quad (22)$$

The solution obtained by the proposed method for $r = 1$ is again found to be exactly the same as that of (crisp result) Godal *et al.*[30]

Case 3 Finally, consider $K = 1$ and $g(x) = \mathrm{e}^{-x}$ with fuzzy initial condition in the parametric form as

$$\tilde{v}(x,0;r) = \left[1-\sqrt{-0.08\log_e r}, 1+\sqrt{-0.08\log_e r}\right].$$

Similarly we have the following fuzzy fractional diffusion equation:

$$\frac{\partial^\alpha \tilde{v}(x,t)}{\partial t^\alpha} = K\frac{\partial^2 \tilde{v}(x,t)}{\partial x^2} + \frac{\partial}{\partial x}(\mathrm{e}^{-x}\tilde{v}(x,t)). \quad (23)$$

By following the proposed method with ADM, we have the solution in the double parametric form as

$$\tilde{v}(x,t;r,\beta) = \mathrm{e}^x\left\{\beta\left(2\sqrt{-0.08\log_e r}\right)+\left(1-\sqrt{-0.08\log_e r}\right)\right\} \times\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \quad (24)$$

The lower and upper bounds of the fuzzy solution of Eq. (24) are obtained respectively as

$$\tilde{v}(x,t;r,0) = \mathrm{e}^x\left(1-\sqrt{-0.08\log_e r}\right)\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$

and

$$\tilde{v}(x,t;r,1) = \mathrm{e}^x\left(1+\sqrt{-0.08\log_e r}\right)\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}.$$

Results for $r = 1$exactly agree with the exact solution (crisp) of Godal *et al.*[30]

3.3. Numerical results and discussion

Numerical results for the fuzzy fractional diffusion equation with different external forces and fuzzy initial conditions are obtained. The results obtained by the present analysis are compared with the existing solution in Ref. [30] in special cases to show the validation of the proposed analysis. Computed results are depicted in terms of plots.

Triangular fuzzy solutions for Cases 1 and 2 and Gaussian fuzzy solution for Case 3 are depicted in Figs. 1–3 by varying the time from 0 to 0.5, 0 to 1, and 0 to 0.5 respectively with $\alpha = 0.5$. For Cases 2 and 3 we consider $x = 0.5$. Next, interval solutions for $r = 0.5$ and 1 for different cases have been given in Figs. 4–9 for $x = 0.5$ and $\alpha = 0.2$, 0.6, and 1. It may be worth mentioning that for all the cases, present results exactly agree with the solution of Godal *et al.*[30] in the special case of $r = 1$. Also it is interesting to note from Figs. 4–9 that the left and right bounds of the uncertain probability density function, i.e., $\tilde{v}(x,t)$ (with particular values of α, r, and x) gradually decrease with the increase of the fractional-order α and time t.

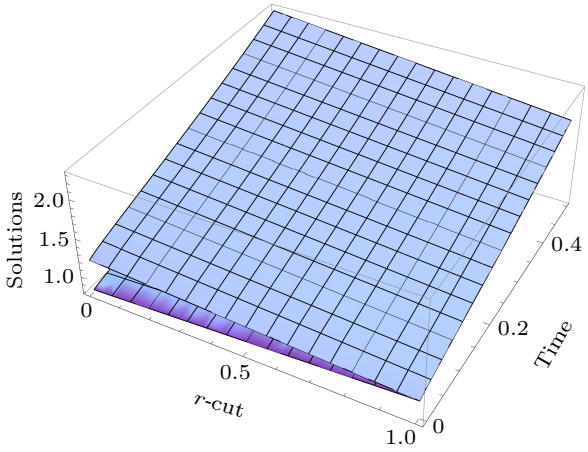


Fig. 1. (color online) Fuzzy solution for Case 1.

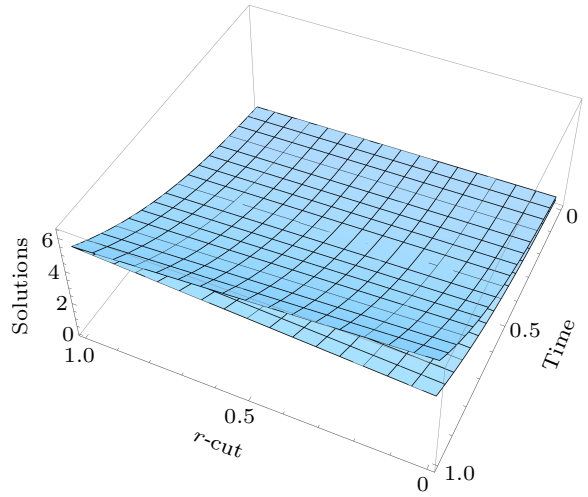


Fig. 2. (color online) Fuzzy solution for Case 2.

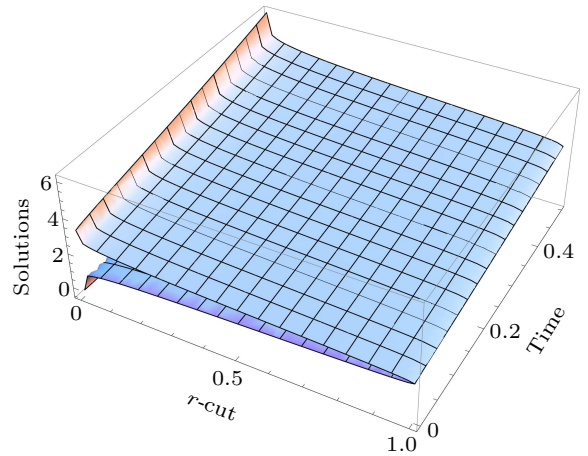


Fig. 3. (color online) Fuzzy solution for Case 3.

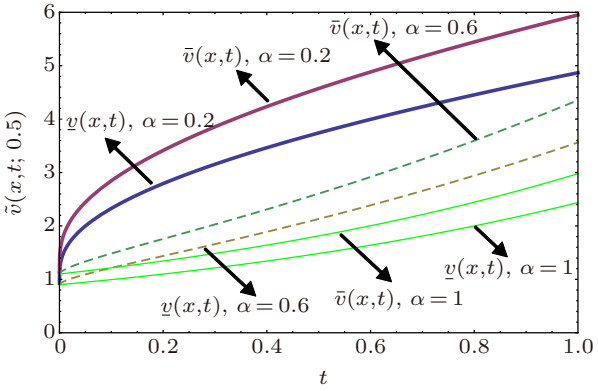


Fig. 4. (color online) Interval solutions for $r = 0.5$ of Case 1.

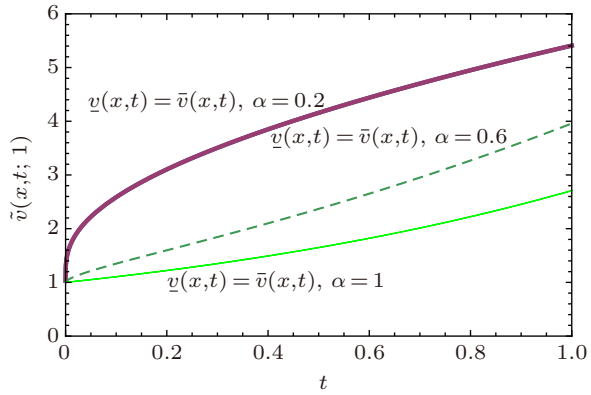


Fig. 5. (color online) Interval solutions for $r = 1$ of Case 1.

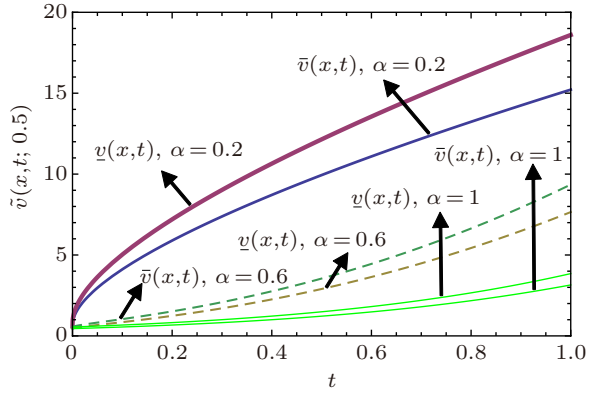


Fig. 6. (color online) Interval solutions for $r = 0.5$ of Case 2.

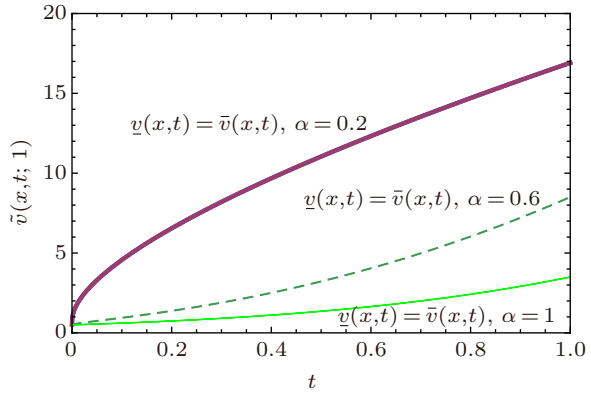


Fig. 7. (color online) Interval solutions for $r = 1$ of Case 2.

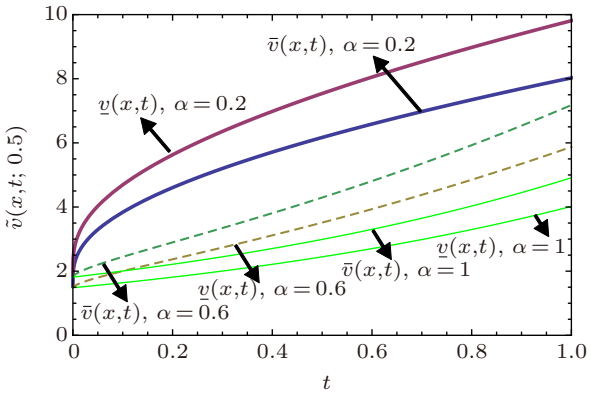


Fig. 8. (color online) Interval solutions for $r = 0.5$ of Case 3.

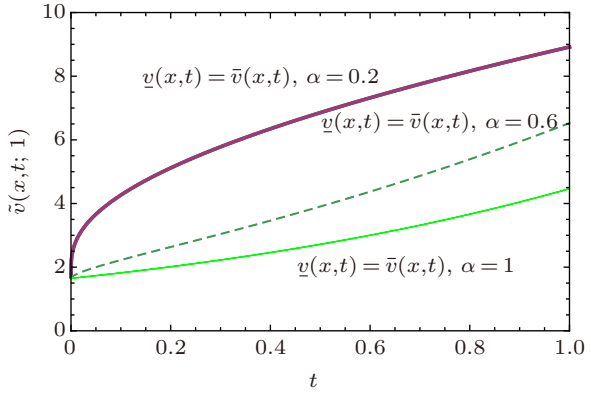


Fig. 9. (color online) Interval solutions for $r = 1$ of Case 3.

4. Conclusions

In this paper, the double parametric form of fuzzy numbers is successfully applied to the solution of fuzzy fractional diffusion equation with different external forces using the Adomian decomposition method. The double parametric form approach is found to be easy and computationally efficient due to the transfer of the governing equation from uncertain to crisp. Here, the performance of the method is shown by using triangular and Gaussian fuzzy numbers. It is interesting to note in all the cases for $r = 1$ that the lower solution is equal to the upper solution. Though the solution by ADM is of the form of an infinite series, it can be written in a closed form. The main advantage of ADM is the ability to achieve an exact solution and rapid convergence with a few terms. The problem discussed here is just for showing the applicability of the proposed computational technique to handle the uncertain fractional-order problems in a straightforward way. The present procedure may very well be extended to other complicated fuzzy fractional differential equations of mathematical physics too.

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