Derivation of baroclinic Ertel–Rossby invariant-based thermally-coupled vorticity equation in moist flow*

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For the potential vorticity (PV) invariant, there is a PV-based complete-form vorticity equation, which we use heuristically in the present paper to answer the following question: for the Ertel–Rossby invariant (ERI), is there a corresponding vorticity tendency equation? Such an ERI-based thermally-coupled vorticity equation is derived and discussed in detail in this study. From the obtained new vorticity equation, the vertical vorticity change is constrained by the vertical velocity term, the term associated with the slope of the generalized momentum surface, the term related to the horizontal vorticity change, and the baroclinic or solenoid term. It explicitly includes both the dynamical and thermodynamic factors' influence on the vorticity change. For the ERI itself, besides the traditional PV term, the ERI also includes the moisture factor, which is excluded in dry ERI, and the term related to the gradients of pressure, kinetic energy, and potential energy that reflects the fast-manifold property. Therefore, it is more complete to describe the fast motions off the slow manifold for severe weather than the PV term. These advantages are naturally handed on and inherited by the ERI-based thermally-coupled vorticity equation. Then the ERI-based thermally-coupled vorticity equation is further transformed and compared with the traditional vorticity equation. The main difference between the two equations is the term which describes the contribution of the solenoid term to the vertical vorticity development. In a barotropic flow, the solenoid term disappears, then the ERI-based thermally-coupled vorticity equation can regress to the traditional vorticity equation.

Keywords: vorticity equation, derivation, Ertel–Rossby invariant

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1. Introduction

Vorticity describes the property of rotation in space. It has been widely applied to quantify the intensity of vortices, such as a typhoon, polar vortex, and cyclone. The corresponding vorticity equation portrays the vorticity change and has an important role in promoting the development of long-wave theory, energy dispersion, and numerical weather prediction.[1] The traditional vorticity equation is derived from the momentum equation, therefore it has apparent dynamic characteristics. However, many intense weather phenomena have close relationships with moisture and thermal processes, e.g., in a typhoon. These thermal factors are not explicitly presented in the traditional vorticity equation, thus how they directly affect the vorticity development is not clear.

To relax the restriction, Wu and Liu derived a complete-form vorticity equation based on the material conserved potential vorticity (PV) theory.[2–4] As an invariant, the PV (expressed as $P_{\rm E} = \alpha\omega_{\rm a}\cdot\nabla\theta$, where $P_{\rm E}$ is the PV, α is the specific volume, $\omega_{\rm a}$ is the absolute vorticity vector, and $\nabla\theta$ is the gradient of the potential temperature) can be used as a tracer of a system's movement. Thus the PV-based complete-form vorticity equation not only reflects the local change of vorticity, but also is constrained by the gradient of the potential temperature because the PV remains constant following the air particles. It is an important advance relative to the traditional vorticity equation, and has been applied to the studies of heavy rainfall, the explosive development of cyclones, and the formation and variation of subtropical highs.[5–11]

With conservation and invertibility, the PV itself and the related PV-thinking[12] have been particularly useful for simplified understanding and interpretation of various large-scale atmospheric processes in which the layered two-dimensional and non-gravity wave parts of the motion are primary.[13] Hoskins *et al.*[12] utilized the two main PV principles (conservation and invertibility), performed isentropic PV map analyses to represent the dynamical processes of quasi-equilibrium motion to illuminate the structure characteristics and evolution trends of upper- and lower-level weather systems induced by upper-level PV anomalies and lower-level potential temperature anomalies. This set of methodology is called PV-thinking. The PV is more effective for the slow-manifold condition and the large-scale flow, for which the PV equation is a better approximation compared to the atmospheric motion equation.[12] The so-called slow-manifold property assumes that the grav-

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ity waves and the inertia-gravity waves are filtered out. Under this assumption, the PV captures only the dynamical information of the non-gravity wave component. For severe weather such as typhoons, however, the pressure gradient and the buoyancy forces vary rapidly, so the slow-manifold balance condition is destroyed. Thus, we wish to find another invariant with the fast-manifold property to describe fast-variation severe weather processes within a baroclinic atmosphere.

Ertel and Rossby obtained an invariant different from the PV, which is called the Ertel–Rossby invariant (ERI) in the barotropic atmosphere.[14] Zdunkowski and Bott revisited the invariant and gave its baroclinic version.[15] By using the Weber transformation and the Lagrangian continuity equation, Yang *et al.* further adapted the baroclinic ERI in moist atmosphere and proved its conservation property.[16] Besides the traditional moist potential vorticity (MPV) term, the invariant includes a moisture factor which is excluded in dry ERI and a term related to the gradients of pressure, kinetic energy, and potential energy that reflects the fast-manifold property. Therefore, it is more complete to describe the fast motions off the slow manifold for severe weather than the MPV term. Case studies and contrastive analyses among moist ERI, dry ERI and MPV also show that moist ERI is a better tool to diagnose the movements and the intensity variations of typhoons.[16]

For the PV, there is a PV-based complete-form vorticity equation. Based on the invariant mentioned above (ERI), could we derive a corresponding vorticity tendency equation that can explicitly include both dynamical and thermodynamic factors and describe the fast motions off the slow manifold for severe weather with a strong rotation property such as a typhoon? Furthermore, what differences are there between the ERI-based thermally-coupled vorticity equation and the traditional vorticity equation? To answer these questions, the ERI-based thermally-coupled vorticity equation is derived and discussed in Section 2. In Section 3, the ERI-based thermally-coupled vorticity equation is further transformed. Then the simplified version of the ERI-vorticity thermally-coupled equation is compared with the traditional vorticity equation to look for their differences in Section 4. A summary is given in the last section.

2. Derivation of ERI-based thermally-coupled vorticity equation in moist baroclinic flow

By following Yang *et al.*,[16] the baroclinic ERI in a moist flow is expressed as

$$\psi_{\mathrm{ER}} = \alpha \boldsymbol{B}_{\mathrm{A}} \cdot \boldsymbol{\nabla} \times \boldsymbol{B}_{\mathrm{A}}, \tag{1}$$

with

$$\boldsymbol{B}_{\mathrm{A}} = \boldsymbol{v}_{\mathrm{A}} - \boldsymbol{\nabla} W_{\mathrm{Am}} - \beta \boldsymbol{\nabla} s_{\mathrm{m}}, \tag{2}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B}_{\mathrm{A}} = \boldsymbol{\nabla} \times \boldsymbol{v}_{\mathrm{A}} - \boldsymbol{\nabla}\beta \times \boldsymbol{\nabla} s_{\mathrm{m}}, \tag{3}$$

where

$$\beta = \int_0^t T \,\mathrm{d}t, \quad W_{\mathrm{Am}} = \int_0^t L_{\mathrm{Am}} \,\mathrm{d}t, \tag{4}$$

$$L_{\mathrm{Am}} = \frac{\boldsymbol{v}_{\mathrm{A}}^2}{2} - \phi_{\mathrm{a}} - h_{\mathrm{m}}. \tag{5}$$

Here L_{Am} is the Lagrangian function in the absolute system, W_{Am} is the action integral of the absolute system, the gradient of the geopotential is $\boldsymbol{\nabla}\phi_{\mathrm{a}} = g\boldsymbol{k}$, g is the gravity acceleration, $\boldsymbol{v}_{\mathrm{A}} = \boldsymbol{v} + \boldsymbol{\Omega} \times r$ is the absolute velocity vector, $\boldsymbol{v}$ is the relative velocity vector, $\boldsymbol{\Omega}$ is the angular velocity of the rotating frame, r is the radius of the Earth, $\boldsymbol{B}_{\mathrm{A}}$ is the generalized momentum, s_{m} ($s_{\mathrm{m}} = C_p \ln \theta_{\mathrm{e}}$) and h_{m} ($h_{\mathrm{m}} = C_p T + L q_{\mathrm{s}}$) are the moist entropy and the moist enthalpy, respectively, θ_{e} is the equivalent potential temperature, q_{s} and T are the saturated specific humidity and the absolute temperature, respectively, C_p is the specific heat of dry air at constant pressure p, and L is the latent heat of vaporization.

With Eq. (3), equation (1) can be rewritten as

$$\begin{aligned} \psi_{\mathrm{ER}} &= \alpha \boldsymbol{B}_{\mathrm{A}} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}_{\mathrm{A}} - \boldsymbol{\nabla}\beta \times \boldsymbol{\nabla} s_{\mathrm{m}}) \\ &= \alpha \boldsymbol{B}_{\mathrm{A}} \cdot (\boldsymbol{\omega}_{\mathrm{a}} - \boldsymbol{\nabla}\beta \times \boldsymbol{\nabla} s_{\mathrm{m}}) = \psi_{\mathrm{ER1}} + \psi_{\mathrm{ER2}}, \end{aligned} \tag{6}$$

where

$$\psi_{\mathrm{ER1}} = \alpha \boldsymbol{B}_{\mathrm{A}} \cdot \boldsymbol{\omega}_{\mathrm{a}} = \alpha B_{\mathrm{Az}} (\zeta + f) + \alpha \boldsymbol{B}_{\mathrm{As}} \cdot \boldsymbol{\xi}_{\mathrm{s}}, \tag{7a}$$

$$\psi_{\mathrm{ER2}} = -\alpha \boldsymbol{B}_{\mathrm{A}} \cdot (\boldsymbol{\nabla}\beta \times \boldsymbol{\nabla} s_{\mathrm{m}}). \tag{7b}$$

Here $\boldsymbol{\omega}_{\mathrm{a}}$ is the absolute vorticity vector, and subscripts z and s represent vertical and horizontal components, respectively. For example, $(\zeta + f)$, B_{Az} and $\boldsymbol{\xi}_{\mathrm{s}}$, $\boldsymbol{B}_{\mathrm{As}}$ are vertical and horizontal components of $\boldsymbol{\omega}_{\mathrm{a}}$ and $\boldsymbol{B}_{\mathrm{A}}$, and f is the Coriolis parameter. Therefore,

$$\alpha(\zeta + f) = \frac{1}{B_{\mathrm{Az}}} [\psi_{\mathrm{ER}} + \alpha \boldsymbol{B}_{\mathrm{A}} \cdot (\boldsymbol{\nabla}\beta \times \boldsymbol{\nabla} s_{\mathrm{m}}) - \alpha \boldsymbol{B}_{\mathrm{As}} \cdot \boldsymbol{\xi}_{\mathrm{s}}]. \tag{8}$$

Using the notation $\mathrm{d}x/\mathrm{d}t = \dot{x}$, we carry out the time differential of Eq. (8) and find

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}[\alpha(\zeta + f)] \\ &= -\frac{\zeta_{\alpha z}}{B_{\mathrm{Az}}} \dot{B}_{\mathrm{Az}} + \frac{1}{B_{\mathrm{Az}}} \frac{\mathrm{d}}{\mathrm{d}t} [\alpha \boldsymbol{B}_{\mathrm{A}} \cdot (\boldsymbol{\nabla}\beta \times \boldsymbol{\nabla} s_{\mathrm{m}})] \\ &\quad - \frac{1}{B_{\mathrm{Az}}} (\dot{\boldsymbol{\xi}}_{\alpha \mathrm{s}} \cdot \boldsymbol{B}_{\mathrm{As}} + \boldsymbol{\xi}_{\alpha \mathrm{s}} \cdot \dot{\boldsymbol{B}}_{\mathrm{As}}), \end{aligned} \tag{9}$$

where $\boldsymbol{\xi}_{\alpha \mathrm{s}} = \alpha \boldsymbol{\xi}_{\mathrm{s}}$ and $\zeta_{\alpha z} = \alpha(\zeta + f)$. From continuity equation $\mathrm{d}\rho/\mathrm{d}t + \rho \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\zeta + f) + (\zeta + f) \boldsymbol{\nabla} \cdot \boldsymbol{v} = \rho \frac{\mathrm{d}}{\mathrm{d}t} [\alpha(\zeta + f)]. \tag{10}$$

Let

$$F(\zeta) = \frac{\mathrm{d}}{\mathrm{d}t}(\zeta + f) + (\zeta + f) \boldsymbol{\nabla} \cdot \boldsymbol{v}, \tag{11}$$

from Eqs. (9)–(11), we have

$$F(\zeta) = \frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}[\alpha(\zeta+f)]$$
$$= -\frac{1}{\alpha B_{\mathrm{Az}}}[\zeta_{\alpha z}\dot{B}_{\mathrm{Az}} + \boldsymbol{\xi}_{\alpha\mathrm{s}}\cdot\dot{\boldsymbol{B}}_{\mathrm{As}} + \dot{\boldsymbol{\xi}}_{\alpha\mathrm{s}}\cdot\boldsymbol{B}_{\mathrm{As}} + \dot{\psi}_{\mathrm{ER2}}]. \quad (12)$$

To eliminate the time derivatives, terms $\dot{\psi}_{\mathrm{ER2}}$, $\dot{B}_{\mathrm{Az}}$, $\dot{\boldsymbol{B}}_{\mathrm{As}}$, and $\dot{\boldsymbol{\xi}}_{\alpha\mathrm{s}}$ are derived (given in Appendixes A–C) respectively. From Eq. (A6) and Eqs. (B5) and (B6), we have

$$\dot{\psi}_{\mathrm{ER2}} = -\alpha\boldsymbol{B}_{\mathrm{A}}\cdot\boldsymbol{\nabla}T\times\boldsymbol{\nabla}s_{\mathrm{m}}, \quad (13)$$
$$\frac{\mathrm{d}B_{\mathrm{Az}}}{\mathrm{d}t} = -\frac{\partial\boldsymbol{v}}{\partial z}\cdot\boldsymbol{B}_{\mathrm{A}}, \quad (14)$$
$$\frac{\mathrm{d}\boldsymbol{B}_{\mathrm{As}}}{\mathrm{d}t} = -\frac{\partial\boldsymbol{v}}{\partial s}\cdot\boldsymbol{B}_{\mathrm{A}}. \quad (15)$$

With Eqs. (13)–(15), equation (12) becomes

$$F(\zeta) = -\frac{1}{\alpha B_{\mathrm{Az}}}\left[\zeta_{\alpha z}\left(-\frac{\partial\boldsymbol{v}}{\partial z}\cdot\boldsymbol{B}_{\mathrm{A}}\right) + \boldsymbol{\xi}_{\alpha\mathrm{s}}\cdot\left(-\frac{\partial\boldsymbol{v}}{\partial s}\cdot\boldsymbol{B}_{\mathrm{A}}\right) + \dot{\boldsymbol{\xi}}_{\alpha\mathrm{s}}\cdot\boldsymbol{B}_{\mathrm{As}} - \alpha\boldsymbol{B}_{\mathrm{A}}\cdot\boldsymbol{\nabla}T\times\boldsymbol{\nabla}s_{\mathrm{m}}\right]$$
$$= \frac{1}{\alpha B_{\mathrm{Az}}}[(\alpha\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla})\boldsymbol{v}\cdot\boldsymbol{B}_{\mathrm{A}}] - \frac{1}{\alpha B_{\mathrm{Az}}}\{\dot{\boldsymbol{\xi}}_{\alpha\mathrm{s}}\cdot\boldsymbol{B}_{\mathrm{As}} - \alpha\boldsymbol{B}_{\mathrm{A}}\cdot\boldsymbol{\nabla}T\times\boldsymbol{\nabla}s_{\mathrm{m}}\}$$
$$= (\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla})w + \frac{1}{B_{\mathrm{Az}}}[(\omega_{\mathrm{a}}\cdot\boldsymbol{\nabla})\boldsymbol{v}_{\mathrm{s}}\cdot\boldsymbol{B}_{\mathrm{As}}] - \frac{1}{\alpha B_{\mathrm{Az}}}\{\dot{\boldsymbol{\xi}}_{\alpha\mathrm{s}}\cdot\boldsymbol{B}_{\mathrm{As}} - \alpha\boldsymbol{B}_{\mathrm{A}}\cdot\boldsymbol{\nabla}T\times\boldsymbol{\nabla}s_{\mathrm{m}}\}. \quad (16)$$

Equations (12) and (16) are the different forms of the ERI-based thermally-coupled vorticity equation. From Eq. (16), the vertical vorticity change is constrained by the vertical velocity term (a term), the term associated with slope $|\boldsymbol{B}_{\mathrm{As}}/B_{\mathrm{Az}}|$ of the generalized momentum surface (b term), the term related to the horizontal vorticity change $\dot{\boldsymbol{\xi}}_{\alpha\mathrm{s}}$ (c term), and the baroclinic or solenoid term (d term).

The relative importance of the terms in Eq. (16) can be estimated. For example, for the a and b terms,

$$|b:a| = \left|\frac{1}{B_{\mathrm{Az}}}[(\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla})\boldsymbol{v}_{\mathrm{s}}\cdot\boldsymbol{B}_{\mathrm{As}} : (\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla})w]\right| = \left|\frac{\boldsymbol{B}_{\mathrm{As}}}{B_{\mathrm{Az}}}\right|\left|\frac{\boldsymbol{v}_{\mathrm{s}}}{w}\right|,$$

and $|b:a|$ is associated with slope $|\boldsymbol{B}_{\mathrm{As}}/B_{\mathrm{Az}}|$ of the generalized momentum surface. Its magnitude is determined by the system scale. Note that the notation : denotes a ratio, which is introduced to estimate the relative importance of the terms and similar to Wu's usage.[1] Another example, for the b and c terms,

$$|b:c| = \left|\frac{1}{B_{\mathrm{Az}}}[(\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla})\boldsymbol{v}_{\mathrm{s}}\cdot\boldsymbol{B}_{\mathrm{As}}] : \frac{1}{\alpha B_{\mathrm{Az}}}[\dot{\boldsymbol{\xi}}_{\alpha\mathrm{s}}\cdot\boldsymbol{B}_{\mathrm{As}}]\right| = \left|\alpha(\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla})\boldsymbol{v}_{\mathrm{s}} : \dot{\boldsymbol{\xi}}_{\alpha\mathrm{s}}\right|.$$

With the derivation of the $\dot{\boldsymbol{\xi}}_{\alpha\mathrm{s}}$ term (Eq. (C4)), we have

$$\dot{\boldsymbol{\xi}}_{\alpha\mathrm{s}} = \alpha[(\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla})\boldsymbol{v}_{\mathrm{s}} + \boldsymbol{N}_{\mathrm{s}}].$$

Therefore, whether terms b and c have the same order of magnitude is determined by the ratio between $(\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla})\boldsymbol{v}_{\mathrm{s}}$ and $\boldsymbol{N}_{\mathrm{s}}$. For different scale systems, the calculation in case studies will further show which term is larger and plays the more important role in limiting and affecting the vorticity development. However, this study just focuses on the theory part. We need to derive the ERI-based thermally-coupled vorticity equation as the first step.

3. Further transformation of ERI-based thermally-coupled vorticity equation

To conveniently compare the traditional vorticity equation with the new ERI-based thermally-coupled vorticity equation, the latter is further transformed. Both terms b and c are associated with the slope $|\boldsymbol{B}_{\mathrm{As}}/B_{\mathrm{Az}}|$ of the generalized momentum surface. Could they be integrated and simplified?

We have

$$b+c = [\alpha(\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla})\boldsymbol{v}_{\mathrm{s}} - \dot{\boldsymbol{\xi}}_{\alpha\mathrm{s}}]\cdot\frac{\boldsymbol{B}_{\mathrm{As}}}{\alpha B_{\mathrm{Az}}}. \quad (17)$$

Substituting Eq. (C4) into Eq. (17) yields

$$b+c = \{\alpha(\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla})\boldsymbol{v}_{\mathrm{s}} - \alpha[(\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla})\boldsymbol{v}_{\mathrm{s}} + \boldsymbol{N}_{\mathrm{s}}]\}\cdot\frac{\boldsymbol{B}_{\mathrm{As}}}{\alpha B_{\mathrm{Az}}} = -\boldsymbol{N}_{\mathrm{s}}\cdot\frac{\boldsymbol{B}_{\mathrm{As}}}{B_{\mathrm{Az}}}.$$

Therefore, equation (16) becomes

$$F(\zeta) = (\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla})w + \frac{1}{B_{\mathrm{Az}}}[(\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla})\boldsymbol{v}_{\mathrm{s}}\cdot\boldsymbol{B}_{\mathrm{As}}] - \frac{1}{\alpha B_{\mathrm{Az}}}\{\dot{\boldsymbol{\xi}}_{\alpha\mathrm{s}}\cdot\boldsymbol{B}_{\mathrm{As}} - \alpha\boldsymbol{B}_{\mathrm{A}}\cdot\boldsymbol{\nabla}T\times\boldsymbol{\nabla}s_{\mathrm{m}}\}$$
$$= (\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla})w - \boldsymbol{N}_{\mathrm{s}}\cdot\frac{\boldsymbol{B}_{\mathrm{As}}}{B_{\mathrm{Az}}} + \frac{1}{B_{\mathrm{Az}}}(\boldsymbol{B}_{\mathrm{A}}\cdot\boldsymbol{\nabla}T\times\boldsymbol{\nabla}s_{\mathrm{m}}). \quad (18)$$

Now, the time tendency completely vanishes in the ERI-based thermally-coupled vorticity equation.

For the last term in Eq. (18), because

$$\boldsymbol{\nabla}s_{\mathrm{m}} = \frac{1}{T}\boldsymbol{\nabla}(C_pT + Lq_{\mathrm{s}}) - \frac{R}{p}\boldsymbol{\nabla}p = \frac{1}{T}\boldsymbol{\nabla}h_{\mathrm{m}} - \frac{\alpha}{T}\boldsymbol{\nabla}p,$$

it becomes

$$\frac{1}{B_{\mathrm{Az}}}\boldsymbol{B}_{\mathrm{A}}\cdot\left\{\boldsymbol{\nabla}T\times\left[\frac{1}{T}\boldsymbol{\nabla}(C_pT + Lq_{\mathrm{s}}) - \frac{R}{p}\boldsymbol{\nabla}p\right]\right\}$$
$$= \frac{1}{B_{\mathrm{Az}}}\boldsymbol{B}_{\mathrm{A}}\cdot\left(\frac{L}{T}\boldsymbol{\nabla}T\times\boldsymbol{\nabla}q_{\mathrm{s}} - \boldsymbol{\nabla}\alpha\times\boldsymbol{\nabla}p\right)$$
$$= \frac{1}{B_{\mathrm{Az}}}\boldsymbol{B}_{\mathrm{A}}\cdot\left(\frac{L}{T}\boldsymbol{\nabla}T\times\boldsymbol{\nabla}q_{\mathrm{s}} + \boldsymbol{N}\right)$$
$$= \frac{1}{B_{\mathrm{Az}}}\boldsymbol{B}_{\mathrm{A}}\cdot\left(\frac{L}{T}\boldsymbol{\nabla}T\times\boldsymbol{\nabla}q_{\mathrm{s}}\right) + N_{\mathrm{z}} + \frac{1}{B_{\mathrm{Az}}}\boldsymbol{B}_{\mathrm{As}}\cdot\boldsymbol{N}_{\mathrm{s}}. \quad (19)$$

With Eq. (19), equation (18) is further simplified as

$$F(\zeta) = (\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla})w - \boldsymbol{N}_{\mathrm{s}}\cdot\frac{\boldsymbol{B}_{\mathrm{As}}}{B_{\mathrm{Az}}} + \left[\frac{1}{B_{\mathrm{Az}}}\boldsymbol{B}_{\mathrm{A}}\cdot\left(\frac{L}{T}\boldsymbol{\nabla}T\times\boldsymbol{\nabla}q_{\mathrm{s}}\right)\right.$$

$$+N_z+\frac{1}{B_{Az}}\boldsymbol{B}_{As}\cdot\boldsymbol{N}_s\Big]$$
$$=(\boldsymbol{\omega}_a\cdot\boldsymbol{\nabla})w+\left[N_z+\frac{1}{B_{Az}}\boldsymbol{B}_A\cdot\left(\frac{L}{T}\boldsymbol{\nabla}T\times\boldsymbol{\nabla}q_s\right)\right]. \quad (20)$$

Because $q_s=f(p,T)$, we have

$$\begin{aligned}
&\frac{1}{B_{Az}}\boldsymbol{B}_A\cdot\left(\frac{L}{T}\boldsymbol{\nabla}T\times\boldsymbol{\nabla}q_s\right)\\
&=\frac{1}{B_{Az}}\boldsymbol{B}_A\cdot\left[\frac{L}{T}\boldsymbol{\nabla}T\times f'(p,T)(\boldsymbol{\nabla}p+\boldsymbol{\nabla}T)\right]\\
&=\frac{1}{B_{Az}}\boldsymbol{B}_A\cdot\left[\frac{L}{T}f'(p,T)(\boldsymbol{\nabla}T\times\boldsymbol{\nabla}p)\right]\\
&=\frac{1}{B_{Az}}\boldsymbol{B}_A\cdot\left[\frac{L}{T}\frac{p}{R}f'(p,T)(\boldsymbol{\nabla}\alpha\times\boldsymbol{\nabla}p)\right]\\
&=-\frac{1}{B_{Az}}\boldsymbol{B}_A\cdot\left[\frac{L}{T}\frac{p}{R}f'(p,T)\boldsymbol{N}\right]\\
&=-\frac{L}{T}\frac{p}{R}f'(p,T)N_z-\frac{L}{T}\frac{p}{R}f'(p,T)\frac{1}{B_{Az}}\boldsymbol{B}_{As}\cdot\boldsymbol{N}_s.
\end{aligned}$$

Thus equation (20) turns into

$$\begin{aligned}
F(\zeta)&=(\boldsymbol{\omega}_a\cdot\boldsymbol{\nabla})w+\left[N_z+\frac{1}{B_{Az}}\boldsymbol{B}_A\cdot\left(\frac{L}{T}\boldsymbol{\nabla}T\times\boldsymbol{\nabla}q_s\right)\right]\\
&=(\boldsymbol{\omega}_a\cdot\boldsymbol{\nabla})w+N_z-\frac{L}{T}\frac{p}{R}f'(p,T)N_z\\
&\quad-\frac{L}{T}\frac{p}{R}f'(p,T)\frac{1}{B_{Az}}B_{As}\cdot\boldsymbol{N}_s\\
&=(\boldsymbol{\omega}_a\cdot\boldsymbol{\nabla})w+\lambda_1N_z+\lambda_2\frac{1}{B_{Az}}\boldsymbol{B}_{As}\cdot\boldsymbol{N}_s, \qquad (21)
\end{aligned}$$

with $\lambda_1=1+\lambda_2$ and $\lambda_2=-L\rho f'(p,T)$.

Equations (18), (20), and (21) are the further transformed forms of the new ERI-based thermally-coupled vorticity equations (12) and (16).

4. Comparison between ERI-based thermally-coupled vorticity equation and the traditional vorticity equation

The traditional frictionless vertical vorticity equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}(\zeta+f)+(\zeta+f)\boldsymbol{\nabla}\cdot v=(\boldsymbol{\omega}_a\cdot\boldsymbol{\nabla})w+N_z, \qquad (22)$$

where

$$N_z=-\left(\frac{\partial\alpha}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial\alpha}{\partial y}\frac{\partial p}{\partial x}\right)$$

is the vertical component of the baroclinic or solenoid term $\boldsymbol{N}$, which reflects the baroclinicity of the atmosphere. Term

$$(\boldsymbol{\omega}_a\cdot\boldsymbol{\nabla})w=(f+\zeta)\frac{\partial w}{\partial z}+\left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\frac{\partial w}{\partial x}\right)$$

is the sum of the vertical stretching term and twisting term.

Comparing the traditional vorticity equation (22) and the new ERI-based thermally-coupled vorticity equation (20), terms a and d_2 ($d_2=N_z$) in ERI-based thermally-coupled vorticity equation (20) are the same as the first and the second terms on the right hand side of Eq. (22). Therefore, the main difference between the two equations is

$$d_1=\frac{1}{B_{Az}}\boldsymbol{B}_A\cdot\left(\frac{L}{T}\boldsymbol{\nabla}T\times\boldsymbol{\nabla}q_s\right)$$

in Eq. (20). The d_1 denotes the mixed product among the saturated specific humidity gradient, the temperature gradient, and the generalized momentum. In the Cartesian coordinate system, for convenience, several conditions are assumed: 1) the y (x) axis points towards north (east); 2) the temperature decreases from south to north; 3) the generalized momentum vector is oriented in the zonal direction; 4) the moisture decreases from lower to upper troposphere and from south to north. These assumptions are easily satisfied in real atmosphere. Thus $-\boldsymbol{\nabla}T$ (along the y axis), $\boldsymbol{B}_A$ (along the x axis), and $-\boldsymbol{\nabla}q_s$ (slantwise upward) constitute an oblique hexahedron in the Cartesian coordinate system. The d_1 has a clear physical meaning with its magnitude denoting the volume of the oblique hexahedron. The direction of d_1 depends on the configuration of three component vectors. Large gradients of moisture and temperature, and generalized momentum are the necessary conditions to derive a strong signal of d_1. However, if certain values of large gradients of moisture and temperature, and generalized momentum are assumed, d_1 may be zero such as in the case when the three component vectors lie on the same horizontal plane. Only when the oblique hexahedron becomes upright and the three vectors are configured to a cube, can the volume reach its maximal value and d_1 have the strongest signal.

By combining with Eq. (21), d_1 describes the contribution of the solenoid term in Eq. (21) to the vertical vorticity development, which implicitly includes the effects of the horizontal vorticity change and the moisture factor by derivation (see $\dot{\xi}_{\alpha s}$ in Eq. (C4) and q_s in Eq. (20)). In a barotropic flow, the solenoid term in Eqs. (20) and (22) disappears. Thus equations (16) and (22) tend to coincide. By analyzing the expression of ψ_{ER} again, and replacing $\boldsymbol{B}_A$ with $\boldsymbol{\nabla}\theta_e$ in Eq. (6), it is not difficult to find that $\psi_{ER}=\alpha\boldsymbol{\nabla}\theta_e\cdot(\boldsymbol{\omega}_a-\boldsymbol{\nabla}\beta\times\boldsymbol{\nabla}s_m)=\alpha\boldsymbol{\nabla}\theta_e\cdot\boldsymbol{\omega}_a$. Thus ψ_{ER} turns into the moist potential vorticity.

5. Summary

In this study, different forms of the ERI-based thermally-coupled vorticity equation (Eqs. (12) and (16)) are derived. From the obtained new vorticity equation (16), the vertical vorticity change is constrained by the vertical velocity term (a term), the term associated with the slope of the generalized momentum surface (b term), the term related to the horizontal vorticity change (c term), and the baroclinic or solenoid term (d term).

To conveniently compare the traditional vorticity equation with the new ERI-based thermally-coupled vorticity equation, the latter is further transformed. Terms b and c can be integrated into one term since both are associated with the slope of the generalized momentum surface. Then it is eliminated by part of term d. Thus the ERI-based thermally-coupled vorticity equation is largely simplified (Eqs. (18), (20), and (21)).

This simplified version of the ERI-based thermally-coupled vorticity equation is compared with the traditional vorticity equation. The main difference between the two equations is d_1 in Eq. (20), which describes the contribution of the solenoid term to the vertical vorticity development. In a barotropic flow, the solenoid term in Eqs. (22) and (20) disappears and the two equations tend to coincide.

As for the theory part, the derivation and discussion of the ERI-based thermally-coupled vorticity equation is performed in this study. The relative importance of the terms in the ERI-based thermally-coupled vorticity equation will be estimated by real case studies in future. For severe weather with strong rotation properties such as a typhoon, whether the ERI-based thermally-coupled vorticity tendency equation can diagnose or even predict the movement and the strength variation of a typhoon will also be illuminated by calculations.

Appendix A: Derivation of $\dot{\psi}_{\mathrm{ER2}}$ in Eq. (12)

During the deriving of moist baroclinic ERI and the proof of its conservation property, the following two relations are utilized:[16]

$$\boldsymbol{v}_{\mathrm{A}} = u_{A_0}\nabla a^1 + v_{A_0}\nabla a^2 + w_{A_0}\nabla a^3 + \nabla W_{\mathrm{Am}} + \beta\nabla s_{\mathrm{m}}, \quad \text{(A1)}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\alpha J_x(q^1,q^2,q^3)] = \alpha[J_x(\dot{q}^1,q^2,q^3) + J_x(q^1,\dot{q}^2,q^3) + J_x(q^1,q^2,\dot{q}^3)], \quad \text{(A2)}$$

where $\boldsymbol{v}_{\mathrm{A}}$ is the absolute velocity derived from absolute motion equation in the moist flow by baroclinic Weber transformation,

$$\nabla a^k = \boldsymbol{i}\frac{\partial a^k}{\partial x^1} + \boldsymbol{j}\frac{\partial a^k}{\partial x^2} + \boldsymbol{k}\frac{\partial a^k}{\partial x^3},$$
$$\boldsymbol{v}_{\mathrm{A}} = \boldsymbol{i}u_A + \boldsymbol{j}v_{\mathrm{A}} + \boldsymbol{k}w_A,$$
$$\left(\frac{\partial x^1}{\partial t}\right)_L = u_A, \quad \left(\frac{\partial x^2}{\partial t}\right)_L = v_{\mathrm{A}}, \quad \left(\frac{\partial x^3}{\partial t}\right)_L = w_A,$$

and subscript 0 denotes $t = 0$. The following relations are observed: $\dot{u}_{\mathrm{A}_0} = 0$, $\dot{v}_{\mathrm{A}_0} = 0$, $\dot{w}_{\mathrm{A}_0} = 0$, $\dot{a}^1 = 0$, $\dot{a}^2 = 0$, $\dot{a}^3 = 0$.

Equation (A2) is the Lagrangian continuity equation, where $J_x(q^1,q^2,q^3) = \mathrm{Det}(\partial q^j/\partial x^i) = \nabla q^1 \cdot \nabla q^2 \times \nabla q^3$. If q^i $(i = 1,2,3)$ is an arbitrary invariant, then this version of the continuity equation involving q^i will be a great advantage. Because q^i is an arbitrary conserved field function and satisfies $\mathrm{d}q^i/\mathrm{d}t = \dot{q}^i = 0$, both the left-side term and the right-side term of Eq. (A2) become zero.[16,17] Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}[\alpha J_x(q^1,q^2,q^3)] = \frac{\mathrm{d}}{\mathrm{d}t}(\alpha\nabla q^1 \cdot \nabla q^2 \times \nabla q^3) = 0. \quad \text{(A3)}$$

Equations (2) and (A1) imply the relation

$$\begin{aligned}\boldsymbol{B}_{\mathrm{A}} &= \boldsymbol{v}_{\mathrm{A}} - \nabla W_{\mathrm{Am}} - \beta\nabla s_{\mathrm{m}} \\ &= u_{\mathrm{A}_0}\nabla a^1 + v_{\mathrm{A}_0}\nabla a^2 + w_{\mathrm{A}_0}\nabla a^3. \end{aligned} \quad \text{(A4)}$$

From Eq. (A4), we have

$$\begin{aligned}\dot{\psi}_{\mathrm{ER2}} &= -\frac{\mathrm{d}}{\mathrm{d}t}[\alpha\boldsymbol{B}_{\mathrm{A}} \cdot (\nabla\beta \times \nabla s_{\mathrm{m}})] \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}[\alpha(u_{\mathrm{A}_0}\nabla a^1 + v_{\mathrm{A}_0}\nabla a^2 \\ &\quad + w_{\mathrm{A}_0}\nabla a^3) \cdot (\nabla\beta \times \nabla s_{\mathrm{m}})]. \end{aligned} \quad \text{(A5)}$$

Based on Eqs. (A2) and (A3), the first term on the right-hand side of the above equation becomes

$$\begin{aligned}&-\frac{\mathrm{d}}{\mathrm{d}t}[\alpha(u_{\mathrm{A}_0}\nabla a^1) \cdot (\nabla\beta \times \nabla s_{\mathrm{m}})] = -\frac{\mathrm{d}}{\mathrm{d}t}[\alpha u_{\mathrm{A}_0} J_x(a^1,\beta,s_{\mathrm{m}})] \\ &= -\alpha u_{\mathrm{A}_0}[J_x(\dot{a}^1,\beta,s_{\mathrm{m}}) + J_x(a^1,\dot{\beta},s_{\mathrm{m}}) + J_x(a^1,\beta,\dot{s}_{\mathrm{m}})] \\ &= -\alpha u_{\mathrm{A}_0}[J_x(a^1,T,s_{\mathrm{m}})] \\ &= -\alpha u_{\mathrm{A}_0}\nabla a^1 \cdot (\nabla T \times \nabla s_{\mathrm{m}}). \end{aligned}$$

Similarly, for the second and the third terms of Eq. (A5), we have

$$\begin{aligned}&-\frac{\mathrm{d}}{\mathrm{d}t}[\alpha(v_{\mathrm{A}_0}\nabla a^2) \cdot (\nabla\beta \times \nabla s_{\mathrm{m}})] \\ &= -\alpha v_{\mathrm{A}_0}\nabla a^2 \cdot (\nabla T \times \nabla s_{\mathrm{m}}), \\ &-\frac{\mathrm{d}}{\mathrm{d}t}[\alpha(w_{\mathrm{A}_0}\nabla a^3) \cdot (\nabla\beta \times \nabla s_{\mathrm{m}})] \\ &= -\alpha w_{\mathrm{A}_0}\nabla a^3 \cdot (\nabla T \times \nabla s_{\mathrm{m}}). \end{aligned}$$

Therefore, equation (A5) becomes

$$\begin{aligned}\dot{\psi}_{\mathrm{ER2}} &= -\frac{\mathrm{d}}{\mathrm{d}t}[\alpha(u_{\mathrm{A}_0}\nabla a^1 + v_{\mathrm{A}_0}\nabla a^2 + w_{\mathrm{A}_0}\nabla a^3) \cdot (\nabla\beta \times \nabla s_{\mathrm{m}})] \\ &= -\alpha u_{\mathrm{A}_0}\nabla a^1 \cdot (\nabla T \times \nabla s_{\mathrm{m}}) - \alpha v_{\mathrm{A}_0}\nabla a^2 \cdot (\nabla T \times \nabla s_{\mathrm{m}}) \\ &\quad - \alpha w_{\mathrm{A}_0}\nabla a^3 \cdot (\nabla T \times \nabla s_{\mathrm{m}}) \\ &= -\alpha(u_{\mathrm{A}_0}\nabla a^1 + v_{\mathrm{A}_0}\nabla a^2 + w_{\mathrm{A}_0}\nabla a^3) \cdot (\nabla T \times \nabla s_{\mathrm{m}}) \\ &= -\alpha\boldsymbol{B}_{\mathrm{A}} \cdot (\nabla T \times \nabla s_{\mathrm{m}}), \end{aligned} \quad \text{(A6)}$$

which is the expression of $\dot{\psi}_{\mathrm{ER2}}$.

Appendix B: Derivations of $\dot{B}_{\mathrm{Az}}$ and $\dot{B}_{\mathrm{As}}$ in Eq. (12)

From Eq. (A4), we have

$$\frac{\mathrm{d}B_{\mathrm{Az}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(u_{\mathrm{A}_0}\frac{\partial a^1}{\partial x^3} + v_{\mathrm{A}_0}\frac{\partial a^2}{\partial x^3} + w_{\mathrm{A}_0}\frac{\partial a^3}{\partial x^3}\right). \quad \text{(B1)}$$

For the first term on the right-hand side of the above equation, because

$$0 = \frac{\partial}{\partial x^3}\frac{\mathrm{d}a^1}{\mathrm{d}t} = \frac{\partial}{\partial x^3}\left[\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)a^1\right]$$

$$= \left[\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\frac{\partial a^1}{\partial x^3}\right] + \frac{\partial \boldsymbol{v}}{\partial x^3}\cdot\boldsymbol{\nabla}a^1,$$

we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(u_{\mathrm{A}_0}\frac{\partial a^1}{\partial x^3}\right)\\ &= u_{\mathrm{A}_0}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial a^1}{\partial x^3}\right)\right] = u_{\mathrm{A}_0}\left[\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\frac{\partial a^1}{\partial x^3}\right]\\ &= -u_{\mathrm{A}_0}\frac{\partial \boldsymbol{v}}{\partial x^3}\cdot\boldsymbol{\nabla}a^1. \end{aligned} \tag{B2}$$

Similarly, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(v_{\mathrm{A}_0}\frac{\partial a^2}{\partial x^3}\right) = -v_{\mathrm{A}_0}\frac{\partial \boldsymbol{v}}{\partial x^3}\cdot\boldsymbol{\nabla}a^2, \tag{B3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(w_{\mathrm{A}_0}\frac{\partial a^3}{\partial x^3}\right) = -w_{\mathrm{A}_0}\frac{\partial \boldsymbol{v}}{\partial x^3}\cdot\boldsymbol{\nabla}a^3. \tag{B4}$$

Therefore, we obtain

$$\begin{aligned}\frac{\mathrm{d}B_{\mathrm{Az}}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(u_{\mathrm{A}_0}\frac{\partial a^1}{\partial x^3} + v_{\mathrm{A}_0}\frac{\partial a^2}{\partial x^3} + w_{\mathrm{A}_0}\frac{\partial a^3}{\partial x^3}\right)\\ &= -u_{\mathrm{A}_0}\frac{\partial \boldsymbol{v}}{\partial x^3}\cdot\boldsymbol{\nabla}a^1 - v_{\mathrm{A}_0}\frac{\partial \boldsymbol{v}}{\partial x^3}\cdot\boldsymbol{\nabla}a^2 - w_{\mathrm{A}_0}\frac{\partial \boldsymbol{v}}{\partial x^3}\cdot\boldsymbol{\nabla}a^3\\ &= -\frac{\partial \boldsymbol{v}}{\partial x^3}\cdot\boldsymbol{B}_{\mathrm{A}} = -\frac{\partial \boldsymbol{v}}{\partial z}\cdot\boldsymbol{B}_{\mathrm{A}}. \end{aligned} \tag{B5}$$

Similarly,

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{B}_{\mathrm{As}}}{\mathrm{d}t} &= \frac{\mathrm{d}\boldsymbol{B}_{\mathrm{A}}}{\mathrm{d}t}\cdot\boldsymbol{i} + \frac{\mathrm{d}\boldsymbol{B}_{\mathrm{A}}}{\mathrm{d}t}\cdot\boldsymbol{j}\\ &= \left(-\frac{\partial \boldsymbol{v}}{\partial x^1}\cdot\boldsymbol{B}_{\mathrm{A}}\right)\boldsymbol{i} + \left(-\frac{\partial \boldsymbol{v}}{\partial x^2}\cdot\boldsymbol{B}_{\mathrm{A}}\right)\boldsymbol{j}\\ &= -\frac{\partial \boldsymbol{v}}{\partial s}\cdot\boldsymbol{B}_{\mathrm{A}}. \end{aligned} \tag{B6}$$

Equations (B5) and (B6) give the expressions of $\dot{B}_{\mathrm{Az}}$ and $\dot{\boldsymbol{B}}_{\mathrm{As}}$, respectively.

Appendix C: Derivation of $\dot{\xi}_{\alpha\mathrm{s}}$ in Eq. (12)

The three-dimensional absolute vorticity equation is

$$\frac{\mathrm{d}\boldsymbol{\omega}_{\mathrm{a}}}{\mathrm{d}t} - (\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla})\boldsymbol{v} + (\boldsymbol{\nabla}\cdot\boldsymbol{v})\boldsymbol{\omega}_{\mathrm{a}} = \boldsymbol{N}, \tag{C1}$$

where $\boldsymbol{N} = -\boldsymbol{\nabla}\alpha\times\boldsymbol{\nabla}p$ is the solenoid term.

From Eq. (C1), the horizontal vorticity equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi}_{\mathrm{s}} = (\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla})\boldsymbol{v}_{\mathrm{s}} - (\boldsymbol{\nabla}\cdot\boldsymbol{v})\boldsymbol{\xi}_{\mathrm{s}} + \boldsymbol{N}_{\mathrm{s}}. \tag{C2}$$

Combining with the continuity equation

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \alpha\boldsymbol{\nabla}\cdot\boldsymbol{v}, \tag{C3}$$

we have

$$\begin{aligned}\dot{\xi}_{\alpha\mathrm{s}} &= \alpha\frac{\mathrm{d}\boldsymbol{\xi}_{\mathrm{s}}}{\mathrm{d}t} + \boldsymbol{\xi}_{\mathrm{s}}\frac{\mathrm{d}\alpha}{\mathrm{d}t}\\ &= \alpha[(\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla})\boldsymbol{v}_{\mathrm{s}} - (\boldsymbol{\nabla}\cdot\boldsymbol{v})\boldsymbol{\xi}_{\mathrm{s}} + \boldsymbol{N}_{\mathrm{s}}] + \boldsymbol{\xi}_{\mathrm{s}}[\alpha\boldsymbol{\nabla}\cdot\boldsymbol{v}]\\ &= \alpha[(\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla})\boldsymbol{v}_{\mathrm{s}} + \boldsymbol{N}_{\mathrm{s}}], \end{aligned} \tag{C4}$$

which is the expression of $\dot{\xi}_{\alpha\mathrm{s}}$.

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